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SPECTRA IN BANACH AND LOCALLY CONVEX ALGEBRAS

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Recommended by A.F. Santos

This paper is dedicated to Professor J.R. Giles

Abstract: We present a counterexample to the converse of results on approximate point spectra. A norm inequality in terms of a distance function is established for a quotient bounded operator on a locally convex Hausdorff space on the set of complex numbers.

1 – Introduction

In this article, we follow the notation and terminology of [1], [3] and [7]. Following [3] and [7], we define the left and right approximate point spectra (LPS and RPS, respectively) of n-tuples $(x_1, x_2, \ldots, x_n) \in X$, where X is a Banach algebra, as:

$$LPS(x_1, \ldots, x_n) = \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \inf_{\|y\|=1} \sum_{i=1}^{n} \|\lambda_i y - x_i y\| = 0 \right\},$$

$$RPS(x_1, \ldots, x_n) = \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \inf_{\|y\|=1} \sum_{i=1}^{n} \|y \lambda_i - y x_i\| = 0 \right\}.$$

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The left and right Harte spectra of $(x_1,\ldots,x_n)\in X$ (LHS and RHS respectively) are defined below as:

$$LHS(x_1,\ldots,x_n) = \left\{(\lambda_1,\ldots,\lambda_n):\ \sum_{i=1}^{n} X(x_i-\lambda_i) \neq X\right\},$$

$$RHS(x_1,\ldots,x_n) = \left\{(\lambda_1,\ldots,\lambda_n):\ \sum_{i=1}^{n} (x_i-\lambda_i)X \neq X\right\}.$$

We note that in [7], $\{LHS\}\cup\{RHS\} = H_s$, where H_s is Harte's spectrum.

Remark 1.1. We present a modified version of Theorem 2 in [3] in the following theorem. □

Theorem 1.2. *Let $x, y\in X$ and $LPS(x)\subseteq RPS(x)$. Then, if $x\,y=1$, $y\,x=1$ where 1 is the unit element of the algebra X.*

Proof: Let $x\,y=1$. Then, $(yx)\,(yx) = y\,(xy)x = y(1)x = yx$ which shows that (yx) is an idempotent. If $yx=\mu$, then we will show that $\mu=1$. Now, for each $x_0\in X$,

$$\begin{aligned}\|x_0\| &= \|x_0\cdot 1\|\\ &= \|x_0 x\,y\| \leq \|x_0 x\|\cdot\|y\|\\ &\implies \|x_0\| - \|x_0 x\|\,\|y\| \leq 0\end{aligned}$$

which imples that zero does not belong to $RPS(x)$ by the definition of $RPS(x)$. Since $LPS(x)\subset RPS(x)$, we have $0\notin LPS(x)$. In this case, there is a positive ϵ with

$$0 < \epsilon \leq \frac{\|x\,x_0\|}{\|x_0\|}, \qquad \forall\, x_0\in X.$$

On the other hand,

$$1\cdot x = (x\,y)\,(x) = x\,(yx) = x\,\mu$$

implying that $x = x\,\mu$. Further simplification yields:

$$\begin{aligned} x - x\mu = 0 &\implies x\,(1-\mu) = 0\\ &\implies \|x\,(1-\mu)\| = 0.\end{aligned}$$

Now, we have

$$\begin{aligned}\|x\,(1-\mu)\| &\geq \epsilon\,\|1-\mu\| \qquad (\text{since } \epsilon\,\|x_0\|\leq\|x\,x_0\|)\\ &\implies \epsilon\,\|1-\mu\|\leq 0\end{aligned}$$

which is only possible if $\mu=1$. ∎

The motivations for the following example are the semigroups of generators (see Remark 12.3 in [8, p. 364]) and the algebra $\ell'(\cdot)$ [2, p. 54]. This example produces a counterexample to the converse of Theorem 1.2.

Example 1.3. Let X be the algebra ℓ' over G, where G is a semigroup of generators g_1, g_2 and g_3. An element of X can be given by

$$x = \sum \lambda\, g \;, \qquad g \in G \;\text{ and }\; \lambda \in \mathbb{C} \;.$$

The norm of x is given by

$$\|x\| = \sum |\lambda|\, \|g\|_G \leq \infty \;.$$

We remark that λ depends on a particular vector g in G. With this construction, X is a Banach algebra with generators g_1, g_2 and g_3. If e is the identity of G, then we let $\|e\|_G = 1$. We also let

$$\|g^k\|_G = (k!)^{-1} \;, \qquad \forall\, g \in G \;.$$

If $k_1 \geq 1$, $k_2 \geq 1$ and $k_3 \geq 1$, then

$$\begin{aligned} \|g_1^{k_1}\| &= (k_1!)^{-1} \\ \|g_2^{k_2}\| &= (k_2!)^{-1} \\ \|g_3^{k_3}\| &= (k_3!)^{-1} \;. \end{aligned}$$

In addition,

$$\|g_1 g_2\|_G \leq \|g_1\|_G\, \|g_2\|_G \,, \qquad \text{for} \quad g_1, g_2 \in G \;.$$

This algebra is a non-trivial algebra since there are no idempotents except for those that are trivial. This is not difficult to prove as we see below. □

Proof: Let $\mu \in X$. Then, by definition,

$$\mu = \sum_{g \in G} \lambda_g\, g \;.$$

Then, $\mu^2 = \mu$ implies either $\mu = 0$ or $\mu = 1$. To see this, let $g_1 g_2 = g$. Then

$$\begin{aligned} \lambda_g &= \sum_{g_1 g_2 = g} \lambda_{g_1} \lambda_{g_2} \;, \qquad \forall\, g \in G \\ &\implies \lambda_g = \lambda_{g_1} \lambda_1 + \lambda_1 \end{aligned}$$

and

$$\lambda_{g_2} = \lambda_{g_3} = 0 \;.$$

By mathematical induction, we have

$$\begin{aligned}\lambda_g &= \lambda_1\lambda_g + \lambda_g\lambda_1 \\ &\implies \lambda_g = 0, \quad \forall\, g \neq 1 \\ &\implies \mu = 0 \;\text{ or }\; \mu = 1\,.\end{aligned}$$

Since $\mu y = y$, it follows that $\mu = yx = 1$, if $xy = 1$. Thus, for all $x \in X$, we have shown that whenever $xy = 1$, $yx = 1$. ∎

Now, consider

$$\begin{aligned}\big\|(g_1+g_2)\,g_2^{k_1}\big\| &= \|g_1\,g_2^{k_1} + g_2^{k_1+1}\| \\ &\le \|g_1\,g_2^{k_1}\| + \|g_2^{k_1+1}\|\,.\end{aligned}$$

A simple calculation shows that

$$\big\|(g_1+g_2)\,g_2^{k_1}\big\| \le \|g_2^{k_1}\|\left(\frac{2}{1+k_1}\right).$$

By the definition of LPS, we have zero belonging to $LPS(g_1+g_2)$. If

$$x - \lambda = x_1\,g_1 + x_2\,g_2 + x_3\,g_3$$

then the following inequality holds

$$2\,\|x\,(g_1+g_2)\| \ge \|x\|$$

implying that $0 \notin RPS(g_1+g_2)$, which shows that the converse of Theorem 1.2 is not true in general.

2 – Spectra of quotient bounded operators

Let E be a locally convex Hausdorff space over the field of complex numbers. Let $S(E)$ be the family of seminorms such that $S(E) = \{s_\alpha\colon \alpha \in I\}$. The topology on E is induced by $S(E)$. For a given S, we denote by $Q_S(E)$ the algebra of quotient bounded operators on E. That is,

$$Q_S(E) = \Big\{T\colon\; s_\alpha(Ta) \le K_\alpha\, s_\alpha(a),\; a \in E,\; \alpha \in I\Big\}\,.$$

If $K_\alpha = K$, then by [5], we have $B_S(E)$ denoting the algebra of bounded operators on E. We say $Q_S(E)$ is a unital l.m.c. algebra where the seminorm

$$S_0 = \Big\{s_{0_\alpha}:\ \alpha \in I\Big\}$$

and

$$s_{0_\alpha}(T) = \sup\Big\{s_\alpha(Ta):\ s_\alpha(a) \le 1,\ a \in E\Big\}\ .$$

Also, the norm of an operator $T \in B_S(E)$ is given by

$$\|T\|_S = \sup\Big\{s_{0_\alpha}(T):\ \alpha \in I\Big\}\ .$$

For each $\alpha \in I$, let N_α denote the null space of s_α. Then, the quotient space $\frac{E}{N_\alpha}$ is denoted by E_α. Through [5] it is known that the algebra E_α is a normed algebra such that for each a_α in E_α, $\|a+N_\alpha\|_\alpha = s_\alpha(a)$. Let $\overline{E}_\alpha$, denote the completion of E_α. We note that $(Ta)_\alpha = T_\alpha\, a_\alpha$, for each $\alpha \in I$ and for each $a \in E$.

If $s_\alpha^2 = (a,a)_\alpha$, $a \in E$, then the adjoint of an operator $T \in Q_S(E)$ is T^*. In other words, for each $\alpha \in I$ and $a, b \in E$, $(Ta, b)_\alpha = (a, T^* b)_\alpha$. Obviously, $\overline{E}_\alpha$ becomes a Hilbert space and $(\overline{T}_\alpha)^*$ is the adjoint operator of $\overline{T}_\alpha$, where $\overline{T}_\alpha$ is the continuous extension of T_α on $\overline{E}_\alpha$. See [5] and [9].

By [1], the spectrum of $T \in Q_S(E)$ is denoted by $SP(T)$. That is,

$$SP(T) = \Big\{\lambda:\ (\lambda I - T) \text{ is not invertible}\Big\}\ .$$

Let $SP_\alpha(\overline{T}_\alpha)$ denote the spectrum of $\overline{T}_\alpha$ in $\overline{E}_\alpha$. Then, by [5], we have

$$SP(T) = \bigcup_\alpha \big\{SP_\alpha(\overline{T}_\alpha)\big\}\ .$$

The following theorem is an easy consequence of the definitions involved. Compare this theorem with Harte's spectrum in [7].

Theorem 2.1. *If SP_a and SP_r are the approximate and residual spectra of $T \in Q_S(E)$ respectively, then $SP(T) = SP_a(T) \cup SP_r(T)$.*

Proof: Let $\lambda \notin SP_a \cup SP_r$. Then

$$\lambda \in \Big(SP_a \cup SP_r\Big)^c$$

which implies that

$$\lambda \in SP_a^c \cap SP_r^c\ .$$

By the definition of an approximate spectrum and since $\lambda \notin SP_a$, there exists an inverse operator which is continuous with

$$s_\alpha\big((T-\lambda I)^{-1}c\big) \le K_\alpha s_\alpha(c)\ , \qquad \alpha \in I$$
$$c \in \text{range of } (T-\lambda I) = R(T-\lambda I)\ .$$

Let $b \in E$. Then, there exists a sequence

$$\{a_n\} \quad \text{s.t.} \quad Ta_n - \lambda a_n = b_n \to b\ .$$

This is possible because the range of $(T-\lambda I)$ is dense. Hence, the sequence $\{a_n\}$ with $a_n = (T-\lambda I)^{-1}b_n$ is convergent. Thus, the continuity of $(T-\lambda I)$ implies that $b = (T-\lambda I)a$. This implies that $R(T-\lambda I) = E$ and $\lambda \in SP^c$. ∎

Theorem 2.2. *Let E be a separated locally convex space and $T \in Q_S(E)$. Then, SP_a is non-empty if and only if for a given $\alpha \in I$ and a sequence $\{a_n\} \in E$ we have $s_\alpha((T-\lambda I)a_n) \to 0$.*

Proof: By the definition of SP_a, for each $\alpha \in I$, provided that $\lambda \notin SP_a$, there exists a $K_\alpha > 0$ with

$$K_\alpha s_\alpha(a) \le s_\alpha(Ta)\ , \quad a \in E\ .$$

This yields the following inequality: for $a_\alpha \in E_\alpha$,

$$K_\alpha \|a_\alpha\|_\alpha \le \|T_\alpha a_\alpha\|_\alpha\ .$$

This shows that zero does not belong to $SP_a(\overline{T}_\alpha)$ for all $\alpha \in I$. In any case, the inequality $K_\alpha s_\alpha(a) \le s_\alpha(Ta)$ holds.

Since $\{a_n\}$ is a sequence in E, we have

$$s_\alpha\big((T-\lambda I)a_n\big) \ge K_\alpha s_\alpha(a_n)$$
$$\implies \quad s_\alpha\big((T-\lambda I)a_n\big) \to 0\ .$$

The other implication follows from the fact that

$$SP_a(T) = \bigcup_\alpha \big\{SP_a(\overline{T}_\alpha)\big\}\ . \ ∎$$

Remark 2.3. $SP_p(T) \subset \bigcup_\alpha \{SP_p(\overline{T}_\alpha)\}$. □

Theorem 2.4. *Let E be a complete, separated, locally convex space and let bSP denote the boundary of the spectrum. Then,*

$$SP_a(T) \supset bSP(T) \cap SP(T) \ .$$

Proof: In the case of a normed algebra, the boundary of the spectrum of an operator is contained in the spectrum if the space is separated. In our case, it is easy to see that if $\lambda \in bSP(T) \cap SP(T)$, then for $\alpha \in I$, $\lambda \in SP(\overline{T}_\alpha)$. Therefore, there exists an open ball B, such that $\lambda \in B$. In fact, B is a subset of $SP(\overline{T}_\alpha)$ which implies that B is contained in $SP(T)$ and hence λ is not a boundary point of $SP(T)$. Thus, $\lambda \in SP_a(\overline{T}_\alpha)$. The proof of the theorem follows from Theorem 2.2 and the fact that $\lambda \in SP_a(\overline{T}_\alpha)$. That is, $\lambda \in SP_a(T)$. ∎

Remark 2.5. If $T \in B_S(E)$ such that $\|T\|_S = |\lambda|$, where λ belongs to the spatial numerical range $V(E,P,T)$ (see [5]), then $\lambda \in SP_a(T)$. This is a straightforward application of Theorem 2.2 and the definition of $V(E,P,T)$. We also remark that for $T \in Q_S(E)$ (where E is separated) that

$$\bigcup_\alpha \bigl\{SP_a(\overline{T}_\alpha)\bigr\} \;=\; SP_a(T) \ . \ \square$$

In the next theorem, we establish a bound for the norm of T^{-1} with respect to the seminorm S in terms of the distance between the origin and the closure of $V(E,S,T) = \overline{V(T)}$. If A is a subset of $\mathbb{C}$, then we denote the distance between the origin and A by $d(A,0)$.

Theorem 2.6. *Let E be a complete, locally convex space and let $T \in Q_s(E)$. Suppose that the set $\overline{V(E,S,T)} = \overline{V(T)}$ does not contain zero. Then, $T^{-1} \in B_S(E)$ and*

$$\|T^{-1}\|_S \left(d\bigl(\overline{V(T)},0\bigr)\right) \;\le\; 1 \ .$$

Proof: Let $0 \notin \overline{V(T)}$. Then, by known results in [4] and [5], zero is in the spectral radius $\rho(\overline{T}_\alpha)$ of $\overline{T}_\alpha$ for each $\alpha \in I$. Also, for $a_\alpha \in \overline{E}_\alpha$, the following inequality holds:

$$\|\overline{T_\alpha^{-1}}\, a_\alpha\|_\alpha \;\le\; \|\overline{T_\alpha^{-1}}\|\ \|a_\alpha\|_\alpha \ .$$

In fact, by the relation between $\|\cdot\|_\alpha$ and seminorms s_α, we have for $\alpha \in I$

$$s_\alpha(T^{-1}) \;\le\; \|\overline{T}_\alpha^{-1}\|_\alpha \ .$$

Hence, by the connection between the spatial and the algebraic numerical ranges in $\overline{E}_\alpha$ (see [1]), we have zero not belonging to $\overline{V(\overline{T}_\alpha)}$. Combining the above facts, we have

$$\|\overline{T}_\alpha^{-1}\|_\alpha \left(d\big(\overline{V(\overline{T}_\alpha)},0\big)\right) \leq 1 \leq d\big(\overline{V(\overline{T}_\alpha)},0\big)\cdot\left(d\Big(\bigcup_\alpha \overline{V(\overline{T}_\alpha)},\,0\Big)\right)^{-1}$$
$$\leq \left(d\big(\overline{V(T)},0\big)\right)^{-1}.$$

Now $s_\alpha(T^{-1}) \leq \|\overline{T}_\alpha^{-1}\|_\alpha \implies s_\alpha(T^{-1})\cdot d\big(\overline{V(T)},0\big) \leq 1,$ for each $\alpha,$ which shows that $T^{-1}\in B_S(E)$ and the required inequality holds. ∎

Remark 2.7. If $r(T)$ and $\rho(T)$ denote the numerical and spectral radii of T respectively, then

$$r \in \left[\frac{\|T\|_S}{2},\, \|T\|_S\right].$$

This can easily be verified by previously known results and the following relation

$$\|\overline{T}_\alpha\|_\alpha \leq 2\,r(\overline{T}_\alpha)\,.$$ □

Remark 2.8. For more on these inequalities and estimates, refer to [6] and [10]. □

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