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NOTES ON PERIODIC SOLUTIONS OF DISCRETE STEADY STATE SYSTEMS

Gen-Qiang Wang and Sui Sun Cheng

Abstract: A three term discrete system is considered and periodic solutions are found by means of a mountian pass theorem in the critical point theory.

1 – Introduction

In [5], discrete systems of the form

$$\Delta^2 X_{n-1} + g(n, X_n) = 0 , \quad n \in \mathbb{Z} = \{0, \pm 1, \pm 2, ...\} , \tag{1}$$

where $g \in C(\mathbb{Z}\times\mathbb{R}^k, \mathbb{R}^k)$ and there is a positive integer ω such that $g(t+\omega, X) = g(t, X)$ for any $(t, X) \in \mathbb{Z}\times\mathbb{R}^k$, is considered and ω-periodic solutions are found by means of critical point theory. Since the difference operator Δ is defined by $\Delta X_n = X_{n+1} - X_n$, the above system reminds us of a second order differential system. Therefore, the above system emphasizes the importance of relative changes of the state variables with respect to spatio or temporal changes. Such an emphasis may, however, not be necessary in general. Consider for example ω artificial neuron units placed on the vertices of a regular ω polygon. Let $x_n^{(t)}$ denote the state value of the n-th neuron unit during the time period $t \in \{0, 1, 2, ...\}$. Assume that each neuron unit is activated by its two neighbors so that the change of state values between two consecutive time period is given by

$$x_n^{(t+1)} - x_n^{(t)} = x_{n-1}^{(t)} + x_{n+1}^{(t)} - h(n, x_n^{(t)}) ,$$

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where f stands for the bias mechanism inherent in the n-th neuron unit, then we have an evolutionary system of the form

$$\begin{aligned} x_n^{(t+1)} - x_n^{(t)} &= x_{n-1}^{(t)} + x_{n+1}^{(t)} - h\big(n, x_n^{(t)}\big) \ , \qquad n \in \{1, 2, ..., \omega\} \ , \\ x_0^{(t)} &= x_\omega^{(t)} \ , \\ x_1^{(t)} &= x_{\omega+1}^{(t)} \ , \end{aligned}$$

for $t = 0, 1, 2, \ldots$. If we try to seek a 'steady state' solution $\big\{(x_1^{(t)}, ..., x_\omega^{(t)})\big\}_{t=0}^{\infty}$ such that $x_n^{(t)} = x_n$ for all t, then we need to find a solution of the steady state system

$$\begin{aligned} 0 &= x_{n-1} + x_{n+1} - h(n, x_n) \ , \qquad n \in \{1, 2, ..., \omega\} \ , \\ x_0 &= x_\omega \ , \\ x_1 &= x_{\omega+1} \ , \end{aligned}$$

or equivalently, to find an ω-periodic solution $\{x_n\}_{n\in\mathbb{Z}}$ of

$$x_{n-1} + x_{n+1} - h(n, x_n) = 0 \ , \qquad n \in \mathbb{Z} \ .$$

For this reason, we need to consider three-term discrete systems of the form

$$X_{n+1} + X_{n-1} - f(n, X_n) = 0 \ , \qquad n \in \mathbb{Z} \ , \tag{2}$$

where $f = (f_1, f_2, ..., f_k)^\dagger \in C(\mathbb{Z}\times\mathbb{R}^k, \mathbb{R}^k)$ and there is a fixed positive integer ω such that $f(n+\omega, U) = f(n, U)$ for all $(n, U) \in \mathbb{Z}\times\mathbb{R}^k$. We will assume throughout that there exists a continuously differentiable function $F \in C^1(\mathbb{R}\times\mathbb{R}^k, \mathbb{R})$ such that $\nabla_U F(n, U) = f(n, U)$ and $F(n+\omega, U) = F(n, U)$ for all $(n, U) \in \mathbb{R}\times\mathbb{R}^k$, where ∇_U denotes the gradient operator in U. As usual, a solution $\{X_n\}_{n\in\mathbb{Z}}$ of (2) is a real vector sequence that renders (2) into an identity after substitution. It is said to be ω-periodic if $X_{n+\omega} = X_n$ for all $n \in \mathbb{Z}$.

As in [5], we will employ some of the well known results in the critical point theory to find ω-periodic solutions of (2). For general information on critical point theory, we refer to [13, 14]. For additional information on discrete systems, we refer to [1–12].

Note that if we let $I\colon \mathbb{R}^k \to \mathbb{R}^k$ be the identity map and let $f(n, X_n) = g(n, X_n) - 2\,I(X_n)$, then (2) is reduced to (1). However, as will be seen in Example 2, our results cannot be deduced from that in [5]. This is not surprising since emphasis is shifted to interactions between state variables.

2 – Existence criteria

The main result of this paper is the following.

Theorem 1. *Suppose $F(n,U) \geqslant 0$ for $n \in \mathbb{Z}$ and $U \in \mathbb{R}^k$. Suppose further that*

(G_1) *there are constants $\delta > 0$ and $\alpha \in (0,1)$ such that for any $n \in \mathbb{Z}$ and $U \in \mathbb{R}^k$ satisfying $|U| \leq \delta$, we have $F(n,U) \leq \alpha\,|U|^2$ and*

(G_2) *there are constants $\rho > 0$, $\gamma > 0$ and $\beta \in (1,\infty)$ such that for any $n \in \mathbb{Z}$ and $U \in \mathbb{R}^k$ satisfying $|U| \geqslant \rho$, we have $F(n,U) \geqslant \beta\,|U|^2 - \gamma$.*

Then (2) has at least one nontrivial ω-periodic solution.

Example 1. Let $k=1$ and $\omega=2$. Set $f(x) = 2a\,(2+\sin\pi t)\,(x-\sin x)$ where $a>1$, and set $F(t,x) = 2a\,(2+\sin\pi t)\,(\frac{x^2}{2}+\cos x - 1)$. Then $f, F \in C(\mathbb{R},\mathbb{R})$ and $\nabla_x F(t,x) = f(t,x)$. It is easy to see from $\lim\limits_{x\to 0}\max\limits_{t\in\mathbb{R}}|F(t,x)/x^2| = 0$ that (G_1) is satisfied. Since $F(t,x) = 2a\,(2+\sin\pi t)\,(\frac{x^2}{2}+\cos x-1) \geqslant a x^2 - 4a$ for $(t,x)\in\mathbb{R}^2$, if we take $\rho>0$, $\beta=a$ and $\gamma=4a$, then (G_2) is satisfied. Thus by Theorem 1, (2) has at least one nontrivial 2-periodic solution. □

Corollary 1. *Suppose $F(n,U) \geqslant 0$ for any $n \in \mathbb{Z}$ and $U \in \mathbb{R}^k$. Suppose further that*

(G_3) *$F(n,U) = o(|U|^2)$ as $U \to 0$ and*

(G_4) *there are constants $R_1 > 0$ and $\alpha_1 > 2$ such that for any $n \in \mathbb{Z}$ and $U \in \mathbb{R}^k$ satisfying $|U| \geqslant R_1$,*

$$\Big\langle U, \nabla_U F(n,U)\Big\rangle \;\geqslant\; \alpha_1\,F(n,U) \;>\; 0\;.$$

Then (2) has at least one nontrivial ω-periodic solution.

Corollary 2. *Suppose $k=1$. Suppose further that*

(G_5) $\displaystyle\int_0^x f(n,t)\,dt \geqslant 0$ *for any $n \in \mathbb{Z}$ and $x \in \mathbb{R}$,*

(G_6) *$f(n,x) = o(x)$ as $x \to 0$ and*

(G_7) *there are constants $R_1 > 0$ and $\alpha_1 > 2$ such that for any $n \in \mathbb{Z}$ and $x \in \mathbb{R}$ satisfying $|x| \geqslant R_1$,*

$$x\,f(n,x) \;\geqslant\; \alpha_1 \int_0^x f(n,u)\,du \;>\; 0\;.$$

Then (2) has at least one nontrivial ω-periodic solution.

Before turning to the proof of our results, let us first show that our results are different from those in [5].

Example 2. Consider a scalar equation of the form

$$x_{n+1} + x_{n-1} - 3(x_n)^{3/2} = 0\ , \qquad n \in \mathbb{Z}\ . \tag{3}$$

We assert that (3) has at least one nontrivial 4-periodic solution. Indeed, here $f(n,x) = 3x^{3/2}$ and $F(n,x) = \frac{6}{5}x^{5/2}$. It is easy to see that $\lim\limits_{x\to 0}\max\limits_{n\in\mathbb{Z}}|F(n,x)/x^2| = 0$ and that (G_1) is satisfied. If we take $\rho > 0$, $\beta = 6/5$ and $\gamma > 0$, then (G_2) is satisfied. Thus by Theorem 1, (3) has at least one nontrivial 4-periodic solution. However, if we rewrite (3) in the form (1), then $g(n,x) = 3x^{3/2} - 2x$. In this case, the conditions of the main Theorem in [5] are not satisfied, for otherwise there is G, such that $\nabla_U G(n,U) = g(n,U)$, $G(n,x) \geqslant 0$ for $n \in \mathbb{Z}$ and $x \in \mathbb{R}$ and $G(n,0) = 0$ for $n \in \mathbb{Z}$. Thus $G(n,x) = \frac{6}{5}x^{\frac{5}{2}} - x^2$, which is contrary to the fact that $G(n,x) \geqslant 0$ for $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. □

The spirit of the proof of Theorem 1 is similar to that of the main theorem in [5] but we use a mountain pass theorem instead of a linking theorem. For the sake of completeness, we give a complete proof as follows. First, for any $U = (U_1, ..., U_k)$, $V = (V_1, ..., V_k) \in \mathbb{R}^k$, their inner product is $\langle U, V\rangle = \sum_{i=1}^k U_i V_i$ and the norm of V is $|V| = \langle V, V\rangle^{1/2}$. Let S be the set of all real vector sequences $X = \{X_n\}_{n\in\mathbb{Z}}$ where $X_n = (X_{n1}, X_{n2}, ..., X_{nk})^\dagger \in \mathbb{R}^k$. For any $X, Y \in S$ and $a, b \in \mathbb{R}$, $aX + bY$ is defined by $aX + bY = \{aX_n + bY_n\}_{n\in\mathbb{Z}}$. Then S is a linear space. Let E_ω be the set of all ω-periodic vector sequences in S. When endowed with the norm $\|\cdot\|_{E_\omega}$ and inner product $\langle\cdot,\cdot\rangle_{E_\omega}$ defined by

$$\|X\|_{E_\omega} = \left(\sum_{n=1}^{\omega} |X_n|^2\right)^{1/2}, \tag{4}$$

and

$$\langle X, Y\rangle_{E_\omega} = \sum_{n=1}^{\omega}\langle X_n, Y_n\rangle$$

for any $X = \{X_n\}_{n\in\mathbb{Z}}$, $Y = \{Y_n\}_{n\in\mathbb{Z}}$ in E_ω, the pair $\left(E_\omega, \langle\cdot,\cdot\rangle_{E_\omega}\right)$ is a Hilbert space.

We now formulate our problem as a critical point problem. Consider the functional I defined on E_ω by

$$I(X) = \sum_{n=1}^{\omega}\Big\{\langle X_{n+1}, X_n\rangle - F(n, X_n)\Big\}\ , \qquad X \in E_\omega\ . \tag{5}$$

Since E_ω is linearly homeomorphic to $\mathbb{R}^{\omega k}$, I can be viewed as a continuously differentiable functional defined on the finite dimensional Hilbert space $\mathbb{R}^{\omega k}$ by taking $X_0 = X_\omega$ and $X_{\omega+1} = X_1$. In particular, the Frechet derivative $I'(X)$ is zero if, and only if, $\frac{\partial I(X)}{\partial X_{n,l}} = 0$ for all $n \in \{1, ..., \omega\}$ and $l \in \{1, ..., k\}$. Since

$$\frac{\partial I(X)}{\partial X_{n,l}} = \Big\{X_{n,l} + X_{n-1,l} - f_l(n, X_n)\Big\} , \qquad 1 \le n \le \omega, \;\; 1 \le l \le k , \tag{6}$$

we see that $I'(X) = 0$ if, and only if,

$$X_{n+1} + X_{n-1} - f(n, X_n) \;=\; 0 , \qquad n \in \{1, ..., \omega\} . \tag{7}$$

That is, $X \in E_\omega$ is a critical point of I (i.e. $I'(X) = 0$) if, and only if, X is an ω-periodic solution of (2).

Let H be a real Banach space. A continuously differentiable functional $J \in C^1(H, \mathbb{R})$ is said to satisfy the Palais–Smale condition (P-S condition) if any sequence $\{u_n\} \subset H$ for which $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence in H.

Lemma 1 (14, Mountain Pass Theorem 2.7)**.** *Let H be a real Banach space and I a real continuously differentiable functional that satisfies the P-S condition. If I is bounded from above in E, then I possesses a critical value $c = \sup_{x \in E} I(x)$.* ∎

Lemma 2. *If (G_2) holds, then the functional I defined by (5) is bounded from above in E_ω.*

Proof: According to (G_2), if we let

$$\gamma_1 \;=\; \max\Big\{\big|F(n, U) - \beta|U|^2 + \gamma\big| :\; n \in \mathbb{Z},\;\; |U| \le \rho\Big\}$$

and $\gamma' = \gamma + \gamma_1$. Then for any $n \in \mathbb{Z}$ and $U \in \mathbb{R}^k$, we have

$$F(n, U) \;\geqslant\; \beta|U|^2 - \gamma' . \tag{8}$$

By (5) and Cauchy's inequality we see that for any $X \in E_\omega$,

$$\begin{aligned} I(X) \;&=\; \sum_{n=1}^{\omega}\Big\{\langle X_{n+1}, X_n\rangle - F(n, X_n)\Big\} \;\le\; \sum_{n=1}^{\omega}|\langle X_{n+1}, X_n\rangle| \;-\; \sum_{n=1}^{\omega}F(n, X_n) \\ &\le\; \|X\|_{E_\omega}^2 - \sum_{n=1}^{\omega}\big(\beta|X_n|^2 - \gamma'\big) \;=\; \|X\|_{E_\omega}^2 - \sum_{n=1}^{\omega}\big(\beta|X_n|^2 - \gamma'\big) \\ &=\; (1 - \beta)\,\|X\|_{E_\omega}^2 + \gamma'\omega\ . \end{aligned}$$

Since $\beta > 1$ by (G_2), for any $X \in E_\omega$, we have $I(X) \le \gamma'\omega$. The proof is complete. ∎

Lemma 3. *If (G_2) holds, then the functional I defined by (5) satisfies the P-S condition.*

Proof: Let $\{I(X^{(i)})\}_{i=1}^{\infty}$ be a sequence bounded from below, that is, there exists a positive constant M such that

$$-M \leq I(X^{(i)}) \ , \qquad i = 1, 2, \dots \ .$$

In view of the proof Lemma 2, it is easy to see that

$$-M \leq I(X^{(i)}) \leq (1-\beta)\, \|X^{(i)}\|_{E_\omega}^2 + \gamma'\omega \ ,$$

which implies that

$$\|X^{(i)}\|_{E_\omega}^2 \leq (\beta-1)^{-1}(M+\gamma'\omega) \ .$$

That is, $\{X^{(i)}\}$ is a bounded sequence in the finite dimensional space E_ω. Hence $\{X^{(i)}\}$ has a convergent subsequence. The proof is complete. ∎

We now turn to the **proof of Theorem 1**. First of all, we can easy to see that $I(0) = 0$. It suffices to find a nontrivial critical point of the functional I defined by (5). By Lemma 2 and Lemma 3, we see that the conditions of Lemma 1 hold. Thus I possesses a critical value $c_0 = \sup_{x\in E_\omega} I(x)$. Let X^0 be a critical point of I in E_ω such that $I(X^0) = c_0$. We claim that $c_0 > 0$. Indeed, by the assumption (G_1) and Lemma 2, for any constant vector sequence $X = \{V\}_{n\in\mathbb{Z}}$ satisfying $\|X\|_{E_\omega} = \delta$,

$$\begin{aligned} I(X) &= \sum_{n=1}^{\omega}\Big\{X_{n+1}X_n - F(n,X_n)\Big\} \geqslant \|X\|_{E_\omega}^2 - \sum_{n=1}^{\omega} F(n,X_n) \\ &\geqslant \|X\|_{E_\omega}^2 - \alpha\sum_{n=1}^{\omega}|X_n|^2 = (1-\alpha)\,\|X\|_{E_\omega}^2 \\ &= \sigma > 0 \ , \end{aligned}$$

where $\sigma = (1-\alpha)\,\delta^2 > 0$. Thus $c_0 = \sup_{X\in E_\omega} I(X) \geqslant \sigma > 0$. Since $I(0) = 0$, $X^0 \neq 0$. Thus X^0 is a nontrivial ω-periodic solution of (2). The proof is complete. ∎

We now turn to the **proof of Corollary 1**. It is easy to see that if (G_3) holds, then the condition (G_1) of Theorem 1 is true. From (G_4), we have

$$\left\langle \frac{U}{|U|}, \frac{\nabla_U F(n,U)}{F(n,U)} \right\rangle \geqslant \frac{\alpha_1}{|U|} \qquad \text{for} \quad n\in\mathbb{Z} \quad \text{and} \quad |U| \geqslant R_1 \ . \tag{9}$$

Thus

$$\frac{d\ \ln F(n,U)}{d\,|U|} \geqslant \frac{\alpha_1}{|U|}\ ,$$

which implies

$$\frac{d}{d\,|U|}\Big(\ln F(n,U) - \alpha_1 \ln|U|\Big) \geqslant 0 \qquad \text{for} \quad n\in\mathbb{Z} \quad \text{and} \quad |U|\geqslant R_1\ . \tag{10}$$

Let

$$\phi \;=\; \min\Big\{\ln F(n,U) - \alpha_1 \ln|U| :\ n\in\mathbb{Z}\ \text{and}\ \ |U| = R_1\Big\}\ .$$

By (9),

$$\ln F(n,U) - \alpha_1 \ln|U| \;\geqslant\; \phi \qquad \text{for} \quad n\in\mathbb{Z} \quad \text{and} \quad |U|\geqslant R_1\ .$$

That is

$$F(n,U) \;\geqslant\; \beta_1\,|U|^{\alpha_1} \qquad \text{for} \quad n\in\mathbb{Z} \quad \text{and} \quad |U|\geqslant R_1\ ,$$

where $\beta_1 = \exp(\phi)$. Let $\rho_1 \geqslant R_1$ satisfying $\beta_1\,\rho_1^{\alpha_1-2} > 1$. Then for $n \in \mathbb{Z}$ and $|U| \geqslant \rho_1$,

$$F(n,U) \;\geqslant\; \beta_1\,|U|^{\alpha_1-2}\,|U|^2 \;\geqslant\; \beta\,|U|^2\ ,$$

where $\beta = \beta_1\,\rho_1^{\alpha_1-2} > 1$. Thus the condition ($G_2$) of Theorem 1 holds. The proof is complete. ∎

Finally, the **proof of Corollary 2** follows from Corollary 1 by taking $F(t,x) = \int_0^x f(t,x)\,dx$. ∎

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Gen-Qiang Wang,
Department of Computer Science, Guangdong Polytechnic Normal University,
Guangzhou, Guangdong 510665 – P.R. CHINA

and

Sui Sun Cheng,
Department of Mathematics, Tsing Hua University,
Hsinchu, Taiwan 30043 – R.O. CHINA