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Partial groupoid representations and a relation with the Birget–Rhodes expansion

Wesley G. Lautenschlaeger and Thaísa Tamusiunas

Abstract. We introduce partial groupoid representations of a finite groupoid $\mathcal{G}$ on an algebra $\mathcal{A}$. We also show that the partial groupoid representations of $\mathcal{G}$ are in one-to-one correspondence with the representations of the algebra generated by the Birget–Rhodes expansion $\mathcal{G}^{\mathrm{BR}}$ of $\mathcal{G}$.

1. Introduction

Many studies concerning actions and partial actions of groupoids have been investigated in the last few years. For instance, the relation between partial and global actions, Galois theory, generalizations of classic theorems of group theory, Morita theory, crossed products and duality theorems were research topics addressed in [3–7, 9, 14, 15]. The idea of classification of something partial in terms of something global gives us conditions to understand the behavior of the partial theory.

The Birget–Rhodes expansion $\mathcal{G}^{\mathrm{BR}}$ of an ordered groupoid $\mathcal{G}$ was constructed by Gilbert in [10] and it was proven that $\mathcal{G}^{\mathrm{BR}}$ has an ordered groupoid structure [10, Proposition 3.1]. Also, there is a one-to-one correspondence between partial actions of $\mathcal{G}$ and actions of $\mathcal{G}^{\mathrm{BR}}$, which can be viewed as a partial-to-global result achieved by enlarging the acting groupoid.

The construction of $\mathcal{G}^{\mathrm{BR}}$ can be used to generalize the work developed by Exel, Dokuchaev and Piccione in [8], regarding partial group representations of a group G. In that work, the authors presented the partial group algebra of a group G, called $K_{\mathrm{par}}(G)$, which is the algebra whose representations correspond to the partial group representations of G. The algebra $K_{\mathrm{par}}(G)$ was shown to be a groupoid algebra $K\Gamma(G)$, where $\Gamma(G)$ is a determined finite groupoid associated to G [8, Corollary 2.7]. However, the groupoid $\Gamma(G)$ has a very rich structure. It is, in fact, the groupoid associated via the ESN Theorem [12, Theorem 4.1.8] with the inverse semigroup which is the Birget–Rhodes expansion of the group G [11, Theorem 2.4].

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Based on this, our major purpose in this paper is to complement and to extend the work of Exel, Dokuchaev and Piccione, establishing a relation between the partial groupoid representations of a groupoid $\mathcal{G}$ and representations of the algebra $K\mathcal{G}^{\mathrm{BR}}$ generated by its Birget–Rhodes expansion $\mathcal{G}^{\mathrm{BR}}$. Indeed, we shall prove that there is a one-to-one correspondence between the partial groupoid representations of $\mathcal{G}$ and the representations of $K\mathcal{G}^{\mathrm{BR}}$. This agrees with the idea of enlarging the groupoid $\mathcal{G}$ to characterize partial groupoid representations in terms of "global" representations.

The paper is organized as follows. We start by fixing some terminology concerning groupoids and we introduce the concept of partial groupoid representation of a finite groupoid $\mathcal{G}$ on an algebra. Next we present the algebra $K_{\mathrm{par}}(\mathcal{G})$, whose representations are in one-to-one correspondence with the partial groupoid representations of $\mathcal{G}$. The last section aims to prove that the algebra $K_{\mathrm{par}}(\mathcal{G})$ is the algebra generated by the Birget–Rhodes expansion of $\mathcal{G}$.

Throughout this paper, rings and algebras are associative and unital. All algebra homomorphisms are unital.

2. Partial groupoid representations and the algebra $K_{\mathrm{par}}(\mathcal{G})$

2.1. Partial groupoid representations

We recall that a *groupoid* $\mathcal{G}$ is a small category in which every morphism is an isomorphism. We denote by $\mathcal{G}_0$ the set of objects of $\mathcal{G}$. Observe that $\mathrm{id} : \mathcal{G}_0 \to \mathcal{G}$, given by $\mathrm{id}(x) = \mathrm{id}_x$, is an injective map and whence we identify $\mathcal{G}_0 \subset \mathcal{G}$. Given $g \in \mathcal{G}$, the *domain* and the *range* of g will be denoted by $d(g)$ and $r(g)$, respectively. Hence, $d(g) = g^{-1}g$ and $r(g) = gg^{-1}$. For all $g, h \in G$, we write $\exists gh$ whenever the product gh is defined. We fix the notation $\mathcal{G}_2 := \{(g, h) \in \mathcal{G} \times \mathcal{G} : \exists gh\}$.

For the rest of the paper, let $\mathcal{G}$ be a finite groupoid, K be a field and $\mathcal{A}$ be a K-algebra. We start this section by defining a partial groupoid representation on $\mathcal{A}$.

Definition 2.1. A *partial groupoid representation* of $\mathcal{G}$ on $\mathcal{A}$ is a map $\pi : \mathcal{G} \to \mathcal{A}$ such that

(i) $\pi(g)\pi(h)\pi(h^{-1}) = \pi(gh)\pi(h^{-1})$, $\forall (g, h) \in \mathcal{G}_2$;

(ii) $\pi(g^{-1})\pi(g)\pi(h) = \pi(g^{-1})\pi(gh)$, $\forall (g, h) \in \mathcal{G}_2$;

(iii) $\pi(g)\pi(g^{-1})\pi(g) = \pi(g)$, $\forall g \in \mathcal{G}$;

(iv) $\sum_{e\in\mathcal{G}_0} \pi(e) = 1_A$ and $\pi(e)\pi(f) = 0$ for $e, f \in \mathcal{G}_0$ such that $e \neq f$.

Remark 2.2. Observe that

$$\pi(g) = \pi(g)\pi(g^{-1})\pi(g) = \pi(g)\pi(g^{-1}g) = \pi(g)\pi(d(g)),$$

for all $g \in \mathcal{G}$. Analogously, $\pi(g) = \pi(r(g))\pi(g)$, for all $g \in \mathcal{G}$. So, if $(g,h) \notin \mathcal{G}_2$, then $\pi(g)\pi(h) = \pi(g)\pi(d(g))\pi(r(h))\pi(h) = \pi(g)0\pi(h) = 0$.

Lemma 2.3. *Let $g \in \mathcal{G}$. Define $\varepsilon(g) = \pi(g)\pi(g^{-1})$. It follows that*

(a) *$\varepsilon(g)$ is an idempotent of A;*

(b) *if $r(g) = r(h)$, then $\varepsilon(g)\varepsilon(h) = \varepsilon(h)\varepsilon(g)$.*

Proof. (a) It is straightforward.

(b) If $r(g) = r(h)$,

$$\begin{aligned}\pi(h^{-1})\varepsilon(g) &= \pi(h^{-1})\pi(g)\pi(g^{-1}) = \pi(h^{-1}g)\pi(g^{-1})\\ &= \pi(h^{-1}g)\pi(g^{-1}h)\pi(h^{-1}g)\pi(g^{-1}) = \pi(h^{-1}g)\pi(g^{-1}h)\pi(h^{-1}gg^{-1})\\ &= \varepsilon(h^{-1}g)\pi(h^{-1}r(g)) = \varepsilon(h^{-1}g)\pi(h^{-1}r(h))\\ &= \varepsilon(h^{-1}g)\pi(h^{-1}), \end{aligned} \tag{1}$$

from where it follows that

$$\begin{aligned}\varepsilon(h)\varepsilon(g) &= \pi(h)\pi(h^{-1})\varepsilon(g) = \pi(h)\varepsilon(h^{-1}g)\pi(h^{-1})\\ &= \varepsilon(r(h)g)\pi(h)\pi(h^{-1}) = \varepsilon(g)\varepsilon(h).\end{aligned}$$

■

2.2. The algebra $K_{\mathrm{par}}(\mathcal{G})$

As in the case of partial group representations of a group G [8], which can be characterized by algebra homomorphisms defined on the partial group algebra $K_{\mathrm{par}}(G)$, in the case of partial groupoid representations of a groupoid $\mathcal{G}$, it is possible to construct an algebra associated to $\mathcal{G}$ which characterizes partial groupoid representations by algebra homomorphisms.

Definition 2.4. We define the partial groupoid K-algebra $K_{\mathrm{par}}(\mathcal{G})$ as the universal K-algebra with unit $1_{K_{\mathrm{par}}(\mathcal{G})}$ generated by the set of symbols $\{[g] : g \in \mathcal{G}\}$ and relations

(i) $[g^{-1}][g][h] = [g^{-1}][gh]$, $\forall (g,h) \in \mathcal{G}_2$;

(ii) $[g][h][h^{-1}] = [gh][h^{-1}]$, $\forall (g,h) \in \mathcal{G}_2$;

(iii) $[r(g)][g] = [g] = [g][d(g)]$, $\forall g \in \mathcal{G}$;

(iv) $[g][h] = 0$, $\forall (g,h) \notin \mathcal{G}_2$.

Notice that $\sum_{e\in\mathcal{G}_0}[e] = 1_{K_{\mathrm{par}}(\mathcal{G})}$. Indeed,

$$\left(\sum_{e\in\mathcal{G}_0}[e]\right)[g] = \sum_{e\in\mathcal{G}_0}[e][g] = [r(g)][g] = [g].$$

Similarly, $[g](\sum_{e\in\mathcal{G}_0}[e]) = [g]$.

Example 2.5. Let $\mathcal{G} = \mathcal{K} \cup \mathcal{H}$ (disjoint union), where $\mathcal{K} = \{g, g^{-1}, r(g), d(g)\}$ and $\mathcal{H} = \{r(h), h\}$ with $h = h^{-1}$. Then $K_{\text{par}}(\mathcal{G})$ has basis $\{[g], [g^{-1}], [r(g)], [d(g)], [g][g^{-1}], [g^{-1}][g], [h], [r(h)], [h][h]\}$ as a K-vector space. It is easy to see that $K_{\text{par}}(\mathcal{G}) \simeq K_{\text{par}}(\mathcal{K}) \oplus K_{\text{par}}(\mathcal{H})$. More generally, if $\mathcal{G}$ is a finite groupoid with connected components $\mathcal{G}^{(1)}, \dots, \mathcal{G}^{(n)}$, then $K_{\text{par}}(\mathcal{G}) \simeq K_{\text{par}}(\mathcal{G}^{(1)}) \oplus \cdots \oplus K_{\text{par}}(\mathcal{G}^{(n)})$.

The next theorem shows that there exists a one-to-one correspondence between partial groupoid representations of $\mathcal{G}$ and representations of $K_{\text{par}}(\mathcal{G})$.

Theorem 2.6. *Let $\pi : \mathcal{G} \to \mathcal{A}$ be a partial groupoid representation of $\mathcal{G}$ on $\mathcal{A}$. Then there exists a unique homomorphism of K-algebras $\phi : K_{\text{par}}(\mathcal{G}) \to \mathcal{A}$ such that $\phi([g]) = \pi(g)$ for all $g \in \mathcal{G}$. Conversely, if $\phi : K_{\text{par}}(\mathcal{G}) \to \mathcal{A}$ is a homomorphism of K-algebras, then $\pi(g) = \phi([g])$ is a partial groupoid representation of $\mathcal{G}$ on $\mathcal{A}$.*

Proof. Let $\pi : \mathcal{G} \to A$ be a partial groupoid representation of $\mathcal{G}$ on $\mathcal{A}$. Define

$$\begin{aligned} \phi : K_{\text{par}}(\mathcal{G}) &\to \mathcal{A}, \\ \sum_{i=1}^{m} k_i \prod_{j=1}^{n} [g_{i,j}] &\mapsto \sum_{i=1}^{m} k_i \prod_{j=1}^{n} \pi(g_{i,j}), \end{aligned}$$

where $k_i \in K$ and $g_{i,j} \in \mathcal{G}$ for all $1 \le i \le m$ and $1 \le j \le n$. Then $\phi([g]) = \pi(g)$, for all $g \in \mathcal{G}$. Furthermore, if $\exists gh$,

$$\begin{aligned} \phi([g][h]) &= \phi([r(g)][g][h]) = \phi([gg^{-1}][g][h]) = \phi([g][g^{-1}][g][h]) \\ &= \phi([g][g^{-1}][gh]) = \pi(g)\pi(g^{-1})\pi(gh) = \pi(g)\pi(g^{-1})\pi(g)\pi(h) \\ &= \pi(g)\pi(h) = \phi([g])\phi([h]), \end{aligned}$$

and $\phi([g][h]) = 0 = \pi(g)\pi(h)$ otherwise.

Moreover, $\phi(1_{K_{\text{par}}(\mathcal{G})}) = \phi(\sum_{e \in \mathcal{G}_0} [e]) = \sum_{e \in \mathcal{G}_0} \pi(e) = 1_A$. Clearly, ϕ is unique.

Conversely, let $\phi : K_{\text{par}}(\mathcal{G}) \to \mathcal{A}$ be a homomorphism of K-algebras. Define $\pi : \mathcal{G} \to K_{\text{par}}(\mathcal{G})$ by $\pi(g) = \phi([g])$, for all $g \in \mathcal{G}$. We shall prove that π is a partial groupoid representation of $\mathcal{G}$ on $\mathcal{A}$. In particular,

(i) if $(g, h) \in \mathcal{G}_2$, then

$$\begin{aligned} \pi(g)\pi(h)\pi(h^{-1}) &= \phi([g])\phi([h])\phi([h^{-1}]) = \phi([g][h][h^{-1}]) \\ &= \phi([gh][h^{-1}]) = \phi([gh])\phi([h^{-1}]) = \pi(gh)\pi(h^{-1}); \end{aligned}$$

(ii) analogous to (i);

(iii) we have that

$$\begin{aligned} \pi(g)\pi(g^{-1})\pi(g) &= \phi([g])\phi([g^{-1}])\phi([g]) = \phi([g][g^{-1}][g]) \\ &= \phi([g]) = \pi(g); \end{aligned}$$

(iv) we have that

$$\sum_{e\in\mathscr{G}_0}\pi(e)=\sum_{e\in\mathscr{G}_0}\phi([e])=\phi\Bigg(\sum_{e\in\mathscr{G}_0}[e]\Bigg)=\phi(1_{K_{\mathrm{par}}(\mathscr{G})})=1_A$$

and if $e,f\in\mathscr{G}_0$ with $e\neq f$, then $\pi(e)\pi(f)=\phi([e])\phi([f])=\phi([e][f])=\phi(0)=0$. ■

3. The relation with the Birget–Rhodes expansion

In this section, we shall describe $K_{\mathrm{par}}(\mathscr{G})$ in terms of the Birget–Rhodes expansion of $\mathscr{G}$.

Define $X_g=\{h\in\mathscr{G}: r(h)=r(g)\}$, for all $g\in\mathscr{G}$. Observe that $X_g=X_{r(g)}$. Define also $Y_g=\{h\in\mathscr{G}: r(h)=d(g)\}=X_{g^{-1}}$. We set the finite groupoid, constructed from $\mathscr{G}$,

$$\mathscr{G}^{\mathrm{BR}}=\big\{(A,g): d(g),g^{-1}\in A\subseteq Y_g,\ g\in\mathscr{G}\big\}$$

as the groupoid with partial multiplication given by

$$(A,g)\cdot(B,h)=\begin{cases}(B,gh), & \text{if } (g,h)\in\mathscr{G}_2 \text{ and } A=hB,\\ \text{undefined}, & \text{otherwise.}\end{cases}$$

The inverse of the pair (A,g) is (gA,g^{-1}). Also $d(A,g)=(A,d(g))$ and $r(A,g)=(gA,r(g))$. This groupoid is the Birget–Rhodes expansion of $\mathscr{G}$ (see [10, Proposition 3.1]).

An easy calculation shows that the elements of the form (A,e), $e\in\mathscr{G}_0$, are idempotents in the groupoid algebra $K\mathscr{G}^{\mathrm{BR}}$, that is, $(A,e)^2=(A,e)$. Also, they are mutually orthogonal and their sum is $1_{K\mathscr{G}^{\mathrm{BR}}}$, since for every groupoid $\mathscr{G}$, if $g,h\in\mathscr{G}$ are such that $(g,h)\notin\mathscr{G}_2$, then $gh=0$ in $K\mathscr{G}$.

To simplify the notation, for every $g\in\mathscr{G}$, consider the set $\mathscr{L}_g=\{A\subseteq Y_g: g^{-1},\ d(g)\in A\}$.

Lemma 3.1. *Let $(g,h)\in\mathscr{G}_2$. On the above notations,*

(i) *$C\in\mathscr{L}_g$ if and only if $gC\in\mathscr{L}_{g^{-1}}$.*

(ii) *$C\in\mathscr{L}_{gh}$ and $ghC\in\mathscr{L}_{g^{-1}}$ if and only if $C\in\mathscr{L}_h\cap\mathscr{L}_{gh}$.*

Proof. (i) Let $C\in\mathscr{L}_g$. That means that $C\subseteq Y_g$ is such that $g^{-1},d(g)\in C$. Hence $gC\subseteq gY_g=Y_{g^{-1}}$, $r(g)=gg^{-1}\in gC$ and $g=gd(g)\in gC$, so that $gC\in\mathscr{L}_{g^{-1}}$. The converse follows by symmetry.

(ii) Assume that $C \in \mathcal{L}_{gh}$ and $ghC \in \mathcal{L}_{g^{-1}}$. By (i), the second inclusion is equivalent to $hC \in \mathcal{L}_g$. Since C is already in $\mathcal{L}_{gh}$, we only need to prove that $C \in \mathcal{L}_h$. For this verification, first notice that $C \subseteq Y_{gh} = Y_h$. Hence it only remains for us to show that $h^{-1}, d(h) \in C$. Since $d(g) = r(h) \in hC$, we have that $hx = r(h)$, for some $x \in C$. But $hx = r(h) = hh^{-1}$ implies that $x = h^{-1}$ by the cancellation law. So $h^{-1} \in C$. Since $C \in \mathcal{L}_{gh}$ and $d(gh) = d(h)$, we have that $d(h) \in C$.

For the converse, assume that $C \in \mathcal{L}_h \cap \mathcal{L}_{gh}$. By assumption C is in $\mathcal{L}_{gh}$. To prove that $ghC \in \mathcal{L}_{g^{-1}}$ it is enough to prove that $hC \in \mathcal{L}_g$, by (i). Since $C \subseteq Y_h$, then $hC \subseteq hY_h = Y_{h^{-1}}$. Now, $h^{-1}, h^{-1}g^{-1} \in C$, from where it follows that $d(g) = r(h) = hh^{-1} \in hC$ and $g^{-1} = h(h^{-1}g^{-1}) \in hC$, ending the proof. ∎

Lemma 3.2. *Define the map $\lambda : \mathcal{G} \to K\mathcal{G}^{\mathrm{BR}}$ by $\lambda(g) = \sum_{A \in \mathcal{L}_g} (A, g)$. Then λ is a partial groupoid representation of $\mathcal{G}$ on $K\mathcal{G}^{\mathrm{BR}}$.*

Proof. (i) Given $(g, h) \in \mathcal{G}_2$, we have

$$\begin{aligned}
\lambda(g^{-1})\lambda(g)\lambda(h) &= \sum_{\substack{A \in \mathcal{L}_{g^{-1}},\\ B \in \mathcal{L}_g, C \in \mathcal{L}_h}} (A, g^{-1})(B, g)(C, h) \\
&= \sum_{\substack{C \in \mathcal{L}_h,\\ hC \in \mathcal{L}_g}} (ghC, g^{-1})(hC, g)(C, h) \\
&\overset{\text{Lemma 3.1}}{=} \sum_{C \in \mathcal{L}_h \cap \mathcal{L}_{gh}} (C, d(g)h) = \sum_{C \in \mathcal{L}_h \cap \mathcal{L}_{gh}} (C, r(h)h) \\
&= \sum_{C \in \mathcal{L}_h \cap \mathcal{L}_{gh}} (C, h).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\lambda(g^{-1})\lambda(gh) &= \sum_{\substack{A \in \mathcal{L}_{g^{-1}},\\ C \in \mathcal{L}_{gh}}} (A, g^{-1})(C, gh) \overset{\text{Lemma 3.1}}{=} \sum_{C \in \mathcal{L}_h \cap \mathcal{L}_{gh}} (ghC, g^{-1})(C, gh) \\
&= \sum_{C \in \mathcal{L}_h \cap \mathcal{L}_{gh}} (C, h).
\end{aligned}$$

Consequently, $\lambda(g^{-1})\lambda(g)\lambda(h) = \lambda(g^{-1})\lambda(gh)$, for all $(g, h) \in \mathcal{G}_2$. The equality $\lambda(g)\lambda(h)\lambda(h^{-1}) = \lambda(gh)\lambda(h^{-1})$ is proved similarly.

(iii) Now that we have proved (i), the equality $\lambda(g)\lambda(g^{-1})\lambda(g) = \lambda(g)$ is equivalent to $\lambda(g)\lambda(d(g)) = \lambda(g)$ by Remark 2.2. We shall show the second equality. For

$g \in \mathcal{G}$,

$$\lambda(g)\lambda(d(g)) = \Big(\sum_{A\in\mathcal{L}_g}(A,g)\Big)\Big(\sum_{B\in\mathcal{L}_{d(g)}}(B,d(g))\Big)$$
$$= \sum_{A\in\mathcal{L}_g}(A,g)(A,d(g)) = \sum_{A\in\mathcal{L}_g}(A,g) = \lambda(g).$$

(iv) We have $\sum_{e\in\mathcal{G}_0}\lambda(e) = \sum_{e\in\mathcal{G}_0}\sum_{A\in\mathcal{L}_e}(A,e) = 1_{K\mathcal{G}^{\mathrm{BR}}}$, and if $e, f \in \mathcal{G}_0$, $e \neq f$,

$$\lambda(e)\lambda(f) = \Big(\sum_{A\in\mathcal{L}_e}(A,e)\Big)\Big(\sum_{B\in\mathcal{L}_f}(B,f)\Big) = \sum_{\substack{A\in\mathcal{L}_e,\\ B\in\mathcal{L}_f}}(A,e)(B,f) = 0. \qquad \blacksquare$$

Theorem 3.3. *There is a one-to-one correspondence between the partial groupoid representations of $\mathcal{G}$ and the representations of $K\mathcal{G}^{\mathrm{BR}}$. More precisely, if $\mathcal{A}$ is any unital K-algebra, then $\pi : \mathcal{G} \to \mathcal{A}$ is a partial groupoid representation of $\mathcal{G}$ if and only if there is an algebra homomorphism $\tilde{\pi} : K\mathcal{G}^{\mathrm{BR}} \to \mathcal{A}$ such that $\pi = \tilde{\pi}\circ\lambda$. Moreover, such a homomorphism $\tilde{\pi}$ is unique.*

Proof. If $\tilde{\pi} : K\mathcal{G}^{\mathrm{BR}} \to \mathcal{A}$ is a homomorphism of K-algebras, then clearly $\pi = \tilde{\pi}\circ\lambda : \mathcal{G} \to \mathcal{A}$ is a partial groupoid representation of $\mathcal{G}$ on $\mathcal{A}$.

Conversely, assume that $\pi : \mathcal{G} \to \mathcal{A}$ is a partial groupoid representation of $\mathcal{G}$. For all $g \in \mathcal{G}$, denote by $\varepsilon(g) = \pi(g)\pi(g^{-1}) \in \mathcal{A}$. Recall from Remark 2.2 and Lemma 2.3 that $\varepsilon(g)\varepsilon(h) = \delta_{r(g),r(h)}\varepsilon(h)\varepsilon(g)$. Also, from (1), if $d(g) = r(h)$ then $\pi(g)\varepsilon(h) = \varepsilon(gh)\pi(g)$, and if $d(g) \neq r(h)$, then $\pi(g)\varepsilon(h) = 0$. Similarly, if $r(g) = r(h)$, then $\varepsilon(h)\pi(g) = \pi(g)\varepsilon(g^{-1}h)$, and $\varepsilon(h)\pi(g) = 0$ otherwise.

For $(A,g) \in \mathcal{G}^{\mathrm{BR}}$, we define

$$\tilde{\pi}(A,g) = \pi(g)\Big(\prod_{h\in A}\varepsilon(h)\Big)\Big(\prod_{h\in Y_g\setminus A}\big(\pi(d(g)) - \varepsilon(h)\big)\Big).$$

For $(A,g), (B,h) \in \mathcal{G}^{\mathrm{BR}}$, we have

$$\tilde{\pi}(A,g)\tilde{\pi}(B,h) = \pi(g)\cdot\prod_{k\in A}\varepsilon(k)\cdot\prod_{k\in Y_g\setminus A}\big(\pi(d(g)) - \varepsilon(k)\big)\cdot\pi(h)$$
$$\cdot\prod_{\ell\in B}\varepsilon(\ell)\cdot\prod_{\ell\in Y_h\setminus B}\big(\pi(d(h)) - \varepsilon(\ell)\big).$$

If $d(g) \neq r(h)$, then $\pi(d(g))\pi(h) = 0$ and $\varepsilon(k)\pi(h) = 0$, for all $k \in Y_g \setminus A$, since $\varepsilon(k) = \pi(k)\pi(k^{-1})$ and $d(k^{-1}) = r(k) = d(g) \neq r(h)$. So in this case,

$$\tilde{\pi}(A,g)\tilde{\pi}(B,h) = 0.$$

Suppose now that $d(g) = r(h)$. Then

$$\begin{aligned}\tilde{\pi}(A,g)\tilde{\pi}(B,h) = \pi(g)\pi(h) \cdot \prod_{k\in A} \varepsilon(h^{-1}k) \cdot \prod_{k\in Y_g\setminus A} (\pi(d(h)) - \varepsilon(h^{-1}k)) \\ \cdot \prod_{\ell\in B} \varepsilon(\ell) \cdot \prod_{\ell\in Y_h\setminus B} (\pi(d(h)) - \varepsilon(\ell)) \\ = \pi(g)\pi(h) \cdot \prod_{k\in h^{-1}A} \varepsilon(k) \cdot \prod_{k\in Y_h\setminus h^{-1}A} (\pi(d(h)) - \varepsilon(k)) \\ \cdot \prod_{\ell\in B} \varepsilon(\ell) \cdot \prod_{\ell\in Y_h\setminus B} (\pi(d(h)) - \varepsilon(\ell)).\end{aligned}$$

If $h^{-1}A \neq B$, that is, if $A \neq hB$, then either there is $k \in h^{-1}A$ such that $k \in Y_h \setminus B$ or there is $k \in B$ such that $k \in Y_h \setminus h^{-1}A$. In either case, the factor $\varepsilon(k)(\pi(d(h)) - \varepsilon(k)) = 0$ appears in the expression of $\tilde{\pi}(A,g)\tilde{\pi}(B,h)$, from where it follows that $\tilde{\pi}(A,g)\tilde{\pi}(B,h) = 0$.

On the other hand, if $h^{-1}A = B$, then

$$\begin{aligned}\tilde{\pi}(A,g)\tilde{\pi}(B,h) &= \pi(g)\pi(h) \cdot \prod_{k\in h^{-1}A} \varepsilon(k) \cdot \prod_{k\in Y_h\setminus h^{-1}A} (\pi(d(h)) - \varepsilon(k)) \\ &= \pi(g)\pi(h)\varepsilon(h^{-1}) \cdot \prod_{\substack{k\in h^{-1}A,\\ k\neq h^{-1}}} \varepsilon(k) \cdot \prod_{k\in Y_h\setminus h^{-1}A} (\pi(d(h)) - \varepsilon(k)) \\ &= \pi(gh)\varepsilon(h^{-1}) \cdot \prod_{\substack{k\in h^{-1}A,\\ k\neq h^{-1}}} \varepsilon(k) \cdot \prod_{k\in Y_h\setminus h^{-1}A} (\pi(d(h)) - \varepsilon(k)) \\ &= \pi(gh) \cdot \prod_{k\in h^{-1}A} \varepsilon(k) \cdot \prod_{k\in Y_h\setminus h^{-1}A} (\pi(d(h)) - \varepsilon(k)) \\ &= \tilde{\pi}(h^{-1}A, gh) = \tilde{\pi}(B, gh) = \tilde{\pi}((A,g)\cdot(B,h)).\end{aligned}$$

Therefore, in all cases we have $\tilde{\pi}(A,g)\tilde{\pi}(B,h) = \tilde{\pi}((A,g)\cdot(B,h))$. This shows that by extending $\tilde{\pi}$ linearly from $\mathcal{G}^{\mathrm{BR}}$ to $K\mathcal{G}^{\mathrm{BR}}$ we obtain a homomorphism of $K\mathcal{G}^{\mathrm{BR}}$ on $\mathcal{A}$.

Now let $S \subseteq X_e$ be a subset, for some $e \in \mathcal{G}_0$. We set

$$P_S = \prod_{h\in S} \varepsilon(h) \prod_{h\in X_e\setminus S} (\pi(e) - \varepsilon(h)). \tag{2}$$

Using (1) it is easy to see that $\pi(\ell)P_S = P_{\ell S}\pi(\ell)$, for all $\ell \in \mathcal{G}$ with $d(\ell) = e$ and $S \subseteq X_e$.

Observe that if $e \notin S$, then $P_S = 0$. Moreover, if $\ell \in X_e \setminus S$, then $P_S\pi(\ell) = 0$, because $\varepsilon(\ell)\pi(\ell) = \pi(\ell)$, so

$$(\pi(e) - \varepsilon(\ell))\pi(\ell) = \pi(e)\pi(\ell) - \varepsilon(\ell)\pi(\ell) = \pi(r(\ell))\pi(\ell) - \pi(\ell) = \pi(\ell) - \pi(\ell) = 0.$$

Furthermore,

$$\pi(e) = \sum_{S \subseteq X_e} P_S, \tag{3}$$

since we have the combinatorial formula

$$\begin{aligned}\pi(e) &= \prod_{h \in X_e} \pi(e) = \prod_{h \in X_e} (\pi(e) - \varepsilon(h) + \varepsilon(h)) \\ &= \sum_{S \subseteq X_e} \left(\left(\prod_{h \in S} \varepsilon(h) \right) \cdot \left(\prod_{h \in X_g \setminus S} (\pi(e) - \varepsilon(h)) \right) \right).\end{aligned}$$

Now recall that $1_{K\mathcal{G}^{\mathrm{BR}}} = \sum_{e \in \mathcal{G}_0} \sum_{A \in \mathcal{L}_e} (A, e)$. Then

$$\begin{aligned}\widetilde{\pi}(1_{K\mathcal{G}^{\mathrm{BR}}}) &= \widetilde{\pi}\left(\sum_{e \in \mathcal{G}_0} \sum_{A \in \mathcal{L}_e} (A, e) \right) = \sum_{e \in \mathcal{G}_0} \sum_{A \in \mathcal{L}_e} \widetilde{\pi}(A, e) \\ &= \sum_{e \in \mathcal{G}_0} \sum_{A \in \mathcal{L}_e} \pi(e) \cdot \prod_{A \in \mathcal{L}_g} \varepsilon(g) \cdot \prod_{g \in Y_e \setminus A} (\pi(e) - \varepsilon(g)) \\ &= \sum_{e \in \mathcal{G}_0} \sum_{A \in \mathcal{L}_e} \prod_{A \in \mathcal{L}_g} \varepsilon(g) \cdot \prod_{g \in Y_e \setminus A} (\pi(e) - \varepsilon(g)) \\ &\overset{(2)}{=} \sum_{e \in \mathcal{G}_0} \sum_{A \in \mathcal{L}_e} P_A = \sum_{e \in \mathcal{G}_0} \sum_{A \subseteq X_e} P_A \overset{(3)}{=} \sum_{e \in \mathcal{G}_0} \pi(e) = 1_{\mathcal{A}}.\end{aligned}$$

Moreover,

$$\begin{aligned}\widetilde{\pi} \circ \lambda(g) &= \widetilde{\pi}\left(\sum_{A \in \mathcal{L}_g} (A, g) \right) = \sum_{A \in \mathcal{L}_g} \widetilde{\pi}(A, g) \\ &= \pi(g) \cdot \sum_{A \in \mathcal{L}_g} \sum_{h \in A} \varepsilon(h) \cdot \prod_{h \in Y_g \setminus A} (\pi(d(g)) - \varepsilon(h)) \\ &= \pi(g)\varepsilon(g^{-1}) \cdot \sum_{A \in \mathcal{L}_g} \sum_{\substack{h \in A, \\ h \neq g^{-1}}} \varepsilon(h) \cdot \prod_{h \in Y_g \setminus A} (\pi(d(g)) - \varepsilon(h)) \\ &= \pi(g) \cdot \sum_{A \in \mathcal{L}_g} \sum_{\substack{h \in A \\ h \neq g^{-1}}} \varepsilon(h) \cdot \prod_{h \in Y_g \setminus A} (\pi(d(g)) - \varepsilon(h)) \\ &= \pi(g) \cdot \sum_{A \in \mathcal{L}_g} \pi(d(g)) \sum_{\substack{h \in A, \\ h \neq g^{-1}}} \varepsilon(h) \cdot \prod_{h \in Y_g \setminus A} (\pi(d(g)) - \varepsilon(h)) \\ &= \pi(g) \cdot \sum_{A \in \mathcal{L}_g} (\varepsilon(g^{-1}) + \pi(d(g)) - \varepsilon(g^{-1})) \sum_{\substack{h \in A, \\ h \neq g^{-1}}} \varepsilon(h) \\ &\quad \cdot \prod_{h \in Y_g \setminus A} (\pi(d(g)) - \varepsilon(h))\end{aligned}$$

$$
\begin{aligned}
&= \pi(g)\cdot\sum_{B}\prod_{h\in B}\varepsilon(h)\cdot\prod_{h\in Y_g\setminus B}(\pi(d(g))-\varepsilon(h))\\
&= \pi(g)\tilde{\pi}\Big(\sum_{B}(B,d(g))\Big) = \pi(g)\tilde{\pi}(d(g)) = \pi(g)\cdot\Big(\sum_{e\in\mathscr{G}_0}\tilde{\pi}(e)\Big)\\
&= \pi(g)1_{K\mathscr{G}^{\mathrm{BR}}} = \pi(g).
\end{aligned}
$$

Now it only remains for us to show the uniqueness of the homomorphism $\tilde{\pi}$. To prove this claim, we shall show that $\lambda(\mathscr{G})$ generates $K\mathscr{G}^{\mathrm{BR}}$.

Let $(B,h)\in\mathscr{G}^{\mathrm{BR}}$, where $B=\{b_1^{-1},b_2^{-1},\ldots,b_{k-1}^{-1},h^{-1}\}$ is a subset of Y_h containing $d(h)$. The set of such pairs forms a vector space basis for $K\mathscr{G}^{\mathrm{BR}}$. Let us denote by $\mathfrak{A}$ the subalgebra of $K\mathscr{G}^{\mathrm{BR}}$ generated by $\lambda(\mathscr{G})$. Let $\{g_1,\ldots,g_k\}\subseteq\mathscr{G}$ be such that

$$
\begin{gathered}
g_1=b_1,\quad g_1g_2=b_2,\quad g_1g_2g_3=b_3,\ldots,\\
g_1g_2\cdots g_{k-1}=b_{k-1},\quad g_1\cdots g_k=h.
\end{gathered}
$$

These elements are well defined. In fact, from $g_1=b_1$, $d(b_1^{-1})=r(b_1)=d(h)=r(b_2)$ and $g_1g_2=b_2$ we obtain $g_2=b_1^{-1}b_2$. Inductively, we obtain $g_i=b_{i-1}^{-1}b_i$, for all $2\le i\le k-1$ and $g_k=b_{k-1}^{-1}h$.

Consider the element

$$
\begin{aligned}
\lambda(g_1)\cdots\lambda(g_k) &= \sum_{\substack{A_1\in\mathscr{L}_{g_1},\\ \vdots\\ A_k\in\mathscr{L}_{g_k}}}(A_1,g_1)\cdots(A_k,g_k) = \sum_{\substack{A_k\in\mathscr{L}_{g_k},\\ g_kA_k\in\mathscr{L}_{g_{k-1}},\\ \vdots\\ g_2\cdots g_kA_k\in\mathscr{L}_{g_1}}}(A_1,g_1\cdots g_k)\\
&= \sum_{A\supseteq B}(A,h).
\end{aligned}
$$

Thus, for all $(B,h)\in\mathscr{G}^{\mathrm{BR}}$, $\sum_{A\supseteq B}(A,h)\in\mathfrak{A}$. Suppose that $Y_g\setminus B=\{x_1,x_2,\ldots,x_n\}$. We have that

$$
\sum_{A\supseteq B}(A,h)-\sum_{A\supseteq B\cup\{x_1\}}(A,h)=\sum_{\substack{A\supseteq B,\\ A\not\ni x_1}}(A,h),
$$

from which it follows inductively that

$$
(B,h)=\sum_{\substack{A\supseteq B,\\ A\not\ni x_1,\ldots,x_{n-1}}}(A,h)-\sum_{\substack{A\supseteq B,\\ A\not\ni x_1,\ldots,x_n}}(A,h)\in\mathfrak{A},
$$

ending the proof. ∎

Remark 3.4. Many partial-to-global results for the case of groupoids have appeared in the literature. For example, in [2], Bagio and Paques proved that a global groupoid action can be constructed from a partial action on a ring under some hypotheses. For this goal, they expanded the ring in which the groupoid acts to then create a global action of the same groupoid on a new ring. In [13], Marín and Pinedo presented a similar globalization construction for groupoids acting partially on sets and topological spaces. Usually the term "globalization" refers to an expansion of the structure on which the groupoid acts, and the term "enlargement" refers to an expansion of the groupoid itself. The latter is our case, since we expanded the groupoid $\mathcal{G}$ to the algebra $K\mathcal{G}^{\mathrm{BR}}$ while the algebra $\mathcal{A}$ was fixed. Other cases of enlargements can also be found in [1] and [10], however in the case of ordered groupoids.

Corollary 3.5. *The groupoid algebra $K\mathcal{G}^{\mathrm{BR}}$ is isomorphic to the partial groupoid algebra $K_{\mathrm{par}}(\mathcal{G})$.*

Proof. The maps $[\,]: \mathcal{G} \to K_{\mathrm{par}}(\mathcal{G})$, $g \mapsto [g]$, and $\lambda : \mathcal{G} \to K\mathcal{G}^{\mathrm{BR}}$ are partial groupoid representations of $\mathcal{G}$. By Theorems 2.6 and 3.3, there exist K-algebra homomorphisms $\tilde{\pi} : K\mathcal{G}^{\mathrm{BR}} \to K_{\mathrm{par}}(\mathcal{G})$ and $\phi : K_{\mathrm{par}}(\mathcal{G}) \to K\mathcal{G}^{\mathrm{BR}}$ such that $\tilde{\pi}(\lambda(g)) = [g]$ and $\phi([g]) = \lambda(g)$, for all $g \in \mathcal{G}$. It is easy to check that $\tilde{\pi}$ and ϕ are inverses of each other, since their compositions are the identity on the generators; hence $\tilde{\pi}$ and ϕ are isomorphisms. ∎

Remark 3.6. The structure of $K_{\mathrm{par}}(\mathcal{G})$ regarding Bernoulli partial actions was studied extensively in [16], and several examples were presented.

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Wesley G. Lautenschlaeger
Instituto de Matemática, Universidade Federal do Rio Grande do Sul, Avenida Bento Gonçalves, 9500, 91509-900 Porto Alegre, Brazil; wesleyglautenschlaeger@gmail.com

Thaísa Tamusiunas
Instituto de Matemática, Universidade Federal do Rio Grande do Sul, Avenida Bento Gonçalves, 9500, 91509-900 Porto Alegre, Brazil; thaisa.tamusiunas@gmail.com