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LOCAL ENERGY DECAY FOR THE NONLINEAR DISSIPATIVE WAVE EQUATION IN AN EXTERIOR DOMAIN

Moez Daoulatli

Recommended by E. Zuazua

Abstract: In odd dimension space, under a microlocal geometric condition, we give the rate of decay of the local energy for solutions of the wave equation on exterior domain, with localized nonlinear damping.

Résumé: En dimension impaire d'espace, on détermine sous une condition géométrique microlocale, le taux de décroissance de l'énergie locale des solutions de l'équation des ondes dans un domaine extérieur, en présence d'un dissipateur non linéaire localisé.

1 – Introduction and Statement of the result

Let O be a compact domain of $\mathbb{R}^d$ ($d \geq 3$ is odd) with C^∞ boundary $\Gamma = \partial\Omega$ and $\Omega = \mathbb{R}^d \backslash O$. Consider the following wave equation with a nonlinear internal damping

$$\text{(1.1)} \qquad \begin{cases} \partial_t^2 u - \Delta u + a(x)\, f(\partial_t u) \;=\; 0 & \text{in } \mathbb{R}_+ \times \Omega\,, \\ u = 0 & \text{on } \mathbb{R}_+ \times \Gamma\,, \\ u(0,x) = \varphi_1 \quad \text{and} \quad \partial_t u(0,x) = \varphi_2\;. \end{cases}$$

Here Δ denotes the Laplace operator in the space variables and $a(x)$ is a non-negative function in $L^\infty(\Omega)$ with compact support. $f\colon \mathbb{R} \to \mathbb{R}$ is a nondecreasing,

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continuous function, such that $f(0) = 0$ and satisfying the following polynomial growth near the origin

$$c_1|s|^r \leq |f(s)| \leq c_2|s|^{\frac{1}{r}} , \qquad |s| \leq 1 , \tag{1.2}$$

where $c_1, c_2 > 0$ and $r \geq 1$; moreover we suppose the growth condition at infinity

$$c_3|s| \leq |f(s)| \leq c_4|s|^p , \qquad |s| > 1 , \tag{1.3}$$

with $c_3, c_4 > 0$ and $p \geq 1$.

The problem with linear or nonlinear dissipation in bounded domain has been intensively investigated in [3], [16], [17], [7], [12], [18], [11], [28], [19], [14], [24], [29], [5], etc.. One can find results with damping terms effective everywhere, or localized on a suitable subset of the domain or on the boundary, under more or less strong geometrical conditions like the Lions condition, or the microlocal Bardos–Lebeau–Rauch condition. Various rates of decay (from exponential decay to logarithmic decay) are then obtained depending on the geometry and the nonlinear behavior of the damping term.

When Ω is an exterior domain, we define the local energy by

$$\begin{aligned} E_\rho(u)(t) &= \frac{1}{2}\int_{\Omega\cap B_\rho}\Big(|\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2\Big)\,dx \\ &= \big\|\big(u(t), \partial_t u(t)\big)\big\|^2_{H(B_\rho)} , \end{aligned}$$

where

$$B_\rho = \Big\{x \in \mathbb{R}^d,\ |x| < \rho\Big\}$$

contains the obstacle O, and our goal is to give a decay estimate for this energy.

For the linear undamped wave equation (i.e. $a = 0$) outside a compact obstacle in odd dimension space, the study of the local energy goes up to the pioneering works of Lax–Phillips [15], Morawetz, Strauss, Ralston. When the obstacle is trapping Ralston [27] proved that there is no uniform decay rate, and Morawetz–Ralston–Strauss [22], Melrose [20] obtained the exponential decay for non-trapping obstacle. On the other hand, without any assumption on the dynamics Burq [6] proved the logarithmic decay of the local energy with respect to any Sobolev norm larger than the energy norm.

When $a(x)\, f(\partial_t u) = a(x)\, \partial_t u$, Nakao in [25] proved that the local energy decay exponentially if d is odd and polynomially if d is even under the Lions's geometric condition. More recently, combining the definition of a non-trapping obstacle and the geometric control condition of Bardos–Lebeau–Rauch [3], Aloui and Khenissi [1] introduced the exterior geometric control condition:

Definition 1.1. Let $R>0$ such that $O\subset B_R$, $T_R>0$ and $\omega=\{x\in\Omega;\ a(x)>0\}$. We say that (ω,T_R) verifies the exterior geometric control condition on B_R $(E.G.C)$, if every geodesic γ starting from B_R at time $t=0$, is such that

- γ leaves $\mathbb{R}_+\times B_R$ before the time T_R, or
- γ meets $\mathbb{R}_+\times\omega$ between the times 0 and T_R. □

So they prove the exponential decay with a localized linear damping term.

To our knowledge very few results seem to be known for the wave equation with nonlinear dissipation in the whole space or in exterior domain. In the whole space, when $a=1$, K. Mochizuki and T. Motai [21] prove the logarithmic decay of the global energy, and K. Ono [26] prove the polynomial decay when the dissipative term is equal to $\partial_t u+|\partial_t u|^{p-1}\partial_t u$, $1<p\le 3$. Finally, we mention the work of M. Nakao and I. Hyo Jung [13] in exterior domain, where they obtained the polynomial decay of the energy with dissipation which is nonlinear in a bounded region and linear far from the obstacle.

Before stating the results of this paper, we need to precise some definitions and notations. $H=H_D(\Omega)\times L^2(\Omega)$, is the completion of $(C_0^\infty(\Omega))^2$ with respect to the norm

$$\|\varphi\|_H^2 \;=\; \|(\varphi_1,\varphi_2)\|_H^2 \;=\; \frac{1}{2}\int_\Omega |\nabla\varphi_1|^2+|\varphi_2|^2\,dx\;.$$

H_0 is the completion of $(C_0^\infty(\mathbb{R}^d))^2$ with respect to the norm

$$\|\varphi\|_{H_0}^2=\;\frac{1}{2}\int_{\mathbb{R}^d}\Big(|\nabla\varphi_1|^2+|\varphi_2|^2\Big)\,dx\;.$$

Let $A=\begin{pmatrix}0 & I\\ \Delta & 0\end{pmatrix}$ be the unbounded operator on H, with domain

$$D(A)\;=\;\Big\{\varphi\in H;\ \Delta\varphi_1\in L^2(\Omega),\ \ \varphi_2\in H_D(\Omega)\Big\}\;.$$

Let us consider the wave equation in exterior domain

$$\begin{cases}\partial_t^2 u-\Delta u\;=\;0 & \text{in }\ \mathbb{R}\times\Omega\,,\\ u=0 & \text{on }\ \mathbb{R}\times\partial\Omega\,,\\ u(0,\cdot)=\varphi_1\quad\text{and}\quad \partial_t u(0,\cdot)=\varphi_2\;, & \end{cases}\tag{1.4}$$

where $(\varphi_1,\varphi_2)\in H$. We denote $U_L(t)$ the isometric linear wave group, defining the solution of (1.4)

$$\begin{array}{rccl} U_L(t): & H & \longrightarrow & H\\ & (\varphi_1,\varphi_2) & \longmapsto & U_L(t)(\varphi_1,\varphi_2)\;=\;\big(u(t),\partial_t u(t)\big)\;. \end{array}\tag{1.5}$$

Let $R>0$ such that B_R contains the obstacle and the support of the function $a(x)$. Following Lax–Phillips we denote by

$$D_+^R = \Big\{\varphi = (\varphi_1,\varphi_2) \in H\,;\ U_L(t)\,\varphi = 0 \ \text{ on } |x| < t+R,\ t \geq 0\Big\}\ .$$

We will precise this space in Section 3.1.1.

Let G the subspace of H, defined by

$$G = D(A) + D_+^R\ . \tag{1.6}$$

Theorem 1. *For every $\varphi \in D(A)$, the problem (1.1) admits a unique solution*

$$u \in C(\mathbb{R}_+, H_D(\Omega))\ , \qquad \partial_t u \in C(\mathbb{R}_+, L^2(\Omega))\ . \tag{1.7}$$

The energy of u is defined by

$$E(u)(t) = \big\|\big(u(t),\partial_t u(t)\big)\big\|_H^2 = \frac{1}{2}\int_\Omega |\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2\,dx \tag{1.8}$$

and satisfies

$$E(u)(t) - E(u)(0) = -\int_0^t\!\!\int_\Omega a(x)\,f(\partial_t u(\tau,x))\,\partial_t u(\tau,x)\ dx\,d\tau \tag{1.9}$$

for every $t \geq 0$. In addition we have

$$\big(u(t),\partial_t u(t)\big) \in D(A) \quad \text{for every } \ t \geq 0 \tag{1.10}$$

and

$$\big\|\big(\partial_t u(t),\partial_t^2 u(t)\big)\big\|_H \leq \big\|\big(\varphi_2,\,\Delta\varphi_1 - af(\varphi_2)\big)\big\|_H\,, \qquad \text{a.e. } \ t \geq 0\ . \tag{1.11}$$

Theorem 2.

(**1**) $p=1$. *For every φ in H, the problem (1.1) admits a unique solution u verifying (1.7) and (1.9).*

(**2**) $1 \leq p \leq \frac{d}{d-2}$. *For every φ in G, the problem (1.1) admits a unique solution u verifying (1.7) and (1.9). In addition we have*

$$\big(u(t),\partial_t u(t)\big) \in G, \quad \text{for every } \ t \geq 0\ . \tag{1.12}$$

Let

$$X = \begin{cases} H & \text{if } p = 1\,, \\ G & \text{if } p > 1\,. \end{cases} \tag{1.13}$$

For every $t \in \mathbb{R}_+$, we define the evolution operator $U(t)$ by

$$\begin{aligned} U(t): \quad X \quad &\longrightarrow \quad X \\ (\varphi_1, \varphi_2) &\longmapsto U(t)(\varphi_1, \varphi_2) = \big(u(t), \partial_t u(t)\big) \end{aligned}$$

where u is the solution of (1.1). $(U(t))_{t\in\mathbb{R}_+}$ forms a one parameter semi-group on X. Moreover, for every $\varphi \in X$, the map

$$\begin{aligned} \mathbb{R}_+ &\to H \\ t &\mapsto U(t)\,\varphi \end{aligned}$$

is continuous.

Take $D_0 > 0$ and denote by

$$B_H(D_0) = \Big\{\varphi \in H;\ \|\varphi\|_H < D_0\Big\} \tag{1.14}$$

and

$$B_{D(A)}(D_0) = \Big\{\varphi \in D(A);\ \|\varphi\|_H^2 + \big\|\big(\varphi_2, \Delta\varphi_1 - a\,f(\varphi_2)\big)\big\|_H^2 < D_0^2\Big\}\,. \tag{1.15}$$

We come now to the main result of this paper.

Theorem 3. *Let $R > 0$ and $T_R > 0$, such that $\big(\{x \in \Omega;\ a(x) > 0\}, T_R\big)$ satisfies the exterior geometric condition on B_R. Then for every $D_0 > 0$*

(**1**) $p = 1$. *There exist $c > 0$ and $\delta > 0$ such that inequality*

$$E_R(u)(t) \le c\,e^{-\delta t}\|\varphi\|_H^2\,, \qquad t \ge 0 \quad \text{if } r = 1\,,$$

$$E_R(u)(t) \le \Big(\|\varphi\|_H^{1-r} + c(t-T)\Big)^{-\frac{2}{r-1}}\,, \qquad t \ge T = T_R + 9R \quad \text{if } r > 1\,,$$

holds for every solution u of (1.1), if the initial data $\varphi \in B_H(D_0)$ and is supported in B_R.

(**2**) $1 < p < \frac{d}{d-2}$. *There exists $c > 0$ such that inequality*

$$E_R(u)(t) \le \Big(\|\varphi\|_H^{2(1-s)} + c(t-T)\Big)^{-\frac{1}{s-1}}\,, \qquad t \ge T = T_R + 9R\,,$$

holds for every solution u of (1.1), if the initial data $\varphi \in B_{D(A)}(D_0)$ and is supported in B_R, with

$$s = \max\left(\frac{r+1}{2}, \frac{(d+2) - p(d-2)}{2\big(d - p(d-2)\big)}\right)\,.$$

Remark 1.1. This theorem is a local stabilization result. The constant c depends on the ball $B_H(D_0)$ or $B_{D(A)}(D_0)$ in which we choose the initial data. However, it is uniform on every ball. The question of the existence of a global decay rate, is still open. □

The tools used in the proof are of microlocal nature and essentially inspired from [1] and [4]. We first built a nonlinear Lax–Phillips semi-group $Z(t)$ which characterizes the local energy and operates on a smooth subspace of the energy space. Using then microlocal defect measures we obtain an a priori estimate on the norm of $Z(t)$ which yields the desired decay of the local energy.

The rest of this article is organized as follows.

2. Proof of Theorems 1 and 2

3. Rate of decay of the local energy

3.1. Lax–Phillips theory

3.2. Reduction to an a priori estimate

3.3. Proof of Theorem 3

2 – Proof of Theorems 1 and 2

2.1. Proof of Theorem 1

To prove this result we use the nonlinear version of Hille–Yosida Theorem. Let $\tilde{A}\colon D(\tilde{A}) \subset H \to H$ the operator defined by

$$\tilde{A}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \Delta u - a(x)\, f(v) \end{pmatrix},$$

with domain

$$D(\tilde{A}) = \Big\{ \varphi \in H;\ \varphi_2 \in H_D(\Omega),\ \Delta\varphi_1 - a(x)\, f(\varphi_2) \in L^2(\Omega) \Big\}\ .$$

We remark that by virtue of the continuity of the embedding $H_D(\Omega) \hookrightarrow L^q_{loc}(\Omega)$ for $1 \le q \le \frac{2d}{d-2}$, we have

$$D(\tilde{A}) = D(A)\ .$$

So if we set $v = \partial_t u$ (u the solution of (1.1)) then formally we have

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \tilde{A}\begin{pmatrix} u \\ v \end{pmatrix}\ , \qquad \begin{pmatrix} u \\ v \end{pmatrix} \in D(A)\ .$$

According to [2, Chapter 3, p. 118], it is clear that in order to prove that the system (1.1) with initial data $\varphi = (\varphi_1, \varphi_2)$ in $D(A)$, admits a unique solution u verifying (1.7), (1.10) and (1.11), it suffices to prove that $\tilde{A}$ is a maximal dissipative operator in H. f is a nondecreasing function then $\tilde{A}$ is dissipative. Indeed

$$\left\langle \tilde{A}\begin{pmatrix} u \\ v \end{pmatrix} - \tilde{A}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \right\rangle = -\int_\Omega a(x)\Big(f(v) - f(v_1)\Big)(v - v_1)\, dx \le 0 .$$

To prove that $\tilde{A}$ is maximal, we need to show that, for every $h \in H$ the equation

$$\tilde{A}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = h .$$

admits a solution $\begin{pmatrix} u \\ v \end{pmatrix}$ in $D(A)$. Equivalently, for every $h_1 \in H_D(\Omega)$ and $h_2 \in L^2(\Omega)$ there exists u and v in $H_D(\Omega)$ such that $\Delta u \in L^2(\Omega)$ and $v \in L^2(\Omega)$ verifying the system

$$\begin{cases} u = v - h_1 & \text{in } D'(\Omega)\,, \\ \Delta v - v - a(x) f(v) = \Delta h_1 + h_2, & \text{in } D'(\Omega)\ . \end{cases}$$

In other words, it is sufficient to show that, there exist v in $H_0^1(\Omega) = H_D(\Omega) \cap L^2(\Omega)$ such that

$$\int_\Omega \nabla v\, \nabla\chi + v\chi\, dx + \int_\Omega a(x) f(v)\, \chi\, d\sigma = \int_\Omega \nabla h_1 \nabla\chi - h_2\chi\, dx . \tag{2.1}$$

for every χ in $H_0^1(\Omega)$. Now we prove (2.1). Let

$$\begin{aligned} A_1\colon\ H_0^1(\Omega) &\longrightarrow H^{-1}(\Omega) \\ v &\longmapsto A_1 v \end{aligned}$$

defined by

$$\forall\, \varphi \in H_0^1(\Omega)\ , \qquad \langle A_1 v, \varphi\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega \nabla v\, \nabla\varphi\, dx\ .$$

and

$$\begin{aligned} B\colon\ H_0^1(\Omega) &\longrightarrow H^{-1}(\Omega) \\ v &\longmapsto Bv \end{aligned}$$

defined by

$$\forall\, \varphi \in H_0^1(\Omega)\ , \qquad \langle Bv, \varphi\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega a(x) f(v)\, \varphi\, dx\ .$$

Thus (2.1) is equivalent to $I+A_1+B$ is onto. We have

- $\big\langle (I+A_1+B)v,\, v\big\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \|v\|^2_{H_0^1(\Omega)} + \displaystyle\int_\Omega a(x)\, f(v)\, v\; dx \;\geq\; 0\,.$
- $\dfrac{\big\langle (I+A_1+B)v,\, v\big\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}}{\|v\|_{H_0^1(\Omega)}} \xrightarrow[\|v\|_{H_0^1(\Omega)}\to+\infty]{} +\infty\,.$

These properties means that the operator $I+A_1+B$ is monotone and coercive, in addition it is clear that $I+A_1+B\colon H_0^1(\Omega)\to H^{-1}(\Omega)$ is hemicontinuous ([1]). Using then [2, Chapter 2, Theorem 1.3, p. 40] we deduce that $I+A_1+B$ is onto.

Finally, multiplying the equation by $\partial_t u$ and integrating by part we obtain (1.9). ∎

2.2. Proof of Theorem 2

(**1**) We remind that $p=1$. Let $\varphi\in H$, then there exists a sequence (φ_n) in $D(A)$ such that φ_n converges strongly to φ in H. Let $(U(t)\,\varphi_n) = (u_n(t), \partial_t u_n(t))$. Using the fact that f is a nondecreasing function and the classical energy estimate, we deduce that $(u_n, \partial_t u_n)$ is a Cauchy sequence in $C(\mathbb{R}_+;\, H_D(\Omega)\times L^2(\Omega))$. Then there exists

$$(u,v) \,\in\, C\big(\mathbb{R}_+;\, H_D(\Omega)\times L^2(\Omega)\big)$$

such that

$$(u_n, \partial_t u_n) \xrightarrow[n\to+\infty]{} (u,v) \quad \text{in } \; C\big(\mathbb{R}_+;\, H_D(\Omega)\times L^2(\Omega)\big)\;.$$

Then, for every $\chi\in C_0^\infty(\Omega)$ and $t\geq 0$

$$\big\langle u(t),\chi\big\rangle_{L^2(\Omega)} = \big\langle \varphi_1,\chi\big\rangle_{L^2(\Omega)} + \int_0^t \big\langle v(s),\chi\big\rangle_{L^2(\Omega)}\, ds$$

so

$$\big\langle \partial_t u(t),\chi\big\rangle_{L^2(\Omega),L^2(\Omega)} = \big\langle v(t),\chi\big\rangle_{L^2(\Omega),L^2(\Omega)}\;,$$

which yields

$$(u_n, \partial_t u_n) \xrightarrow[n\to+\infty]{} (u,\partial_t u) \quad \text{in } \; C\big(\mathbb{R}_+;\, H_D(\Omega)\times L^2(\Omega)\big)\;. \tag{2.2}$$

(1.9) gives

$$\int_0^t\!\!\int_\Omega a(x)\, f(\partial_t u_n(t,x))\, \partial_t u_n(t,x)\; dx\, dt \;\leq\; \|\varphi_n\|^2_H\;, \tag{2.3}$$

([1]) We say that C is hemicontinuous on $H_0^1(\Omega)$ if $D(C)=H_0^1(\Omega)$ and $C(x+ty) \underset{t\to 0}{\rightharpoonup} C(x)$ weakly in $H_0^1(\Omega)$.

for every $T \geq 0$. Using (1.2), (1.3) and (2.3), we deduce that $af(\partial_t u_n)$ is bounded in $L^2([0,T]\times\Omega)$, for every $T \geq 0$. Then there exists ψ in $L^2([0,T]\times\Omega)$ such that

$$a\,f(\partial_t u_n) \underset{n\to+\infty}{\rightharpoonup} \psi \qquad \text{weakly in } \; L^2([0,T]\times\Omega)\ .$$

On the other hand, using (2.2), (1.3) and (1.2) we obtain

$$a\,f(\partial_t u) \;\in\; L^2([0,T]\times\Omega)\ . \tag{2.4}$$

In order to finish the proof of the existence, it remains to show that $\psi = a\,f(\partial_t u)$ a.e. on $[0,T]\times\Omega$.

The classical energy estimates gives

$$\lim_{n\to+\infty} \int_0^t\!\!\int_\Omega a(x)\,f(\partial_t u_n)\,\partial_t u_n \; dx\,ds \;=\; \int_0^t\!\!\int_\Omega \psi\,\partial_t u \; dx\,ds\ ,$$

which yields

$$\begin{aligned}\lim_{n\to+\infty} \int_0^t\!\!\int_\Omega a(x)\,\Big(f(\partial_t u_n) - f(\phi)\Big)\,(\partial_t u_n - \phi)\; dx\,ds \;&=\\ = \int_0^t\!\!\int_\Omega \Big(\psi - a(x)\,f(\phi)\Big)\,(\partial_t u - \phi)\; dx\,ds&\end{aligned} \tag{2.5}$$

for every $t \in [0,T]$ and $\phi \in L^2([0,T]\times\Omega)$. Using the fact that f is monotone, we obtain:

$$\int_0^T\!\!\int_\Omega \Big(\psi - a(x)\,f(\phi)\Big)\,(\partial_t u - \phi)\; dx\,ds \;\geq\; 0\ , \qquad \forall\,\phi \in L^2([0,T]\times\Omega)\ . \tag{2.6}$$

Taking $\phi = \partial_t u - \lambda\xi,\ \lambda > 0,\ \xi \in L^2([0,T]\times\Omega)$ in (2.6) we obtain

$$\lambda \int_0^T\!\!\int_\Omega \Big(\psi - a(x)\,f(\partial_t u - \lambda\,\xi)\Big)\,\xi \; dx\,ds \;\geq\; 0\ ,$$

for every $\xi \in L^2([0,T]\times\Omega)$ and $\lambda > 0$. So

$$\int_0^T\!\!\int_\Omega \Big(\psi - a(x)\,f(\partial_t u - \lambda\,\xi)\Big)\,\xi \; dx\,ds \;\geq\; 0\ ,$$

then letting $\lambda \to 0$

$$\int_0^T\!\!\int_\Omega \Big(\psi - a(x)\,f(\partial_t u)\Big)\,\xi \; dx\,ds \;\geq\; 0\ , \qquad \forall\,\xi \in L^2([0,T]\times\Omega)\ ,$$

and the result follows.

Uniqueness of the solution.

Let u_1 and u_2 two solutions of the problem (1.1), with the same initial data φ in H. For $w = u_1 - u_2$, we can write

$$\begin{cases} \partial_t^2 w - \Delta w \;=\; -a(x)\left[f(\partial_t u_1) - f(\partial_t u_2)\right] & \text{in } \mathbb{R}_+ \times \Omega\,, \\ w = 0 & \text{on } \mathbb{R}_+ \times \Gamma\,, \\ w(0,x) = 0 \quad \text{and} \quad \partial_t w(0,x) = 0\;. \end{cases}$$

Hence

$$E(w)(t) \;=\; -\int_0^t\!\!\int_\Omega a(x)\left[f(\partial_t u_1) - f(\partial_t u_2)\right]\partial_t(u_1 - u_2)\; dx\, ds \;\le\; 0$$

since f is monotone; then $w = 0$.

(**2**) We remind that $1 \le p \le \frac{d}{d-2}$. Let φ in G, then there exist ψ in $D(A)$ and χ in D_+^R, such that $\varphi = \psi + \chi$.

Let v and w verifying

$$\begin{cases} \partial_t^2 v - \Delta v + a(x)\,f(\partial_t v) \;=\; 0 & \text{in } \mathbb{R}_+ \times \Omega\,, \\ v = 0 & \text{on } \mathbb{R}_+ \times \Gamma\,, \\ v(0,x) = \psi_1 \quad \text{and} \quad \partial_t v(0,x) = \psi_2\;, \end{cases}$$

$$\begin{cases} \partial_t^2 w - \Delta w \;=\; 0 & \text{in } \mathbb{R}_+ \times \Omega\,, \\ w = 0 & \text{on } \mathbb{R}_+ \times \Gamma\,, \\ w(0,x) = \chi_1 \quad \text{and} \quad \partial_t w(0,x) = \chi_2\;. \end{cases}$$

Since $\chi \in D_+^R$, Supp $a(x)f \subset B_R$ and using the fact that $U_L(t)$ operates on D_+^R (see [15]), we obtain

$$a(x)\,f(\partial_t v) \;=\; a(x)\,f\big(\partial_t(v+w)\big)\;. \tag{2.7}$$

So $u = v + w$ is a solution of (1.1). It is then clear that u satisfies (1.7) and $\big(u(t), \partial_t u(t)\big) \in G$ for every $t \ge 0$.

Finally, for the uniqueness we argue as in the case $p = 1$. ∎

2.3. Some properties of the solutions of the system (1.1)

In this section we give several properties of the solutions of (1.1), usefull in the sequel.

Corollary 2.1. *Let $\varphi \in G$ and $(\psi,\chi) \in D(A)\times D_+^R$, such that $\varphi = \psi + \chi$. We have*

$$U(t)\,\varphi \;=\; U(t)\,\psi + U_L(t)\,\chi\,, \qquad \text{for all } \; t \geq 0\;. \tag{2.8}$$

and

$$\Big\|\frac{d}{dt}U(t)\,\varphi\Big\|^2_{H(B_{R+t})} \leq\; \big\|\big(\psi_2,\,\Delta\psi_1 - a\,f(\psi_2)\big)\big\|^2_H \qquad \text{a.e. } \; t \geq 0\;. \tag{2.9}$$

Proof: Let $\varphi \in G$ then there exists $(\psi,\chi) \in D(A)\times D_+^R$, such that $\varphi = \psi+\chi$. From the proof of the existence of the solutions of (1.1), we have

$$U(t)\,\varphi \;=\; U(t)\,\psi + U_L(t)\,\chi\,, \qquad \text{for all } \; t \geq 0\;.$$

On the other hand, $U_L(t)\,\chi \in D_+^R$, for all $t \geq 0$ then

$$U(t)\,\varphi \;=\; U(t)\,\psi \qquad \text{in } \; B_{R+t}\cap\Omega\;.$$

Using then (1.11), we obtain (2.9). ∎

In the sequel we denote $U(t)\,\varphi = \big(u(t), \partial_t u(t)\big)$ the solution of the system (1.1) with initial data φ.

Proposition 2.1.

(**1**) *For every $\varphi \in X$, we have*

$$\begin{aligned}\big\|a\,\partial_t u\big\|_{L^2([0,T]\times\Omega)} \;&\leq \\ &\leq\; C\big(\|a\|_{L^\infty}, T\big)\,\Big[\big\|a\,f(\partial_t u)\,\partial_t u\big\|^{1/2}_{L^1([0,T]\times\Omega)} + \big\|a\,f(\partial_t u)\,\partial_t u\big\|^{\frac{1}{r+1}}_{L^1([0,T]\times\Omega)}\Big]\end{aligned} \tag{2.10}$$

for every $T \geq 0$.

(**2**) *For every $\varphi \in G$, we have*

$$\begin{aligned}\big\|a\,f(\partial_t u)\big\|_{L^2([0,T]\times\Omega)} \;&\leq \\ &\leq C\big(\|a\|_{L^\infty}, T, R\big)\,\Big[\big\|a\,f(\partial_t u)\,\partial_t u\big\|^{\frac{1}{r+1}}_{L^1([0,T]\times\Omega)} \\ &\quad + \big\|a\,f(\partial_t u)\,\partial_t u\big\|^{\frac{(d-p(d-2))}{(d+2)-p(d-2)}}_{L^1([0,T]\times\Omega)}\;\operatorname*{ess\,sup}_{[0,T]}\big\|\nabla\partial_t u(t,\cdot)\big\|^{\frac{d(p-1)}{(d+2)-p(d-2)}}_{L^2(B_R\cap\Omega)}\Big]\end{aligned} \tag{2.11}$$

for every $T \geq 0$.

(**3**) $p=1$. *For every* $\varphi \in H$, *we have*

$$\big\|a\,f(\partial_t u)\big\|_{L^2([0,T]\times\Omega)} \ \leq \tag{2.12}$$
$$\leq\ C\big(\|a\|_{L^\infty},T\big)\left[\big\|a\,f(\partial_t u)\,\partial_t u\big\|^{\frac{1}{r+1}}_{L^1([0,T]\times\Omega)}+\big\|a\,f(\partial_t u)\,\partial_t u\big\|^{\frac{1}{2}}_{L^1([0,T]\times\Omega)}\right]$$

for every $T\geq 0$.

Proof: Throughout this proof we use the following notations

$$Q_1=\Big\{(t,x)\in[0,T]\times\Omega;\ |\partial_t u(t,x)|\leq 1\Big\}\ ,\qquad Q_2=\big([0,T]\times\Omega\big)\backslash Q_1\ .$$

(**1**) (1.2) gives

$$|s|\ \leq\ \left(\frac{1}{c_1}\right)^{\frac{1}{r+1}}\big(f(s)\,s\big)^{\frac{1}{r+1}}\ ,\qquad \text{pour}\ \ |s|\leq 1\ ,$$

which yields

$$\int_{Q_1}|a(x)\,\partial_t u|^2\ dx\,dt\ \leq\ \left(\frac{1}{c_1}\right)^{\frac{2}{r+1}}\int_{Q_1}a^2(x)\,\big(f(\partial_t u)\,\partial_t u\big)^{\frac{2}{r+1}}\ dx\,dt\ .$$

Then using Hölder inequality we obtain

$$\int_{Q_1}|a(x)\,\partial_t u|^2\ dx\,dt\ \leq\ C\big(\|a\|_{L^\infty},T,R\big)\left(\int_{[0,T]\times\Omega}a(x)\,f(\partial_t u)\,\partial_t u\ dx\,dt\right)^{\frac{2}{r+1}}.$$

On the other hand, (1.3) gives

$$\int_{Q_2}|a(x)\,\partial_t u|^2\ dx\,dt\ \leq\ C\big(\|a\|_{L^\infty}\big)\int_{[0,T]\times\Omega}a(x)\,f(\partial_t u)\,\partial_t u\ dx\,dt$$

and the result follows.

(**2**) (1.2) gives

$$|f(s)|\ \leq\ c_2^{\frac{1}{r+1}}\big(f(s)\,s\big)^{\frac{1}{r+1}}\ ,\qquad \text{for}\ \ |s|\leq 1\ ,$$

so

$$\int_{Q_1}\big|a(x)\,f(\partial_t u)\big|^2\,dx\,dt\ \leq\ C_2^{\frac{2}{r+1}}\int_{Q_1}\big(a(x)\big)^2\,\big(f(\partial_t u)\,\partial_t u\big)^{\frac{2}{r+1}}\ dx\,dt\ .$$

Then using Hölder inequality we obtain

(2.13) $$\big\|a\,f(\partial_t u)\big\|_{L^2([0,T]\times Q_1)} \;\le\; C\big(\|a\|_{L^\infty},T,R\big)\,\big\|a\,f(\partial_t u)\,\partial_t u\big\|^{\frac{1}{r+1}}_{L^1([0,T]\times\Omega)}\;.$$

Let $0<\theta\le 1$, such that $\dfrac{1}{2}=\dfrac{\theta p}{p+1}+\dfrac{p(d-2)\,(1-\theta)}{2\,d}$. We have

$$\big\|a\,f(\partial_t u)\big\|_{L^2(Q_2)} \;\le\; C\big(\|a\|_{L^\infty}\big)\,\big\|a^{\frac{p}{p+1}}f(\partial_t u)\big\|^{\theta}_{L^{\frac{p+1}{p}}(Q_2)}\,\big\|f(\partial_t u)\big\|^{(1-\theta)}_{L^{\frac{2d}{p(d-2)}}(([0,T]\times B_R)\cap Q_2)}\;.$$

On the other hand (1.3), gives

$$|f(s)|^{\frac{2d}{p(d-2)}} \;\le\; c_4^{\frac{2d}{d-2}}\,|s|^{\frac{2d}{d-2}}\,, \qquad \text{for}\quad |s|>1\;.$$

and

$$|f(s)|^{\frac{p+1}{p}} \;\le\; c_4^{\frac{1}{p}}\big(f(s)\,s\big)\,, \qquad \text{for}\quad |s|>1\;.$$

So

$$\big\|a\,f(\partial_t u)\big\|_{L^2(Q_2)} \;\le\; C\big(\|a\|_{L^\infty}\big)\,\big\|a\,f(\partial_t u)\,\partial_t u\big\|^{\frac{(d-p(d-2))}{(d+2)-p(d-2)}}_{L^1([0,T]\times\Omega)}\,\big\|\partial_t u\big\|^{(1-\theta)p}_{L^{\frac{2d}{(d-2)}}([0,T]\times(B_R\cap\Omega))}\;.$$

The continuity of the embedding $H_0^1(\Omega)\hookrightarrow L^r_{loc}(\Omega)$ for $2\le r\le \dfrac{2d}{d-2}$, gives

(2.14) $$\begin{aligned}\big\|a\,f(\partial_t u)\big\|_{L^2(Q_2)} \;&\le\\ \le\; C\big(\|a\|_{L^\infty},T,R\big)\,&\big\|a\,f(\partial_t u)\,\partial_t u\big\|^{\frac{(d-p(d-2))}{(d+2)-p(d-2)}}_{L^1([0,T]\times\Omega)}\,\operatorname*{ess\,sup}_{[0,T]}\big\|\nabla\partial_t u(t,\cdot)\big\|^{\frac{d-dp}{dp-2p-d-2}}_{L^2(B_R\cap\Omega)}\;.\end{aligned}$$

Combining then (2.13) and (2.14) we obtain (2.11).

(**3**) Taking $p=1$ in (2.14), we obtain

$$\big\|a\,f(\partial_t u)\big\|_{L^2(Q_2)} \;\le\; C\big(\|a\|_{L^\infty},T\big)\,\big\|a\,f(\partial_t u)\,\partial_t u\big\|^{\frac{1}{2}}_{L^1([0,T]\times\Omega)}\;.$$

The result then follows from (2.13). ∎

3 – Rate of decay of the local energy

3.1. Lax–Phillips theory

3.1.1. Definitions and preliminary results

Let us consider the free wave equation

(3.1) $$\begin{cases}\partial_t^2 u-\Delta u \;=\; 0 & \text{in}\quad \mathbb{R}\times\mathbb{R}^d\,,\\ u(0,\cdot)=\varphi_1\,, & \partial_t u(0,\cdot)=\varphi_2\;.\end{cases}$$

We recall that the solution of (3.1) is given by the propagator

$$U_0(t)\colon\ \ H_0 \ni \varphi = (\varphi_1, \varphi_2) \ \to\ U_0(t)\varphi = (u, \partial_t u) \,\in\, H_0 \ . \tag{3.2}$$

Following Lax and Phillips, we denote:

$$D^0_+ = \Big\{\varphi = (\varphi_1, \varphi_2) \in H_0\,;\ \ U_0(t)\varphi = 0 \ \text{ on } |x| < t,\ t \geq 0\Big\}$$

the space of outgoing data, and

$$D^0_- = \Big\{\varphi = (\varphi_1, \varphi_2) \in H_0\,;\ \ U_0(t)\varphi = 0 \ \text{ on } |x| < -t,\ t \leq 0\Big\}\ ,$$

the space of incoming data associated to the solutions of (3.1).

We choose $R > 0$ such that B_R contains the obstacle and the support of the function $a(x)$.

According to the Lax–Phillips theory, we define spaces of outgoing and incoming data associated to solutions of problem (1.4) by

$$D^R_+ = \Big\{\varphi = (\varphi_1, \varphi_2) \in H\,;\ \ U_L(t)\varphi = 0 \ \text{ on } |x| < t + R,\ t \geq 0\Big\}\ , \tag{3.3}$$

$$D^R_- = \Big\{\varphi = (\varphi_1, \varphi_2) \in H\,;\ \ U_L(t)\varphi = 0 \ \text{ on } |x| < -t + R,\ t \leq 0\Big\}\ . \tag{3.4}$$

We identify H to a subspace of H_0 with the help of the following extension operator

$$E\colon H \to H_0 : \quad E\varphi = \begin{cases} \varphi & \text{on } \ \Omega\,, \\ 0 & \text{on } \ \mathbb{R}^d \backslash \Omega\ . \end{cases}$$

Then we remark that the subspaces of outgoing and incoming data associated to (1.4) coincide respectively with $U_0(R)D^0_+$ and $U_0(-R)D^0_-$. Moreover, they satisfy the following properties:

(**1**) D^R_+ and D^R_- are closed in H.

(**2**) D^R_+ and D^R_- are orthogonal and

$$D^R_+ \oplus D^R_- \oplus \big((D^R_+)^\perp \cap (D^R_-)^\perp\big) \;=\; H\ . \tag{3.5}$$

Remark 3.1.

(**1**) Solutions of (3.1), (1.4) and (1.1) verify the finite speed propagation property.

(**2**) The nonlinearity being localized in a ball B_R, it's easy to see that

$$U(t) = U_L(t) \qquad \text{on} \ \ D^R_+ \ \ \text{for every} \ \ t \geq 0\ . \tag{3.6}$$

(**3**) Following [15] we denote P_+ [resp. P_-] the orthogonal projection of H onto the orthogonal complement of D_+^R [resp. D_-^R]. Thanks to (3.5), it's clear that

$$P_+\varphi \in (D_+^R)^\perp \cap (D_-^R)^\perp \qquad \text{if } \varphi \in (D_-^R)^\perp \ . \tag{3.7}$$

(**4**) If $\varphi \in G$ then $P_+\varphi \in G$.

(**5**) $U(t)$ operates on D_+^R for $t \geq 0$, so $(\mathrm{Supp}\, U(t)\varphi) \cap \mathrm{Supp}\, a(x) = \varnothing$ for every $t \geq 0$ and $\varphi \in D_+^R$. Using then the fact that the Cauchy problem admits a unique solution, we obtain: for every φ in X and for every $t \in \mathbb{R}_+$,

$$\begin{aligned} U(t)\varphi &= U(t)\,P_+\varphi + U(t)\,(I - P_+)\,\varphi \\ &= U(t)\,P_+\varphi + U_L(t)\,(I - P_+)\,\varphi \ . \ \square \end{aligned} \tag{3.8}$$

3.1.2. Lax–Phillips semi-group

Notation 3.1. We remind that

$$X = \begin{cases} H & \text{if } p = 1\,, \\ G & \text{if } p > 1\ . \end{cases}$$

If $p = 1$, we denote

$$N_X(\varphi) = \|\varphi\|_H \ .$$

If $p > 1$, we denote

$$\big(N_X(\varphi)\big)^2 = \inf_{\{\psi=(\psi^1,\psi^2)\in D(A);\ \exists\chi\in D_+^R,\ \varphi=\psi+\chi\}} \Big\{\|\varphi\|_H^2 + \big\|\big(\psi^2, \Delta\psi^1 - a\,f(\psi^2)\big)\big\|_H^2\Big\} \ . \ \square$$

We remind that $K = (D_+^R)^\perp \cap (D_-^R)^\perp$, and we define the nonlinear Lax–Phillips operator on $K \cap X$ by

$$Z(t) = P_+\,U(t)\,P_- \qquad \text{for } t \geq 0\ . \tag{3.9}$$

In order to prove that $Z(t)$ is a semi-group operating on $K \cap X$, we need the following lemma.

Lemma 3.1. *Let $(\varphi, \psi) \in X \times H$, then for every $t \geq 0$,*

$$\big\langle U(t)\,\varphi, \psi\big\rangle_H - \big\langle \varphi, U_L(-t)\,\psi\big\rangle_H = -\int_0^t \Big\langle a\,f(\partial_t u(s)),\, \partial_t v(s-t)\Big\rangle_{L^2} ds \tag{3.10}$$

where we denoted $U(t)\,\varphi = \big(u(t), \partial_t u(t)\big)$ and $U_L(t)\,\psi = \big(v(t), \partial_t v(t)\big)$.

Proof: Since $U_L(t)$ is a group, it's clear that (3.10) is equivalent to

$$\Big\langle U(t)\varphi,\, U_L(t)\psi\Big\rangle_H - \langle\varphi,\psi\rangle_H \;=\; -\int_0^t \Big\langle a\, f(\partial_t u(s)),\, \partial_t v(s)\Big\rangle_{L^2} ds\ .$$

By density argument, it suffices to prove the result for (φ,ψ) in $D(A)\times(C_0^\infty(\Omega))^2$.

Thanks to Green formula

$$\begin{aligned}\frac{d}{dt}\Big\langle U(t)\varphi,\, U_L(t)\psi\Big\rangle_H &= \frac{d}{dt}\Big(\langle\nabla u,\nabla v\rangle_{L^2} + \langle\partial_t u,\partial_t v\rangle_{L^2}\Big)\\ &= \langle\partial_t^2 u,\partial_t v\rangle_{L^2} + \langle\partial_t u,\partial_t^2 v\rangle_{L^2} - \langle\partial_t u,\Delta v\rangle_{L^2} - \langle\Delta u,\partial_t v\rangle_{L^2}\ .\end{aligned}$$

Since u and v verify respectively the system (1.1) and (1.4), with initial data φ, respectively ψ, we obtain

$$\frac{d}{dt}\Big\langle U(t)\varphi, U_L(t)\psi\Big\rangle_H = -\Big\langle a\, f(\partial_t u(t)), \partial_t v(t)\Big\rangle_{L^2}\ ,$$

and the result follows. ∎

Proposition 3.1. $\big(Z(t)\big)_{t\geq 0}$ *is a semi-group on* $K\cap X$*. Moreover, for every* $\varphi\in K\cap X$*, the map*

$$\begin{aligned}\mathbb{R}_+ &\to H\\ t &\mapsto Z(t)\varphi\end{aligned}\tag{3.11}$$

is continuous.

Proof: Let $\varphi\in K\cap X$, and $t\geq 0$. We begin by proving that $Z(t)\varphi\in K$. Indeed according to (3.7), it suffices to verify that $U(t)\varphi\in(D_-^R)^\perp$. Let $\psi\in D_-^R$, by Lemma 3.1, we have

$$\Big\langle U(t)\varphi,\psi\Big\rangle_H - \Big\langle\varphi, U_L(-t)\psi\Big\rangle_H = -\int_0^t \Big\langle a\, f(\partial_t u(s)), \partial_t v(s-t)\Big\rangle_{L^2} ds\ .$$

where $U_L(t)\psi = \big(v(t),\partial_t v(t)\big)$. Since $U_L(s-t)\psi\in D_-^R$ for $s\leq t$ and the support of $a(x)$ is contained in B_R, we obtain

$$\Big\langle U(t)\varphi,\psi\Big\rangle_H = \Big\langle\varphi, U_L(-t)\psi\Big\rangle_H = 0\ .$$

When $X=G$, we need also to prove that $Z(t)\varphi\in G$. According to Corollary 2.1 we have

$$\exists\,\psi\in D(A)\ \text{ and }\ \chi\in D_+^R;\qquad U(t)\varphi = U(t)\psi + U_L(t)\chi\ .$$

Since P_+ is the orthogonal projection of H onto the orthogonal complement of D_+^R

$$\begin{aligned} Z(t)\varphi &= P_+ U(t)\psi \\ &= U(t)\psi + (P_+ - I)\, U(t)\psi \ , \end{aligned}$$

and the result follows.

Finally, it is clear that for every φ in X

$$Z(t_1 + t_2)\,\varphi = Z(t_1)\, Z(t_2)\,\varphi\,, \qquad \text{for every } \; t_1, t_2 \geq 0 \ .$$

The continuity of $Z(t)$ is a consequence of (1.7). ∎

3.2. Reduction to an a priori estimate

In this section, we prove that the rate of decay of the local energy is equivalent to an a priori estimate for $\|Z(t)\|_H$. First we give the following lemma.

Lemma 3.2. *Let $s \geq 1$ and χ a nonincreasing positive function of $C(\mathbb{R},\mathbb{R})$. We suppose that there exist $T>0$ and $c>1$, such that*

$$\chi(t)^s \leq c\Big(\chi(t) - \chi(t+T)\Big) \ , \qquad t \geq 0 \ . \tag{3.12}$$

Then

$$\begin{aligned} \chi(t) &\leq \frac{c}{c-1}\, e^{-(\frac{1}{T}\log\frac{c}{c-1})t}\,\chi(0) \ , \qquad t \geq 0 \quad \textit{if } s=1 \ , \\ \chi(t) &\leq \left(\chi(0)^{(1-s)} + \frac{s-1}{cT}\,(t-T)\right)^{-\frac{1}{(s-1)}} \ , \qquad t \geq T \quad \textit{if } s>1 \ . \end{aligned}$$ ∎

For the proof of this lemma we refer the interested reader to Nakao [23].

Lemma 3.3. *Let $\varphi \in H$ (resp. $D(A)$) with support in $B_R \cap \Omega$, then $\varphi \in K$ (resp. $\varphi \in K \cap G$) and*

$$E_R(u)(t) \leq \|Z(t)\varphi\|_H^2 \ , \qquad t \geq 0 \ ,$$

with $U(t)\varphi = \big(u(t), \partial_t u(t)\big)$.

Proof: Let $\varphi\in H$ (resp. $D(A)$); $\operatorname{Supp}\varphi\subset B_R\cap\Omega$. By (3.3) and (3.4), we have,

if $\psi\in D_+^R$ [resp. $\chi\in D_-^R$] then $\operatorname{Supp}\psi$ [resp. $\operatorname{Supp}\chi$] is contained in $\mathbb{R}^d\backslash B_R$,

which yields immediately, $\varphi\in K=(D_+^R)^\perp\cap(D_-^R)^\perp$. On the other hand

$$\forall\, t\geq 0 \qquad U(t)\varphi \;=\; Z(t)\varphi+(I-P_+)\,U(t)\varphi\ .$$

Since the Support of $(I-P_+)\,U(t)\varphi$ is contained in $\mathbb{R}^d\backslash B_R$, we obtain

$$U(t)\varphi=Z(t)\varphi \qquad \text{on} \;\; B_R\cap\Omega\ ,$$

and

$$E_R(u)(t)\;\leq\;\|Z(T)\varphi\|_H^2\ . \blacksquare$$

Take $D_0>0$ and denote by

$$B_X^K(D_0)\;=\;\Big\{\varphi\in X\cap K\,;\;\; N_X(\varphi)<D_0\Big\}\ .$$

Remark 3.2.

(**1**) If $\varphi\in B_{D(A)}(D_0)$, such that $\operatorname{Supp}\varphi\subset B_R$, then $\varphi\in B_G^K(D_0)$.

(**2**) (**a**) It is clear that $Z(t)\varphi\in B_H^K(D_0)$ for all $\varphi\in B_H^K(D_0)$ and $t\geq 0$.

(**b**) If $\varphi\in B_G^K(D_0)$ then $Z(t)\varphi\in B_G^K(D_0)$, a.e. $t\geq 0$.
Indeed, let $\varphi\in B_G^K(D_0)$ then there exists $\psi\in D(A)$ such that

$$\|\varphi\|_H^2+\big\|\big(\psi^2,\,\Delta\psi^1-a\,f(\psi^2)\big)\big\|_H^2\;<\;D_0^2\ ,$$

and

$$Z(t)\varphi\;=\;U(t)\psi+(P_+-I)\,U(t)\,\psi\ .$$

Then using (1.11), we obtain

$$\begin{aligned}\|Z(t)\varphi\|_H^2+\big\|\big(\partial_t v(t),\,\Delta v(t)-a\,f(\partial_t v(t))\big)\big\|_H^2 \;&=\\ =\;\|Z(t)\varphi\|_H^2+\Big\|\frac{d}{dt}U(t)\psi\Big\|_H^2&\\ \leq\;\|\varphi\|_H^2+\big\|\big(\psi^2,\,\Delta\psi^1-a\,f(\psi^2)\big)\big\|_H^2\;<\;D_0^2\,,&\qquad \text{a.e.}\;\; t\geq 0\,,\end{aligned}$$

where we denoted $U(t)\psi=\big(v(t),\partial_t v(t)\big)$.
Finally, $\big(N_X(Z(t)\varphi)\big)<D_0$. $\square$

We remind that, we want to find the rate of decay of the local energy for every solution u of (1.1), if the initial data $\varphi \in B_H(D_0)$ (resp. $\varphi \in B_{D(A)}(D_0)$) and is supported in B_R. Thus Lemma 3.3, combined with the first part of Remark 3.2 and Lemma 3.2, shows that it suffices to prove

$$\|Z(t)\varphi\|_H^{2s} \le c\Big(\|Z(t)\varphi\|_H^2 - \|Z(t+T)\varphi\|_H^2\Big) \tag{3.13}$$

for some $c>0$, some $T>0$, for every $t \ge 0$ and for every φ in $B_X^K(D_0)$. On the other hand, the second part of the Remark 3.1 allows one to apply the semi-group property for every φ in $B_X^K(D_0)$. Hence it suffices to prove the estimate

$$\|\varphi\|_H^{2s} \le c\Big(\|\varphi\|_H^2 - \|Z(T)\varphi\|_H^2\Big)\ , \tag{3.14}$$

for some $c>0$, some $T>0$ and for every φ in $B_X^K(D_0)$. And this is the a priori estimate for $\|Z(t)\|_H$ mentioned in the beginning of this section.

3.3. Proof of Theorem 3

In the sequel, we suppose that $\big(\omega = \{x \in \Omega;\ a(x)>0\},\, T_R\big)$ verifies the exterior geometric condition on B_R.

Let

$$s = \max\left(\frac{r+1}{2},\ \frac{(d+2)-p(d-2)}{2\big(d-p(d-2)\big)}\right)$$

and take $T = T_R + 9R$.

To prove (3.14), we argue by contradiction: We suppose the existence of a sequence $(\varphi_n)_n$ in $B_X^K(D_0)$, such that

$$\|\varphi_n\|_H^{2s} > n\Big(\|\varphi_n\|_H^2 - \|Z(t)\varphi_n\|_H^2\Big)\,, \qquad \text{for every} \quad t \le T\ . \tag{3.15}$$

Let u_n to be the solution of the system (1.1) with initial data φ_n and set

$$\alpha_n = \|\varphi_n\|_H\ , \qquad v_n = \frac{u_n}{\alpha_n}\ .$$

v_n verifies the system

$$\begin{cases} \partial_t^2 v_n - \Delta v_n + \frac{1}{\alpha_n}\, a(x) f(\partial_t u_n) = 0 & \text{in } \mathbb{R}_+ \times \Omega\,, \\ v_n(t,x) = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega\,, \\ \big(v_n(0,x), \partial_t v_n(0,x)\big) = \dfrac{\varphi_n}{\alpha_n} = \psi_n\,, \quad \|\psi_n\|_H = 1\ . \end{cases} \tag{3.16}$$

On the other hand, due to (3.15) and (1.9) the sequence (φ_n) satisfies

$$\|\varphi_n\|_H^{2s} \geq n \int_0^T\int_\Omega a(x)\, f(\partial_t u_n(\tau,x))\, \partial_t u_n(\tau,x)\ dx\, d\tau\ .$$

In order to obtain a contradiction we need the following results

Proposition 3.2. *Let $(\varphi_n)_n \in B_X^K(D_0)$ satisfying*

$$\|\varphi_n\|_H^{2s} \geq n \int_0^T\int_\Omega a(x)\, f(\partial_t u_n(\tau,x))\, \partial_t u_n(\tau,x)\ dx\, d\tau \tag{3.17}$$

where u_n denotes the solution of the system (1.1) with initial data φ_n. Set

$$\alpha_n = \|\varphi_n\|_H\ , \qquad v_n = \frac{u_n}{\alpha_n}\ .$$

Then

$$\Big\|\frac{1}{\alpha_n}\ a(x)\, f(\partial_t u_n)\Big\|_{L^2([0,T]\times\Omega)} \underset{n\to+\infty}{\longrightarrow} 0\ , \tag{3.18}$$

and there exists a subsequence of $(V_n) = ((v_n, \partial_t v_n))$, still denoted $(V_n)_n$ that converges weakly- to $V = (v, \partial_t v)$ in $L^\infty\big([0,T], (D_-^R)^\perp\big)$. Moreover*

$$P_+ V_n(T_R + 9R) \to P_+ V(T_R + 9R) \qquad \text{in } H\,. \tag{3.19}$$

First we finish the proof of Theorem 3, then we give the proof of the Proposition 3.2.

By Proposition 3.2 there exists a subsequence of $(V_n) = ((v_n, \partial_t v_n))$ still denoted (V_n), that converges weakly-* to $V = (v, \partial_t v)$ in $L^\infty\big([0,T], (D_-^R)^\perp\big)$ and verifies (3.18). Passing then to the limit in the system satisfied by v_n, we infer that v the weak limit of (v_n), verifies

$$\begin{cases} \partial_t^2 v - \Delta v = 0 & \text{in }]0,T[\times\Omega\,, \\ v(t,x) = 0 & \text{on }]0,T[\times\partial\Omega\,, \\ \partial_t v = 0 & \text{on }]0,T[\times\omega\,, \\ \big(v(0,x), \partial_t v(0,x)\big) = \psi \in K\ , \end{cases}$$

where ψ denotes the weak limit of $(\psi_n)_n$ in H. Moreover, $V = (v, \partial_t v) \in C([0,T], H)$ and

$$\|V(t)\|_H \leq 1 \qquad \text{for every } t \in [0,T]\ .$$

On the other hand, it is clear that (3.15) gives

$$0 \le 1 - \|P_+ V_n(t)\|_H^2 \le \frac{D_0^{2(s-1)}}{n} \qquad \text{for } t \le T \; . \tag{3.20}$$

Now using (3.19) and then passing to the limit in (3.20), we conclude by the classical energy estimate that there exists ψ in K such that

$$\left\{ \begin{array}{ll} \|P_+ V(t)\|_H = \|P_+ U_L(t)\psi\|_H = \|\psi\|_H = 1\,, & \text{for } t \le T\,, \\ \partial_t v = 0\,, \quad \text{on } \;]0,T[\times\omega & \end{array} \right\} \tag{3.21}$$

and this is in contradiction with the following lemma [1, Lemma 5.1].

Lemma 3.4. *The space*

$$\Big\{\psi \in K;\; \|P_+ U_L(t)\psi\|_H = \|\psi\|_H \;\; \forall\, t \le T,\; \partial_t v = 0,\; \textit{on }]0,T[\times\omega\Big\}$$

is reduced to the null vector. ∎

This finishes the proof of Theorem 3. ∎

Proof of Proposition 3.2: v_n verifies the system

$$\begin{cases} \partial_t^2 v_n - \Delta v_n + \frac{1}{\alpha_n}\, a(x)\, f(\partial_t u_n) \;=\; 0 & \text{in } \mathbb{R}_+ \times \Omega\,, \\ v_n(t,x) \;=\; 0 & \text{on } \mathbb{R}_+ \times \partial\Omega\,, \\ \big(v_n(0,x), \partial_t v_n(0,x)\big) = \dfrac{\varphi_n}{\alpha_n} = \psi_n\;, \quad \|\psi_n\|_H = 1\;, & \end{cases}$$

and satisfies the energy identity:

$$E(v_n)(t) - E(v_n)(0) \;=\; -\,\alpha_n^{-2} \int_0^t\!\!\int_\Omega a(x)\, f(\partial_t u_n)\, \partial_t u_n \; dx\, d\tau\;, \qquad t \ge 0\;.$$

This estimate allows one to show that the sequence $\big(V_n = (v_n, \partial_t v_n)\big)_n$ is bounded in $C\big([0,T], (D_-^R)^\perp\big)$; then it admits a subsequence, $(V_n)_n$ that converges weakly-* to $V = (v, \partial_t v)$ in $L^\infty\big([0,T], (D_-^R)^\perp\big)$. In this way,

$$V_n(t) \rightharpoonup V(t) \;\text{ in } H \;\text{ a.e. } t \in [0,T]\,, \qquad \text{and} \qquad \operatorname*{ess\,sup}_{[0,T]} \|V(t)\|_H \le 1\;,$$

which yields

$$v_n \rightharpoonup v \quad \text{in } \; H^1_{loc}\big([0,T]\times\Omega\big)\;. \tag{3.22}$$

First, we prove (eventually after extracting a subsequence) that

$$v_n \to v \quad \text{in} \;\; H^1_{loc}(\tilde{K}(T_0))\;, \tag{3.23}$$

where $T_0 = T_R + 3R$ and

$$\tilde{K}(T_0) \;=\; \Big\{ (t,x) \in \mathbb{R}_+ \times \Omega\,;\;\; |x| < t - T_0 + R,\;\; T_0 \le t \le T \Big\}\;. \tag{3.24}$$

For that, we use the notion of microlocal defect measures. These measures were introduced by P. Gérard in [9], [10]. They propagate along generalized bicharacteristics of the wave operator under Dirichlet condition on the boundary (G. Lebeau [16]).

(3.22) allows one to associate to the sequence $(v_n - v)_n$ a microlocal defect measure μ in $H^1_{loc}([0,T]\times\Omega)$. So in order to obtain (3.23), we have to show that $\mu = 0$ on $\tilde{K}(T_0)$.

Let w_n be the solution of the system

$$\begin{cases} \partial_t^2 w_n - \Delta w_n \,=\, 0 & \text{in} \;\; \mathbb{R}_+ \times \Omega\,, \\ w_n(t,x) \,=\, 0 & \text{on} \;\; \mathbb{R}_+ \times \Omega\,, \\ \big(w_n(0,x),\, \partial_t w_n(0,x)\big) \,=\, \dfrac{\varphi_n}{\alpha_n} \,=\, \psi_n\;. & \end{cases} \tag{3.25}$$

Then the sequence $(v_n - w_n)$ verifies

$$\sup_{0\le t\le T} E^{1/2}(v_n - w_n)(t) \;\le\; C(T) \left\| \frac{1}{\alpha_n}\, a(x)\, f(\partial_t u_n) \right\|_{L^2([0,T]\times\Omega)}\;. \tag{3.26}$$

Now using (3.17) and (2.12) (resp. (2.11) and (2.9)) when $X = H$ (resp. $X = G$), we get

$$\left\| \frac{1}{\alpha_n}\, a(x)\, f(\partial_t u_n) \right\|_{L^2([0,T]\times\Omega)} \;\le\; C(D_0, T) \left(\frac{1}{n}\right)^{\frac{1}{2s}} \underset{n\to+\infty}{\longrightarrow} 0\;, \tag{3.27}$$

which yields

$$\sup_{0\le t\le T} E(v_n - w_n)(t) \underset{n\to+\infty}{\longrightarrow} 0\;,$$

and this means in particular that $(v_n - w_n) \underset{n\to+\infty}{\longrightarrow} 0$ in $H^1_{loc}([0,T]\times\Omega)$. $(v_n)_n$ is then a “linearizable” sequence according to the terminology of P. Gérard [10].

From this we deduce these two properties of the microlocal defect measure μ:

- The support of μ is contained in the characteristic set of the wave operator $\{(t,x,\tau,\xi);\ \tau^2=|\xi|^2\}$ (^{2}).
- μ propagates along the bicharacteristic flow of the d'Alembertian on $[0,T]\times\Omega$ (^{3}).

Let $q\in T^*(\tilde{K}(T_0))$, and γ a generalized bicharacteristic issued from q. To prove that $\mu=0$ near q, we argue as Aloui–Khenissi [1]. We are in one of the following situations:

1$^{\text{st}}$ case: γ traced backwards in time, does not meet $\partial\Omega$ or meets $\partial\Omega$ at $t_0>2R$.

Consequently, $\gamma_0=\gamma_{/t=0}\notin B_R$. The support of $a(x)$ is contained in B_R, then $v_n=z_n$ near γ_0, which gives $\mu=\mu_0$ near γ_0, where $\big(z_n(t),\partial_t z_n(t)\big)=U_0(t)\,\psi_n$ and μ_0 is the microlocal defect measures associated to the sequence (z_n-z) in $H^1_{loc}([0,T]\times\Omega)$ where $\big(z(t),\partial_t z(t)\big)=U_0(t)\,\psi$, ψ the weak limit of ψ_n (^{4}). On the other hand $(\psi_n-\psi)\in(D^R_-)^\perp$, then using the translation representation of the free wave equation (see [15, Chapter 3]), we obtain $U_0(t)(\psi_n-\psi)=0$ in $\{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d;\ |x|<t-R,\ 2R\le t\}$, so $\mu_0=0$ near

$$q'=\begin{cases} q & 1^{\text{st}}\text{ subcase},\\ \gamma(t_1) & 2^{\text{nd}}\text{ subcase},\end{cases}$$

with $t_1\le t_0$ and $\gamma(t_1)\in T^*\big(\{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d;\ |x|\le t-R,\ 2R<t\}\big)$. Therefore by propagation of the support of μ_0, we deduce that $\mu_0=0$ near γ_0, which gives $\mu_0=\mu=0$ near γ_0. Then $\mu=0$ near q, by propagation of the support of μ.

2$^{\text{nd}}$ case: γ meets $\partial\Omega$ at $t_0\le 2R$ and there exists t_1 such that $\gamma(t_1)\in B_R$ and $t_1-t_0>T_R$.

Using (2.10) and (3.17), it is clear that

$$\text{(3.28)}\qquad \big\|a\,\partial_t v_n(t,x)\big\|_{L^2([0,T]\times\Omega)}\le C(D_0)\left(\frac{1}{n}\right)^{\frac{1}{r+1}}\underset{n\to+\infty}{\longrightarrow}0\ ,$$

(^{2}) This is known as elliptic regularity theorem of the microlocal defect measure and is a direct consequence of the fact that $\partial_t^2 v_n-\Delta v_n\underset{n\to+\infty}{\longrightarrow}0$ in $L^2([0,T]\times\Omega)$.

(^{3}) If some point ω_0 of a generalized bicharacterestic γ is not in $\mathrm{supp}(\mu)$, then $\gamma\cap\mathrm{supp}(\mu)=\varnothing$.

(^{4}) $\|\psi_n\|_H\le 1$ then due to the classical energy estimate the sequence (z_n-z) is bounded in $C([0,T],H_0)$, which allows us to attach to the sequence (z_n-z) a microlocal defect measure.

we remind that $\omega=\{x\in\Omega;\ a(x)>0\}$, then

$$\partial_t(v_n-v)\underset{n\to\infty}{\longrightarrow}0 \qquad \text{in } L^2\big([0,T]\times\omega\big)\ ,$$

and since the Support of μ is contained in $\{(t,x,\tau,\xi);\ \tau^2=|\xi|^2\}$, then $\mu=0$ on $[0,T]\times\omega$. Finally, the condition $(C.G.E.)$ implies that γ meets the region $[0,T_0]\times\omega$. By the propagation of the support of μ, we infer that $\mu=0$ near q.

Then after extracting a subsequence and using (3.27) and the hyperbolic energy inequality, we obtain

$$V_n(T_R+7R)\ \to\ V(T_R+7R) \qquad \text{in } H(B_{5R}\cap\Omega)\ . \tag{3.29}$$

To finish the proof, we need the following lemma

Lemma 3.5. *Let* $M=U(2R)-U_0(2R)$, $M_L=U_L(2R)-U_0(2R)$.

(**1**) $\forall f\in H$, *we have*

$$\operatorname{Supp} M_Lf\subset B_{3R} \quad \text{and} \quad \|M_Lf\|_{H_0}\le 2\|f\|_{H(B_{5R}\cap\Omega)}\ .$$

(**2**) $\forall (f,g)\in X\times H,\ \forall\lambda\neq 0$

$$\Big\|\frac{1}{\lambda}Mf-M_Lg\Big\|_{H_0}\le\Big\|\frac{1}{\lambda}\,af(\partial_tu)\Big\|_{L^1((0,2R),L^2(\Omega))}+2\Big\|\frac{1}{\lambda}f-g\Big\|_{H(B_{5R}\cap\Omega)}\ , \tag{3.30}$$

where $\big(u(t),\partial_tu(t)\big)=U(t)f$.

First we finish the proof of the proposition, then we give the proof of the lemma.

Taking $f=U(T_R+7R)\varphi_n$, $g=V(T_R+7R)$, $\lambda=\alpha_n$ in (3.30), and using (3.29) and (3.27), we get

$$\Big\|\frac{1}{\alpha_n}MU(T_R+7R)\varphi_n-M_LV(T_R+7R)\Big\|_{H_0}\underset{n\to\infty}{\longrightarrow}0\ . \tag{3.31}$$

On the other hand

$$V_n(T)\ =\ \frac{1}{\alpha_n}MU(T_R+7R)\varphi_n+\frac{1}{\alpha_n}U_0(2R)\,U(T_R+7R)\varphi_n\ ,$$

then, by using the translation representation of the free wave equation (see [15]), we obtain

$$\frac{1}{\alpha_n}U_0(2R)\,U(T_R+7R)\varphi_n\ \in\ D_+^R\ .$$

So we deduce that

$$P_+ V_n(T) \;=\; P_+\Big(\frac{1}{\alpha_n}\, M\,U(T_R+7R)\,\varphi_n\Big)\;,$$

and by (3.31)

$$P_+\Big(\frac{1}{\alpha_n}\, M\,U(T_R+7R)\,\varphi_n\Big) \underset{n\to\infty}{\longrightarrow} P_+ M_L V(T_R+7R) \qquad \text{in} \;\; H_0$$

and the result follows. ∎

Proof of Lemma 3.5:

(**1**) Let f in H. By the finite speed propagation property

$$U_L(t)\,f = U_0(t)\,f \qquad \text{on} \;\; |x|>t+R, \;\;\; t\geq 0 \;,$$

then for $t=2R$, $U_L(2R)\,f = U_0(2R)\,f$ on $|x|>3R$. Using the same property, it is clear that

$$\begin{aligned} \|M_L\,f\|_{H_0} \;&=\; \big\|\big(U_L(2R)-U_0(2R)\big)\,f\big\|_{H_0(B_{3R})} \\ &\leq\; 2\,\|f\|_{H(B_{5R})}\;. \end{aligned}$$

(**2**) Let f in X and g in H. We have

$$\frac{1}{\lambda}\,Mf - M_L\,g \;=\; \frac{1}{\lambda}\,U(2R)\,f - \frac{1}{\lambda}\,U_L(2R)\,f + M_L\Big(\frac{1}{\lambda}\,f - g\Big)$$

then due to the hyperbolic inequality

$$\Big\|\frac{1}{\lambda}\,U(2R)\,f - \frac{1}{\lambda}\,U_L(2R)\,f\Big\|_H \;\leq\; \Big\|\frac{1}{\lambda}\,a\,f(\partial_t u)\Big\|_{L^1((0,2R),L^2(\Omega))}\;,$$

and we conclude using the first part of the lemma. ∎

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Moez Daoulatli,
LAMSIN, ENIT, BP 37,
1002 Tunis Le Belvédère – TUNISIE
E-mail: `moez.daoulatli@infcom.rnu.tn`