



Mechanics — *The ephemeral nature of Navier–Stokes- $\alpha\beta$ continua*, by GIANFRANCO CAPRIZ and ELIOT FRIED, communicated on 10 December 2010.

Dedicated to Giovanni Prodi.

ABSTRACT. — Navier–Stokes- $\alpha\beta$ continua are shown to be a special subclass of the recently introduced class of ephemeral continua that arises when particular constraints and constitutive relations are introduced. Beside offering a new endorsement of balance equations already obtained by numerous Authors, our study offers a chance for enlightening remarks, remarks that might possibly lead to further developments.

KEY WORDS: Turbulence, inertia, dispersion, dissipation.

AMS SUBJECT CLASSIFICATION: 76A02, 76A05, 76D99, 76F02, 76F99.

1. INTRODUCTION

Ephemeral continua are members of the vast class of bodies for which, on principle, no single preferred placement, nor even a single preferred configuration, exists, which could be invoked as physically significant reference. The vastness of the class appears to be so great to suspect it to be almost meaningless, particularly in view of the consequent vagueness of the separation between dynamic and thermal phenomena.

The search ensues for subclasses assuring a link with well-established categories to which a right of citizenship has already been accorded in the literature—foremost hypocontinua, a compressed noun invented from two adjectives, hypoelastic and hypoplastic, already of some repute.

A remarkable catch emerges. The differential equation for stress, which in hypoelasticity and hypoplasticity is declared to be constitutive, appears now to be instead an added balanced law for a tensor of equilibrated suffusion or collision. The catch should not be too surprising, however: even in the standard continuum theory of simple materials, conversely, the law of balance of moment of momentum is already treated as if it were a constitutive prescription.

Here we make a further step towards specifics and examine the particular case of Navier–Stokes- $\alpha\beta$ continua, which appear naturally to be a subclass of ephemeral continua obeying certain constraints evident under the circumstances. The connection thus established could be deemed trivial. In fact, it is far from so. To quote only one point, the distinct origin of the dispersive and dissipative terms in the balance equations becomes now clear and can be classified strictly. Precisely,

a term previously attributed to a dispersive contribution to the extra stress is actually inertial and, thus, not constitutive in nature.

2. REPRISÉ OF THE THEORY FOR EPHEMERAL CONTINUA

In Capriz's [3] theory for ephemeral continua, each place x in the region $\mathcal{B}(t)$ occupied by a body at a time t is the mass center of a loculus $e(x, t)$ of subplaces. Aside from the conventional notions of mass density ρ and velocity v , averages with a suitably defined locular number density give rise to a symmetric and positive-definite mesoinertia tensor Y , a mesodistortion tensor, with rate B , a moment of mesomomentum tensor K , and a symmetric and positive-semidefinite mesofluctuation (or Reynolds) tensor H , all of which may depend on place and time. The spatial fields Y , B , K , and H are all measured per unit mass. The mesodistortion tensor is assumed affine, so that $B = K^\top Y^{-1}$ or, equivalently,

$$(1) \quad K = YB^\top.$$

Possible discrepancy between the macroscopic and mesoscopic disfigurements described by the velocity gradient

$$(2) \quad L = \text{grad } v$$

and the mesodistortion rate B is accompanied by suffusion of matter between loculi defined by

$$(3) \quad \sigma = \text{tr}(L - B).$$

The theory generates balance laws for mass, moment of inertia, linear momentum, moment of mesomomentum, and mesofluctuations. In pointwise form, these balance laws read

$$(4) \quad \begin{cases} \dot{\rho} + \rho \text{div } v = \sigma \rho, \\ \rho(\dot{Y} + \sigma Y - YB^\top - BY) = 0, \\ \rho(\dot{v} + \sigma v) = \rho b + \text{div } T, \\ \rho(\dot{K} + \sigma K - KB^\top - BK - H) = \rho M - A + \text{div } m, \\ \rho(\dot{H} + \sigma H - HB^\top - BH) = \rho J - Z + \text{div } j, \end{cases}$$

where a superposed dot indicates time differentiation along paths obtained by retrogression from the average velocity, the full left-hand sides contain the coshaping time derivatives, T is the familiar Cauchy stress tensor, A and Z are second-order tensorial internal supply densities associated, respectively, with moment of mesomomentum and mesofluctuations, m and j are third-order tensors, the former a hyperstress associated with the moment of mesomomentum and the latter a measure of power flux, b and M are applied or noninertial external forces, measured per unit mass, and J is the mesofluctuation supply, also measured per unit mass. In place of the classical requirement that T be symmetric, T and A must

satisfy

$$(5) \quad \text{skw } T = \text{skw } A.$$

The condition (5) can be replaced by the stronger alternative $T = -A^\top$. Here, we rest content with (5).

For a detailed justification of the balance equations (4), the reader is addressed to papers by Brocato and Capriz [2] and Capriz [3]. However, also in view of the goal of this report, we must provide at least a cursory inkling of ideas underlying those equations. Consider a time t and a place x in $\mathcal{B}(t)$. The value $v(x, t)$ of the velocity v at that time and place arises on averaging a mesoscale velocity w over all subplaces in the loculus $e(x, t)$. As such, v and w are justly termed filtered and unfiltered velocities. For the difference $w - v$, statistical mesofiltering is performed: the average being obviously null, the Euler inertia tensor Y , the moment of momentum K , and the variance H are evaluated and their laws of evolution sought. The tensor H is there to account for the intensity of collisions within each loculus, so that ρH provides a sort of anisotropic pressure—more precisely, its spherical component $\frac{1}{3}\rho(\text{tr } H)I$ takes the role of pressure while its deviatoric component $\text{dev}(\rho H) = \rho(H - \frac{1}{3}(\text{tr } H)I)$ is a sort of additional stress. Since H is symmetric and positive-semidefinite, it possesses nonnegative eigenvalues η_i , $i = 1, 2, 3$, and a corresponding orthonormal eigenbasis $\{h_1, h_2, h_3\}$. Additional insight regarding the nature of H arises on expressing it in canonical form

$$(6) \quad H = \sum_{i=1}^3 \eta_i h_i \otimes h_i,$$

which provides a caricature of the primitive definition of H based on averaging as the sum of three terms as though the population of molecules, all having the same mass, were spread between three swarms: within the i -th swarm, all molecules move along the line spanned by h_i with speed η_i ; each swarm is split evenly into two subswarms but with opposing velocities $\pm\eta_i h_i$. Alternatively, one may imagine all molecules to have not only the same mass but also the same speed, but with the fraction of those moving along the line spanned by h_i being $\eta_i/(\eta_1 + \eta_2 + \eta_3)$. With this interpretation in mind, it becomes evident that the square-root $H^{1/2}$ of H regulates the balanced cross-flux of molecules— $H^{1/2}n$ being a measure of the flux of molecules through a plane with unit normal n . Thus, with reference to (4)₄, within a loculus, the tensor $\text{sym } A$ should account for coherence opposing suffusion and dispersal, actions which are promoted, instead, by collisions.

3. CONSEQUENCES OF CONSTRAINING B TO EQUAL THE SKEW PART OF L

Imposing the constraint $B = L$ within the context in the theory of ephemeral continua leads to Capriz's [4] theory of hypocontinua. Here, we consider the stronger constraint

$$(7) \quad B = \text{skw } L = W.$$

Moreover, to procure the simplest account of effects of turbulence on flows of liquids, we supplement (7) by the classical constraint

$$(8) \quad \operatorname{tr} L = \operatorname{div} v = 0$$

of incompressibility.

To deduce the primary implications of the constraints (7) and (8), we proceed as in Capriz's [4] derivation of the theory of hypocontinua. This rests on considerations involving the internal power density, which, in the theory for ephemeral continua, has the general form

$$(9) \quad T \cdot L + A \cdot B^\top + m \cdot \operatorname{grad}(B^\top) + \frac{1}{2} \operatorname{tr} Z,$$

but which, with the constraint (7) and the standard decomposition $L = D + W$, with $D = \operatorname{sym} L$, of the velocity gradient, specializes to

$$(10) \quad (\operatorname{dev} \operatorname{sym} T) \cdot D + (\operatorname{skw} T - \operatorname{skw} A) \cdot W - m \cdot \operatorname{grad} W + \frac{1}{2} \operatorname{tr} Z.$$

Following the traditional approach to dealing with constraints, we suppose that the fields T , A , m , and Z split, additively, into active and reactive components,

$$(11) \quad T = T_a + T_r, \quad A = A_a + A_r, \quad m = m_a + m_r, \quad Z = Z_a + Z_r,$$

and we require that the internal power density obey

$$(12) \quad (\operatorname{dev} \operatorname{sym} T) \cdot D + (\operatorname{skw} T - \operatorname{skw} A) \cdot W - m \cdot \operatorname{grad} W + \frac{1}{2} \operatorname{tr} Z \\ = (\operatorname{dev} \operatorname{sym} T_a) \cdot D + (\operatorname{skw} T_a - \operatorname{skw} A_a) \cdot W - m_a \cdot \operatorname{grad} W + \frac{1}{2} \operatorname{tr} Z_a$$

for all admissible choices of D , W , and $\operatorname{grad} W$, so that the reactions T_r , A_r , m_r , and Z_r are powerless.

Since the power flux j associated with mesofluctuations does not enter the internal power (10), it seems reasonable to assume that it does not react to the imposition of any internal constraint. With this assumption, $j_r = 0$ and

$$(13) \quad j = j_a.$$

Moreover, since the deviatoric component $\operatorname{dev} Z = Z - \frac{1}{3}(\operatorname{tr} Z)I$ of Z is absent from (10), it seems reasonable to assume that it cannot include, under internally constrained circumstances, an additive reactive component. This amounts to assuming that Z_r is spherical, viz.

$$(14) \quad Z_r = \frac{1}{3}(\operatorname{tr} Z_r)I.$$

Using the decompositions (11) of T , A , m , and Z in (12) yields

$$(15) \quad (\operatorname{dev} \operatorname{sym} T_r) \cdot D + (\operatorname{skw} T_r - \operatorname{skw} A_r) \cdot W - m_r \cdot \operatorname{grad} W + \frac{1}{2} \operatorname{tr} Z_r = 0.$$

In view of (8) and (14), if $\text{tr } Z$ is independent of D , W , and $\text{grad } W$, then (15) is satisfied for all choices of D , W , and $\text{grad } W$, if and only if

$$(16) \quad \text{dev sym } T_r = 0, \quad \text{skw } T_r = \text{skw } A_r, \quad m_r = {}^t m_r, \quad \text{tr } Z_r = 0.$$

The third of (16) embodies the requirement that

$$(17) \quad m_r \cdot (a_1 \otimes a_2 \otimes a_3) = m_r \cdot (a_2 \otimes a_1 \otimes a_3)$$

for all vectors a_1 , a_2 , and a_3 (that is, that m_r be symmetric in its first pair of indices). In view of (14), the fourth of (16) implies that

$$(18) \quad Z_r = 0.$$

Consistent with its absence from (15), the symmetric part $\text{sym } A_r$ of the reaction A_r is unrestricted by (16).

A simple calculation shows that if T_r , A_r , m_r , and Z_r satisfy (16), then (15) holds for all choices of D , W , and $\text{grad } W$. To establish the converse, note that since D , W , and $\text{grad } W$ can be prescribed independently at any given point and time, (15) holds for all choices of D , W , and $\text{grad } W$ only if each of its terms vanish separately:

$$(19) \quad \begin{cases} (\text{dev sym } T_r) \cdot D = 0, & (\text{skw } T_r - \text{skw } A_r) \cdot W = 0, \\ m_r \cdot \text{grad } W = 0, & \text{tr } Z_r = 0. \end{cases}$$

To satisfy (19)₁ for all deviatoric and symmetric second-order tensors D , $\text{dev sym } T_r$ must vanish, which establishes (16)₁. To satisfy (19)₂ for all skew second-order tensors W , $(\text{skw } T_r - \text{skw } A_r) = \text{skw}(T_r - A_r)$ must be symmetric and, thus, must vanish, which establishes (16)₂. Since

$$(20) \quad (\text{grad } W) \cdot (a_1 \otimes a_2 \otimes a_3) = -(\text{grad } W) \cdot (a_2 \otimes a_1 \otimes a_3)$$

for all vectors a_1 , a_2 , and a_3 (that is, since $\text{grad } W$ is skew in its first two indices), to satisfy (19)₃ for all $\text{grad } W$ with W skew, m_r must obey (17), which establishes (16)₃. Finally, granted (14), to satisfy (19)₄, Z_r must vanish, which establishes (16)₄. The restrictions (16) are therefore both necessary and sufficient to ensure that the reactions T_r , A_r , m_r , and Z_r are powerless.

Taken together, (16)₁ and (16)₂ imply that there must exist a scalar field φ such that

$$(21) \quad T_r = -\varphi I + \text{skw } A_r.$$

Further, (16)₃ implies that $\text{skw}(\text{div } m_r) = 0$, which leads to the conclusion that

$$(22) \quad \text{div } m_r = \text{sym}(\text{div } m_r).$$

Moreover, by (11)₁ and (16)₂, the basic relation (5) reduces to

$$(23) \quad \text{skw } T_a = \text{skw } A_a.$$

4. REDUCED BALANCE LAWS

The constraints (7) and (8) require that

$$(24) \quad \sigma = 0.$$

Hence, the present subclass of ephemeral continua cannot sustain suffusion of matter between loculi. By (8) and (24), the mass balance (4)₁ reduces to $\dot{\varrho} = 0$, so that the mass density is constant along particle trajectories obtained by retrogression. Consistent with this requirement, we assume that

$$(25) \quad \varrho = \text{constant} > 0.$$

Together, (24) and (25) ensure that the mass balance (4)₁ is satisfied trivially. The requirement (24) leads immediately to a partial simplification of the remaining balances (4)₂₋₅. Somewhat more concise versions of these balances arise on noting that, for any second-order tensor field G , (7) yields

$$(26) \quad \dot{G} - GB^\top - BG = \overset{\circ}{G} + GW - WG.$$

The right side of (26) is the corotational rate of G . For brevity, we use a superposed circle to denote the corotational time differentiation, so that

$$(27) \quad \overset{\circ}{G} = \dot{G} + GW - WG.$$

For the particular choice $G = W$, (27) yields

$$(28) \quad \overset{\circ}{W} = \dot{W} + W^2 - W^2 = \dot{W},$$

whereby, on choosing $G = K$ in (27), we conclude from (1) and (7) that

$$(29) \quad \overset{\circ}{K} = -\overset{\circ}{Y}W - Y\overset{\circ}{W} = -\overset{\circ}{Y}W - Y\dot{W}.$$

In view of (13), (16), (24), (25), (27), and (29), the balances (4)₂₋₅ reduce to

$$(30) \quad \begin{cases} \overset{\circ}{Y} = 0, \\ \varrho\dot{v} = \varrho b - \text{grad } \varphi + \text{div}[T_a + \text{skw}(A_a + A_r)], \\ \varrho(Y\dot{W} + H) = -\varrho M + A_a + A_r - \text{div } m_a - \text{sym}(\text{div } m_r), \\ \varrho\overset{\circ}{H} = \varrho J - Z_a + \text{div } j_a, \end{cases}$$

where (21) and (23) have been used to express T in the form

$$(31) \quad \begin{aligned} T &= T_a + T_r = T_a - \varphi I + \text{skw } A_r \\ &= -\varphi I + \text{sym } T_a + \text{skw } T_a + \text{skw } A_r \\ &= -\varphi I + \text{sym } T_a + \text{skw}(A_a + A_r) \end{aligned}$$

and (30)₁ has been used to reduce the expression (29) for $\overset{\circ}{K}$ to

$$(32) \quad \overset{\circ}{K} = -Y\dot{W}.$$

The balance (30)₃ can be decomposed into symmetric and skew components

$$(33) \quad \varrho(\text{sym}(Y\dot{W}) + H) = -\varrho \text{sym} M + \text{sym}(A_a + A_r) - \text{sym}[\text{div}(m_r + m_a)]$$

and

$$(34) \quad \varrho \text{sym}(Y\dot{W}) = -\varrho \text{skw} M + \text{skw}(A_r + A_a) + \text{skw}(\text{div} m_a).$$

Given v , Y , M , A_a , and m_a , (34) determines $\text{skw} A_r$ while (33) determines a gauge relation for $\text{sym}(A_r - \text{div} m_r)$. Thus, (30)₃ is inconsequential to the present specialization of the theory for ephemeral continua. In view of (23), the skew component $\text{skw} T_a$ of the active component T_a the Cauchy stress is also irrelevant to the theory.

However, solving (34) for $\text{skw}(A_a + A_r)$, substituting the result into (30)₂, and invoking the assumption (25) that the mass density ϱ is constant, reduces the balance of linear momentum to the form

$$(35) \quad \varrho[\dot{v} - \frac{1}{2} \text{div}(Y\dot{W} + \dot{W}Y)] = \varrho[b + \text{div}(\text{skw} M)] - \text{grad} \varphi + \text{div}[\text{sym} T_a + \text{skw}(\text{div} m_a)],$$

which, aside from the reactive pressure φ needed to ensure satisfaction of the constraint (8), shows no influence of the reaction A_r . Importantly, (35) is also independent of the active contribution A_a to A .

In summary, the general balances for an ephemeral continuum constrained according to (7) and (8) are

$$(36) \quad \begin{cases} \overset{\circ}{Y} = 0, \\ \varrho[\dot{v} - \frac{1}{2} \text{div}(Y\dot{W} + \dot{W}Y)] \\ \quad = \varrho[b + \text{div}(\text{skw} M)] - \text{grad} \varphi + \text{div}[\text{sym} T_a + \text{skw}(\text{div} m_a)], \\ \varrho\overset{\circ}{H} = \varrho J - Z_a + \text{div} j_a. \end{cases}$$

Interestingly the balances (36)_{1,2} become decoupled from balance (36)₃ if, merely, the constitutive rules for T_a and m_a fail to involve H . Such absence would be too restrictive, in general, because, surely, the intensity of collisions is bound to influence these tensors of stress and hyperstress. However, for those two tensors, one accepts now the corresponding constitutive laws valid in general, but with the single insertion in them of $\text{skw} L$ for B and the null scalar for $\text{div} v$. In the consequent absence of volume changes, suffusion, and so on, collisions may end up by being expended only via the reactive pressure, and the decoupling ensues.

On the contrary, the balance (36)₃ cannot escape the influence of the gross motion as the corotational rate of H involves the velocity field v and the spin

tensor W . A study of some special flows and the search for simple corollaries pursued after a choice of tentative constitutive laws for Z and j would help clarify doubts.

5. NAVIER–STOKES- $\alpha\beta$ CONTINUA

Provided that Y , T_a , and m_a have the forms

$$(37) \quad Y = 2\alpha^2 I, \quad \text{sym } T_a = 2\rho\nu D, \quad m_a = -2\rho\nu\beta^2 \text{grad } W,$$

where $\alpha > 0$ and $\beta > 0$ are constants with dimensions of length and $\nu > 0$ is the kinematic viscosity, the balances (30)_{1,2} reduce to the equation

$$(38) \quad \rho(\dot{v} - 2\alpha^2 \text{div } \overset{\circ}{D}) = \rho f - \text{grad } p + \rho\nu(1 - \beta^2 \Delta)\Delta v$$

governing the flow of a Navier–Stokes- $\alpha\beta$ continuum. In (38),

$$(39) \quad f = b + \text{div}(\text{skw } M)$$

denotes the effective body force, per unit mass, and

$$(40) \quad p = \varphi - \rho\alpha^2 \text{grad tr}(L^2)$$

denotes the effective pressure.

We verify the foregoing assertion in steps. First, we consider the implications of assuming that the moment of inertia tensor Y is as given by (37)₁, with α constant. By (27),

$$(41) \quad \overset{\circ}{Y} = 2\alpha^2 \overset{\circ}{I} = 2\alpha^2(\dot{I} + IW - WI) = 2\alpha^2(W - W) = 0,$$

so that (30)₁ is satisfied trivially. Further, since

$$(42) \quad 2 \text{div } \dot{W} = \overset{\cdot}{\Delta} \bar{v} + \text{div}(LL^\top) - \frac{1}{2} \text{grad tr}(L^2)$$

and

$$(43) \quad 2 \text{div } \overset{\circ}{D} = \overset{\cdot}{\Delta} \bar{v} + \text{div}(LL^\top) + \frac{1}{2} \text{grad tr}(L^2),$$

it follows that, for Y as given by (37)₁, with α constant,

$$(44) \quad \begin{aligned} \frac{1}{2} \text{div}(Y\dot{W} + \dot{W}Y) &= \alpha^2 \text{div}(I\dot{W} + \dot{W}I) \\ &= 2\alpha^2 \text{div } \dot{W} \\ &= 2\alpha^2 \text{div } \overset{\circ}{D} - \alpha^2 \text{grad tr}(L^2). \end{aligned}$$

Next, for T_a given by (37)₂, with ρ constant,

$$(45) \quad \text{div}(\text{sym } T_a) = 2\rho\nu \text{div } D = \rho\nu\Delta v,$$

while, for m_a given by (37)₃, with ϱ and β constant,

$$\begin{aligned}
 (46) \quad \operatorname{div}[\operatorname{skw}(\operatorname{div} m_a)] &= \operatorname{div}(\operatorname{div} m_a) \\
 &= -2\varrho v\beta^2 \operatorname{div}(\Delta W) \\
 &= -\varrho v\beta^2 [\Delta\Delta v - \operatorname{grad}(\Delta \operatorname{div} v)] \\
 &= -\varrho v\beta^2 \Delta\Delta v.
 \end{aligned}$$

Finally, on using (44)–(46) in (30) and recalling the definition (40) of p , we obtain the flow equation (38).

6. SPECIALIZATION: EULER- α AND NAVIER–STOKES- α CONTINUA

When the kinematic viscosity ν vanishes, so that (37)₂ and (37)₃ become $T_a = 0$ and $m_a = 0$, (30)_{1,2} and (37) yield the flow equation

$$(47) \quad \varrho(\dot{v} - 2\alpha^2 \operatorname{div} \overset{\circ}{D}) = \varrho f - \operatorname{grad} p$$

for an Euler- α continuum. Further, on choosing $\beta = \alpha$ in (37)₃, (30) and (37) yield the flow equation

$$(48) \quad \varrho(\dot{v} - 2\alpha^2 \operatorname{div} \overset{\circ}{D}) = \varrho f - \operatorname{grad} p + \varrho v(1 - \alpha^2 \Delta)\Delta v$$

for a Navier–Stokes- α continuum. More directly, (38) yields (47) when $\nu = 0$ and (48) when $\beta = \alpha$.

The Euler- α equation (47) was first introduced in the Euler–Poincaré variational framework of Holm, Marsden and Ratiu [12, 13]. Alternatively, Holm [14] showed that (47) can be obtained by applying Lagrangian averaging to the Euler equations and invoking a closure based on Taylor’s [19] hypothesis that small rapid fluctuations convect with the mean flow. The equations for Euler- α are ordinarily encountered as a system,

$$\begin{aligned}
 (49) \quad \varrho(u' + (\operatorname{grad} u)v + (\operatorname{grad} v)^\top u) &= -\operatorname{grad} \varpi, \\
 u &= (1 - \alpha^2)v,
 \end{aligned}$$

where a prime is used to denote spatial time-differentiation and ϖ is an effective pressure determined by p , v , and D via

$$(50) \quad \varpi = p - \frac{1}{2}\varrho(|v|^2 + \alpha^2|D|^2).$$

The Navier–Stokes- α equation (48) was first proposed by Chen, Foias, Holm, Olson, Titi, and Wynne [6, 7], who obtained it by augmenting the Euler- α equation with Navier–Stokes viscosity so that (49)₁ becomes

$$(51) \quad \varrho(u' + (\operatorname{grad} u)v + (\operatorname{grad} v)^\top u) = -\operatorname{grad} \varpi + \varrho\nu\Delta v.$$

7. INHOMOGENEOUS AND ANISOTROPIC GENERALIZATIONS

A straightforward generalization of Navier–Stokes- $\alpha\beta$ equation arises on relaxing the assumption that α and β are independent of place and time. If α is a field, then (41) is replaced by

$$(52) \quad \overset{\circ}{Y} = 2(2\alpha\dot{\alpha}I + \alpha^2IW - \alpha^2WI) = 2\alpha(2\dot{\alpha}I + \alpha W - \alpha W) = 2\alpha\dot{\alpha}I,$$

whereby (30)₁ yields

$$(53) \quad \dot{\alpha} = 0,$$

so that α is convected with the mean flow. If β is also a field, then using (37) in (30)₂ yields the inhomogeneous generalization

$$(54) \quad \begin{aligned} \varrho(\dot{v} - 2\alpha^2 \operatorname{div} \overset{\circ}{D} - 4\alpha\overset{\circ}{D} \operatorname{grad} \alpha) \\ = \varrho f - \operatorname{grad} p + \varrho v(1 - \beta^2 \Delta)\Delta v - 2\varrho v\beta(\Delta L) \operatorname{grad} \beta \\ - \varrho v\beta(\Delta L)^\top \operatorname{grad} \beta - 2\varrho v(\operatorname{grad} W)(\operatorname{grad} \operatorname{grad} \beta + \operatorname{grad} \beta \otimes \operatorname{grad} \beta) \end{aligned}$$

of the flow equation (38) for Navier–Stokes- $\alpha\beta$ continua. When the kinematic viscosity ν vanishes, (54) is equivalent to an inhomogeneous Euler- α equation derived by Marsden and Shkoller [17] and Holm [15] on the basis of Taylor’s [19] hypothesis.

Finally, a mildly anisotropic generalization of the Navier–Stokes- $\alpha\beta$ equation arises on allowing Y to evolve according to (30)₁. In this case, granted that $\operatorname{sym} T_a$ and m_a have the simple isotropic forms given in (37)₂ and (37)₃, (30)₂ yields

$$(55) \quad \varrho[\dot{v} - \frac{1}{2} \operatorname{div}(Y\dot{W} + \dot{W}Y)] = \varrho f - \operatorname{grad} p + \varrho v(1 - \beta^2 \Delta)\Delta v.$$

Any anisotropy present in (55) is solely due to the possible asphericity of the inertia tensor Y . A more general equation would arise by replacing (37)₃ by an anisotropic relation for m_a in terms of $\operatorname{grad} W$.

In their anisotropic generalization of the Navier–Stokes- α equation (48), Marsden and Shkoller [17] consider a covariance tensor F and find that this tensor obeys the evolution equation

$$(56) \quad \dot{F} - LF - FL^\top = 0.$$

That is, the Oldroyd, or upper-convected, rate of F must vanish. Although F is inherently symmetric, Marsden and Shkoller [17] allow for the possibility that it may vanish at boundaries. Thus, F need not be positive-definite. If, nevertheless, F is identified with the moment of inertia tensor Y from the theory of ephemeral continua, comparison of (30)₁ and (56) demonstrates that the anisotropic evolu-

tion equations obtained on the basis of the constraints (7) and (8) are not as general as those obtained by Marsden and Shkoller [17]. Interestingly, Capriz's [4] theory of hypocontinua requires that Oldroyd rate of the moment of inertia tensor vanish. It therefore seems possible that the theory of hypocontinua might encompass the anisotropic generalization of the Navier–Stokes- α equations obtained by Marsden and Shkoller [17]. In contrast to Marsden and Shkoller [17], Holm [15] provides an anisotropic generalization of the Navier–Stokes- α equation in which the covariance tensor is convected with the mean flow.

The paper of Marsden and Shkoller [17] calls for some comments on point of view, beyond a mere endeavour to check coincidences or justify minor discrepancies in the final equation. In the Introduction to that paper, the closure problem for the Reynolds stress H is quoted and, as per tradition, translated into a constitutive issue: the expression for H in terms of gradients of the mean velocity field. In the theory of ephemeral continua, the matter is argued differently and a separate balance equation is proposed for H . Of course, the, perhaps deeper, problem then arises of deciding on laws for vigour of the corresponding supply and power flux; but the new setting might even help the imagination. In any case the problem is bypassed here by a decoupling of balance equations. Curiously, the research path crosses now a different one pursued in hypoelasticity, where the constitutive equation for stress (Reynolds or otherwise) is presumed to be, midway, differential, with the final outcome depending on initial data. The second question highlighted in the Introduction of Marsden and Shkoller [17] is the possible synergy, rather than contrast, between two approaches, Lagrangian versus Eulerian (whatever the faults be of that terminology, faults exposed by Truesdell [21, Footnote 2 of Chapter II] in his precise historical investigations). In introducing ephemeral continua, the more modest technique of retrogression (in the vocabulary of Truesdell and Muncaster [20]) is shown to suffice and is conveniently linked with Euler moment of inertia tensor, for which conservation laws are, for the most part, standard.

8. DISTINCTION BETWEEN INERTIAL AND KINETIC TERMS IN THE NAVIER–STOKES- $\alpha\beta$ EQUATION

Fried and Gurtin [11] derived the flow equation (38) for Naver–Stokes- $\alpha\beta$ continua based on a theory for fluids with higher-order gradient dependencies. That theory yields a linear-momentum balance of the form

$$(57) \quad \rho \dot{v} = \rho b - \text{grad } p + \text{div } S + \text{curl div } G,$$

where $S = \text{dev } T = T - \frac{1}{3}(\text{tr } T)I = T + pI$ is the deviatoric component of the symmetric Cauchy stress T and G is a traceless second-order hyperstress, with S and G being power-conjugate to D and $\text{grad curl } v$, respectively. To arrive at (38) on the basis of (57), Fried and Gurtin [11] chose

$$(58) \quad S = 2\rho\nu D + 2\rho\alpha^2 \overset{\circ}{D}, \quad G = \rho\nu\beta^2(\text{curl curl } v + \gamma(\text{curl curl } v)^\top),$$

where, to ensure that the internal dissipation be nonnegative, ν and γ must obey

$$(59) \quad \nu \geq 0, \quad -1 \leq \gamma \leq 1.$$

The expression (58)₁ defines the extra stress of a fluid of second grade. Whereas the Newtonian contribution to that expression is dissipative, the remaining, non-Newtonian, contribution is dispersive. As Dunn and Fosdick [9] show, the latter contribution stems from a specific free-energy ψ which, up to an indeterminate additive constant, must have the form

$$(60) \quad \psi = \alpha^2 |D|^2.$$

Indeed, Fried and Gurtin [11] impose a free-energy inequality with local form $\rho \dot{\psi} - S \cdot D - G \cdot \text{grad curl } v \leq 0$. Granted (59), the choices (58) and (60) guarantee satisfaction of this inequality in all processes. In this sense, said choices are thermodynamically compatible.

Although the flow equation that arises on combining the momentum balance (57) and the constitutive relations (58) for S and G is indeed the Navier–Stokes- $\alpha\beta$ equation (38), the role of S in (57) identifies its origin as constitutive—that is, as related to collisions between molecules. In contrast, the derivation presented here shows that the dispersive term entering the Navier–Stokes- $\alpha\beta$ equation has an inertial origin.

The foregoing interpretation is consistent with observations made by Fried and Gurtin [11, 10], who associated the length scale α entering the dispersive contribution to their extra stress S with a characteristic measure of eddy sizes in the, dissipationless, inertial range of the turbulent energy cascade and the length scale β entering the wholly dissipative hyperstress G with a characteristic measure of eddy scales in the dissipation range. These interpretations stem from heuristic reasoning based on the observation that the dispersive contribution to S is generated by the potential $\rho\psi$. This leads Fried and Gurtin [11, 10] to interpret $\rho\psi$ as a measure of turbulent kinetic-energy. Analytical and numerical support for said heuristics are provided by Chen and Fried [8] and Kim, Cassiani, Albertson, Dolbow, Fried, and Gurtin [16], respectively.

9. RELATION TO THE THEORY FOR CONTINUA WITH AFFINE MICROSTRUCTURE

The balance equations (4) look, formally, not very different from ones that are proposed for continua with affine microstructure, though the latter arise on the implicit, and contrary, assumption that material elements are perfectly identifiable, an assumption which must be intended to apply also for grains within the element. Precisely, the grains are supposed then to belong to their specific element forever, even though a process might remove them a long way from their common centre of mass; the mathematical model does not allow suffusion. Hence the absence, in the equations, of terms implying variable mass ($\sigma = 0$). Besides, the local distortion is supposed to be exactly affine; the mathematical model does not allow chaotic motions either. The peculiar velocities are null and conse-

quently H vanishes. Hence the lack of an evolution equation for H and the absence of H in the law of balance of moment of mesomomentum. Actually the two restrictive conditions have fundamentally distinct character when seen from the different points of view of the two theories. Within the ‘affine’ approach they declare bounds to the model, any deficiency being implicitly transferred, for a redress, to some theory of heat and attendant constitutive choices. Within the ‘ephemeral’ approach they are constraints; their consequences should be explored and, eventually, their influence on balance equations made explicit. If the plain route followed above to deal with perfect constraints would be expedient here is a moot point, to which we may return later. Finally, the definition of affine moment of momentum, say \bar{K} , makes use of a reference stance and thus \bar{K} has the properties of a double vector (or, equivalently, a two-point tensor) rather than of a tensor, as the ephemeral K properly is. In fact, the relation between \bar{K} and K is $\bar{K} = KG^{-\top}$, where G is related to B via

$$(61) \quad \dot{G}G^{-1} = B.$$

In view of this relation, $G^{\top}B^{\top} = \dot{G}^{\top}$ and

$$(62) \quad \begin{aligned} \dot{\bar{K}} &= \dot{\bar{K}}G^{\top} + \bar{K}\dot{G}^{\top} - \bar{K}G^{\top}B^{\top} - B\bar{K}G^{\top} \\ &= \dot{\bar{K}}G^{\top} + \bar{K}\dot{G}^{\top} - \bar{K}\dot{G}^{\top} - B\bar{K}G^{\top} \\ &= (\dot{\bar{K}} - B\bar{K})G^{\top}. \end{aligned}$$

Thus, when expressed in terms of \bar{K} , the coshaping rate of K loses a term and, granted that $\sigma = 0$ and $H = 0$, (4) can be converted to

$$(63) \quad \rho(\dot{\bar{K}} - B\bar{K}) = \rho\bar{M} + T^{\top} + C + \operatorname{div} h,$$

in which $\bar{M} = MG^{-\top}$ and C and h obey

$$(64) \quad C + \operatorname{div} h = -T^{\top} + (-A + \operatorname{div} m)G^{-\top}.$$

In (63), the divergence applied to h must be the covariant divergence; if written with trivial derivatives, without involvement of the local metric, the separation does not make sense and the two addenda are not singly covariant. In any case the sum of the terms $T^{\top} + C$ and $\operatorname{div} h$ is covariant. Computing the covariant divergence involves the metric Y and, consequently, G . Reference to Capriz and Podio-Guidugli [5] or the more recent contributions of Obukhov and Tresguerres [18] and Brocato and Capriz [1] reveals that (63) coincides with the balance of generalized moment of momentum arising in the theory for continua with affine microstructure. The use of \bar{K} instead of K in the affine dynamics is therefore strictly unnecessary. Hence, one can study the evolution of an affine continuum also with the use of the balance law as written for ephemeral continua by simply canceling terms involving σ and H and adapting appropriately the constitutive

laws for sources and fluxes. Thus, the Navier–Stokes- $\alpha\beta$ equation could have been based on the more restricted affine version of the balance laws, in view of the many constraints introduced along the derivation from the ephemeral dynamics.

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