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TOTALLY UMBILICAL SEMI-INVARIANT SUBMANIFOLDS OF A NEARLY TRANS-SASAKIAN MANIFOLD

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Abstract: We have studied some differential geometric aspects of semi-invariant submanifolds of a nearly trans-Sasakian manifold. That have led to the classification of totally umbilical semi-invariant submanifolds of a nearly trans-Sasakian manifold.

1 – Introduction

Geometry of submanifolds of Sasakian and Kenmotsu manifolds has been an active area of research since long (cf. [1], [11], [14], [16], etc.). As the class of trans-Sasakian manifolds includes Sasakian and Kenmotsu manifolds both, the study of geometry of submanifold of trans-Sasakian manifolds becomes more meaningful. The historical background of trans-Sasakian manifolds can be traced back to the classification of almost Hermitian manifolds by A. Gray and L.M. Hervella [7]. One of the classes that appears in this classification, denoted by $\mathcal{W}_4$ is closely related with locally conformal Kaehler manifolds. An almost contact metric structure (ϕ,ξ,η,g) on a manifold $\bar{M}$ is called a trans-Sasakian structure if $\bar{M}\times R$ belongs to class $\mathcal{W}_4$ of almost Hermitian manifolds. D. Blair and J.A. Oubina [5] showed that an almost contact metric manifold $\bar{M}$ with structure tensors (ϕ,ξ,η,g) is a trans-Sasakian manifold if

$$(1.1)\qquad (\bar{\nabla}_X\phi)Y \;=\; \alpha\Big(g(X,Y)\,\xi-\eta(Y)X\Big)+\beta\Big(g(\phi X,Y)\,\xi-\eta(Y)\,\phi X\Big)\;,$$

for each $X,Y\in T\bar{M}$ where α and β are smooth functions on $\bar{M}$ and $\bar{\nabla}$ is the Riemannian connection on $\bar{M}$. A trans-Sasakian structure defined by equation (1.1)

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is termed as a structure of type (α, β). Thus, a trans-Sasakian structure of type $(0,0)$ is cosymplectic [3], a trans-Sasakian structure of type $(0, \beta)$ is β-Kenmotsu [9] and trans-Sasakian structure of type $(\alpha, 0)$ is α-Sasakian [9].

Recently, C. Gherghe [8] introduced a nearly trans-Sasakian structure of type (α, β), which generalizes trans-Sasakian structure in the same sense as nearly Sasakian structure generalizes Sasakian one.

An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is a nearly trans-Sasakian structure [8] if

$$(1.2)\qquad \begin{aligned}(\bar{\nabla}_X \phi) Y + (\bar{\nabla}_Y \phi) X \,=\, & \alpha \Big[2\, g(X,Y)\, \xi - \eta(Y) X - \eta(X) Y \Big] \\ & - \beta \Big[\eta(Y)\, \phi X + \eta(X)\, \phi Y \Big] \,.\end{aligned}$$

A trans-Sasakian structure is always a nearly trans-Sasakian structure. Moreover, a nearly trans-Sasakian structure of type (α, β) is nearly Sasakian [4] or nearly Kenmotsu [10] or nearly cosymplectic [3] accordingly as $\beta = 0$ or $\alpha = 0$ or $\alpha = 0 = \beta$.

The study of semi-invariant submanifold or contact CR-submanifolds of almost contact metric manifold was initiated by A. Bejancu and N. Papaghiuc [1] and was followed up by several other geometers (cf. [11], [14], [16], etc.). In particular, semi-invariant submanifolds of different classes of almost contact metric manifolds have also been studied (cf. [6], [10]). J.S. Kim et al. [10] initiated the study of semi-invariant submanifolds of nearly trans-Sasakian manifolds. They have obtained some basic results concerning the integrability of the distributions involved in this setting and their parallelism in the submanifold. In the present note, we have worked out a classification for totally umbilical semi-invariant submanifolds of a nearly trans-Sasakian manifold.

2 – Preliminaries

Let $\bar{M}$ be an almost contact metric manifold with almost contact metric structure (ϕ, ξ, η, g), that is ϕ is a (1,1) tensor field, ξ is a vector field, η is a 1-form and g is the compatible Riemannian metric such that

$$(2.1)\qquad \phi^2 X = -X + \eta(X)\, \xi \,, \quad \eta(\xi) = 1 \,, \quad \phi(\xi) = 0 \,, \quad \eta \circ \phi = 0 \,,$$

$$(2.2)\qquad g(\phi X, \phi Y) \,=\, g(X,Y) - \eta(X)\, \eta(Y) \,, \quad \eta(X) = g(X, \xi) \,,$$

for each $X, Y \in T\bar{M}$ where $T\bar{M}$ denotes the tangent bundle of $\bar{M}$.

The two known classes of submanifolds namely Sasakian and Kenmotsu manifolds are respectively characterized by the tensorial relations

$$(\bar{\nabla}_X \phi) Y = g(X, Y)\, \xi - \eta(Y) X \tag{2.3}$$

and

$$(\bar{\nabla}_X \phi) Y = g(\phi X, Y)\, \xi - \eta(Y)\, \phi X \ . \tag{2.4}$$

An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called a trans-Sasakian structure [15] if $(\bar{M}\times R, J, G)$ belongs to the class $\mathcal{W}_4$ of the Gray–Hervella classification of almost Hermitian manifolds, where J is the almost complex structure and G is the product metric on $\bar{M}\times R$ defined respectively by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi,\, \eta(X)\, \frac{d}{dt}\right) ,$$

$$G\left(\left(X, f_1\frac{d}{dt}\right), \left(Y, f_2\frac{d}{dt}\right)\right) = g(X, Y) + f_1 f_2 ,$$

for all vector fields X, Y on $\bar{M}$ and smooth functions f, f_1 and f_2 on $\bar{M}\times R$. A trans-Sasakian manifold is characterized by the tensorial relation (1.1) and a nearly trans-Sasakian manifold by the relation (1.2).

An m-dimensional submanifold M of $\bar{M}$ is said to be a semi-invariant submanifold if there exist a pair of orthogonal distributions $(D, D^{\perp})$ satisfying the conditions

(**i**) $TM = D \oplus D^{\perp} \oplus \langle\xi\rangle$.

(**ii**) The distribution D is invariant by ϕ, i.e., $\phi D_x = D_x\ \ \forall\, x \in M$.

(**iii**) The distribution $D^{\perp}$ is anti-invariant, i.e., $\phi D_x^{\perp} \subseteq T_x^{\perp} M,\ \forall\, x \in M$.

where $\langle\xi\rangle$ is the distribution spanned by the structure vector field ξ.

Let TM denote the tangent bundle on M. The orthogonal complement of $\phi D^{\perp}$ in the normal bundle $T^{\perp}M$ is an invariant sub bundle of $T^{\perp}M$ under ϕ and is denoted by μ, i.e.,

$$T^{\perp}M = \phi D^{\perp} \oplus \mu \ . \tag{2.5}$$

For $U, V \in TM$ and $N \in T^{\perp}M$, the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_U V = \nabla_U V + h(U, V) ,$$

$$\bar{\nabla}_U N = -A_N U + \nabla_U^{\perp} N ,$$

where ∇ and $\nabla^{\perp}$ are symbols used for connection on TM and $T^{\perp}M$ respectively. h and A_N denote the second fundamental forms related by $g(h(U,V),N) = g(A_N U, V)$ and g denotes the Riemannian metric on $\bar{M}$ as well as on M.

The transforms ϕU and ϕN are decomposed into tangential and normal parts respectively as

$$\phi U = PU + FU \;, \tag{2.6}$$

$$\phi N = tN + fN \;. \tag{2.7}$$

Note 2.1. It is easy to observe that $PU \in D$, $FU \in \phi D^{\perp}$, $tN \in D^{\perp}$ and $fN \in \mu$. □

Now, denoting by $\mathcal{P}_U V$ and $\mathcal{Q}_U V$ the tangential and normal parts of $(\bar{\nabla}_U \phi)V$ and making use of equations (2.6), (2.7), the Gauss and Weingarten formulae, the following equations may easily be obtained

$$\mathcal{P}_U V = (\nabla_U P)V - A_{FV} U - t h(U,V) \;, \tag{2.8}$$

$$\mathcal{Q}_U V = (\nabla_U F)V + h(U, PV) - f h(U,V) \;, \tag{2.9}$$

where the covariant derivatives of P and F are defined by

$$(\nabla_U P)V = \nabla_U PV - P\,\nabla_U V \;, \tag{2.10}$$

$$(\nabla_U F)V = \nabla^{\perp}_U FV - F\,\nabla_U V \;. \tag{2.11}$$

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be a totally umbilical submanifold if the second fundamental form satisfies

$$h(U,V) = g(U,V)\,H \;,$$

where H is the mean curvature vector.

3 – Semi-invariant submanifolds of a nearly trans-Sasakian manifold

To develop the proof of the main theorem, we start with the following preparatory results.

Proposition 3.1. *Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold $\bar{M}$ with $h(X, \phi X) = 0$ for each $X \in D \oplus \langle\xi\rangle$. If the invariant distribution $D \oplus \langle\xi\rangle$ is integrable, then each of its leaves is totally geodesic in M as well as in $\bar{M}$.*

Proof: For $X, Y \in D \oplus \langle \xi \rangle$, we have

$$h(X, \phi Y) + h(\phi X, Y) \;=\; (\bar{\nabla}_X \phi) Y + (\bar{\nabla}_Y \phi) X + \phi(\bar{\nabla}_X Y + \bar{\nabla}_Y X) \\ -\; (\nabla_X \phi Y + \nabla_Y \phi X)\ .$$

Taking account of the hypothesis and the formula (1.2), the above equation takes the form

$$0 \;=\; 2\,\alpha\, g(X,Y)\xi - \eta(Y)\,(\alpha X + \beta\,\phi X) - \eta(X)\,(\alpha Y + \beta\,\phi Y) \\ +\; \phi(\nabla_X Y + \nabla_Y X) - (\nabla_X \phi Y + \nabla_Y \phi X) + 2\,\phi\, h(X,Y)\ . \tag{3.1}$$

On equating the normal parts in the right hand side of the last equation to zero, we get

$$2\, f h(X,Y) \;=\; F(\nabla_X Y + \nabla_Y X)\ ,$$

from which we deduce that

$$\nabla_X Y + \nabla_Y X \;\in\; D \oplus \langle \xi \rangle\ , \tag{3.2}$$

$$h(X,Y) \;\in\; \phi D^{\perp}\ . \tag{3.3}$$

As $D \oplus \langle \xi \rangle$ is integrable, it follows from the observation (3.2) that

$$\nabla_X Y \in D \oplus \langle \xi \rangle\ . \tag{3.4}$$

Taking account of this fact in equation (3.1), it follows that

$$h(X,Y) = 0\ . \tag{3.5}$$

The assertion is proved by virtue of (3.4) and (3.5). ∎

The above proposition leads to the following consequence which is in itself an important result with geometric point of view.

Corollary 3.1. *Let M be a totally umbilical semi-invariant submanifold of a nearly trans-Sasakian manifold $\bar{M}$. If the invariant distribution $D \oplus \langle \xi \rangle$ on M is integrable, then M is totally geodesic in $\bar{M}$.* ∎

For the integrability of the distributions involved on a semi-invariant submanifold of a nearly trans-Sasakian manifold, Jeong-Sik Kim et al. [10] obtained:

Proposition 3.2 ([10]). *Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold $\bar{M}$, then*

(**i**) *D is not integrable,*

(**ii**) *$D \oplus \langle\xi\rangle$ is integrable on M if and only if*

$$N^{(1)}(X,Y) \in D \oplus \langle\xi\rangle ,$$

$$2\Big(h(X,\phi Y) - h(Y,\phi X)\Big) = \eta(X)\Big(\phi Z\, \nabla_Y \xi + f\, h(Y,\xi)\Big) - \eta(Y)\Big(\phi Z\, \nabla_X \xi + f\, h(X,\xi)\Big) ,$$

for all $X, Y \in D \oplus \langle\xi\rangle$ and $Z \in D^\perp$.

(**iii**) *$D^\perp$ is integrable on M if and only if*

$$N^{(1)}(Z,W) \in D \oplus \phi D^\perp , \qquad Z, W \in D^\perp$$

and

$$A_{\phi Z} W = A_{\phi W} Z , \qquad Z, W \in D^\perp ,$$

(**iv**) *The distribution $D^\perp \oplus \langle\xi\rangle$ on M is integrable if and only if*

$$A_{\phi Z} W = A_{\phi W} Z , \qquad Z, W \in D^\perp \oplus \langle\xi\rangle$$

where $N^{(1)} = [\phi,\phi] + 2\,d\eta \otimes \xi$, $[\phi,\phi]$ being the Nijenhuis tensor of ϕ. ∎

4 – Totally umbilical semi-invariant submanifold of nearly trans-Sasakian manifolds

Throughout this section M denotes a totally umbilical semi-invariant submanifold of a nearly trans-Sasakian manifold $\bar{M}$. For $U \in TM$, by formula (1.2),

$$(\bar{\nabla}_U \phi) U = \alpha \|U\|^2 \xi - \eta(U)\,(\alpha U + \beta\, \phi U) . \tag{4.1}$$

In particular for $Z \in D^\perp$,

$$(\bar{\nabla}_Z \phi) Z = \alpha \|Z\|^2 \xi \tag{4.2}$$

and therefore,

$$\mathcal{P}_Z Z = \alpha \|Z\|^2 \xi , \tag{4.3}$$

$$\mathcal{Q}_Z Z = 0 . \tag{4.4}$$

On applying equation (2.8) and using the fact that $PZ = 0$, equation (4.3) yields,

$$-\alpha\,\|Z\|^2\,\xi - g(H,FZ)\,Z - \|Z\|^2\,tH \;=\; P\,\nabla_Z Z \;. \tag{4.5}$$

Now, in view of the observations made in note (2.1), the above equation yields

$$g(H,FZ)\,Z + \|Z\|^2\,tH \;=\; 0 \;, \tag{4.6}$$

$$\alpha\,\|Z\|^2\,\xi \;=\; 0 \;. \tag{4.7}$$

Equation (4.6) has solutions if either (a) $\dim(D^\perp) = 1$ or (b) $H \in \mu$.

On the other hand equation (4.7) has solutions if either (a) $\alpha = 0$ on M or, (b) $D^\perp = \{0\}$.

Now, we are in a position to prove

Theorem 4.1. *Let M be a totally umbilical semi-invariant submanifold of a nearly trans-Sasakian manifold $\bar{M}$. Then at least one of the following is true*

(i) *M is anti-invariant.*

(ii) *M is totally geodesic.*

(iii) *dim($D^\perp$) $= 1$ and $D \oplus \langle\xi\rangle$ is not integrable.*

Proof: If $D = \{0\}$, then by definition M is anti-invariant which is case (i). If $D \neq \{0\}$ and $D \oplus \langle\xi\rangle$ is integrable then by Corollary 3.1 M is totally geodesic in $\bar{M}$ which accounts for case (ii). If $D \oplus \langle\xi\rangle$ is not integrable and $H \in \mu$ then by virtue of observation (3.3), M is again totally geodesic. If however $H \notin \mu$, then equation (4.6) has solutions if and only if dim $(D^\perp) = 1$ or $D^\perp = \{0\}$. For the case when $\dim(D^\perp) = 1$, M belongs to the class (iii). Lastly, if $H \notin \mu$ and $D^\perp = \{0\}$, then again by Corollary 3.1 M is totally geodesic. This completes the proof. ∎

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