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POSITIVE INTEGERS DIVISIBLE BY THE PRODUCT OF THEIR NONZERO DIGITS

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Abstract: Let $\mathcal{N}_0(x)$ denote the set of positive integers $n \le x$ which are divisible by the product of their nonzero digits. In this note, we show that if x is large, then $x^{.495} < \#\mathcal{N}_0(x) < x^{.654}$.

1 – Introduction

For any positive integer n we write

$$n = \overline{n_{t-1}\, n_{t-2} \ldots n_0}\ , \qquad n_i \in \{0, \ldots, 9\}, \quad n_{t-1} \neq 0\ , \tag{1}$$

for the base 10 representation of n. There are several papers in the literature in which arithmetic properties of those positive integers n which obey certain restrictions with respect to their base 10 digits are investigated. For example, almost primes with missing digits are investigated in [1] and [3], arithmetic properties of integers with a fixed sum of digits are investigated in [4], [5], [6] and [7], while *Niven numbers*, that is positive integers n divisible by the sum of their digits, are investigated in [2].

In this note, for a positive integer n whose base 10 representation is given by (1), we write

$$\mathcal{P}_0(n) = \prod_{\substack{0 \le i \le t-1 \\ n_i \neq 0}} n_i$$

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and

$$\mathcal{P}(n) = \prod_{i=0}^{t-1} n_i \; .$$

Note that $\mathcal{P}_0(n) = \mathcal{P}(n)$ whenever n has only nonzero digits, and $\mathcal{P}(n)$ is zero otherwise.

We write $\mathcal{N}_0$ and $\mathcal{N}$ for the set of all positive integers n such that $\mathcal{P}_0(n) \,|\, n$ and $\mathcal{P}(n) \,|\, n$, respectively. For a subset $\mathcal{A}$ of the set of all positive integers, and for a real $x > 1$ we write $\mathcal{A}(x) = \mathcal{A} \cap [1, x]$. Observe that $\#\mathcal{N}(x) \le \#\mathcal{N}_0(x)$. In this paper, we give upper and lower estimates for $\#\mathcal{N}(x)$ and $\#\mathcal{N}_0(x)$. We have the following results.

Theorem 1.

(i) *There exists* x_0 *such that if* $x > x_0$, *then*

$$x^{.495} < \#\mathcal{N}_0(x) < x^{.654} \; .$$

(ii) *There exists* x_1 *such that if* $x > x_1$, *then*

$$x^{.122} < \#\mathcal{N}(x) < x^{.618} \; .$$

The above theorem shows that if we write

$$\rho_0(x) = \frac{\log\bigl(\#\mathcal{N}_0(x)\bigr)}{\log x} \qquad \text{and} \qquad \rho(x) = \frac{\log\bigl(\#\mathcal{N}(x)\bigr)}{\log x} \; ,$$

then

$$.495 \le \liminf_{x\to\infty} \rho_0(x) \le \limsup_{x\to\infty} \rho_0(x) \le .654 \; ,$$

and

$$.122 \le \liminf_{x\to\infty} \rho(x) \le \limsup_{x\to\infty} \rho(x) \le .618 \; .$$

We believe that

$$\lim_{x\to\infty} \rho_0(x) = \rho_0 \quad \text{and} \quad \lim_{x\to\infty} \rho(x) = \rho \tag{2}$$

exist but we do not have any heuristic for precise values for ρ_0 and ρ, although numerical calculations up to $x = 10^{10}$ seem to indicate that $\rho_0 \sim 0.6$ and $\rho \sim 0.4$.

For a positive integer n we write $P(n)$ for the largest prime factor of n. It is clear that we can regard both $\mathcal{P}_0$ and $\mathcal{P}$ as functions from the set of all positive integers to the set $\{m : P(m) \le 7\}$. Experimentally, we noted that there is no

$n \in \mathcal{N}(x)$ for $x = 10^7$ such that $\mathcal{P}_0(n) \in \{27, 36\}$. However, when increasing x to 10^8 several examples of such n were found. This raised however the question of whether the function $\mathcal{P}_0$ (or $\mathcal{P}$) is surjective when restricted to $\mathcal{N}_0$. Note that it cannot be surjective when restricted to $\mathcal{N}$ since if $d = \mathcal{P}(n)$ is a multiple of 10, then n must end in zero, therefore $\mathcal{P}(n) = 0$ contradicting $\mathcal{P}(n) = d$. We show however that $\mathcal{P}_0$ is surjective when restricted to $\mathcal{N}_0$.

Proposition 1. *The function* $\mathcal{P}_0 \colon \mathcal{N}_0 \longrightarrow \{m \colon P(m) \le 7\}$ *is onto.*

We also looked at strings of consecutive integers contained either in $\mathcal{N}_0$ or in $\mathcal{N}$. The answer is contained in the following result.

Proposition 2.

(i) *There are no strings of* 14 *consecutive integers in* $\mathcal{N}_0$, *but there are infinitely many of* 13.

(ii) *There are no strings of* 4 *consecutive integers in* $\mathcal{N}$, *but there are infinitely many of* 3.

Most of the results in this paper can be extended to other bases g such that g is not a power of a prime. We chose to present our results only in the context of the base 10.

Throughout this paper we write x for a large positive real number. We use the Vinogradov symbols $\gg$ and $\ll$ as well as the Landau symbols with their usual meanings.

2 – The Proof of Theorem 1

2.1. A preliminary result

Throughout the proof of Theorem 1, we shall use the following fact.

Lemma 1. *Let* $v_1, \ldots, v_\ell > 0$ *be fixed and* $u = v_1 + \cdots + v_\ell$. *Then*

$$\frac{(\lfloor us \rfloor)!}{(\lfloor v_1 s \rfloor)! \ldots (\lfloor v_\ell s \rfloor)!} = \exp\left(s \left(u \log u - \sum_{i=1}^{\ell} v_i \log v_i + o(1) \right) \right)$$

as $s \to \infty$.

Proof: Follows immediately from Stirling's formula. ∎

2.2. Lower bounds

The Case $\mathcal{N}_0$. We let α be a constant > 2 to be chosen later. We let s be a large positive integer and put $t = \lfloor \alpha s \rfloor + s$. We set $n_i = 0$ for $i = 0, \ldots, s-1$. The remaining digits n_j for $j \in \{s, \ldots, t-1\}$ will be chosen from the set $\{0, 1, 2, 5\}$ in such a way that exactly s of them are 2 and exactly s of them are 5. There are $\binom{\lfloor \alpha s \rfloor}{s}$ ways of putting 2 in s locations of the totality of $\lfloor \alpha s \rfloor$ available locations, and $\binom{\lfloor \alpha s \rfloor - s}{s}$ ways of putting 5 is s of the remaining $\lfloor \alpha s \rfloor - s$ locations. Finally, there are $2^{\lfloor \alpha s \rfloor - 2s}$ ways of filling in the remaining positions with 0 or 1. This shows that if we write $x = 5(10^t - 1)/9$, then every one of the numbers we have constructed above is in $\mathcal{N}_0(x)$, therefore with Lemma 1 for $\ell = 3$, $v_1 = v_2 = 1$ and $v_3 = \alpha - 2$, we get

$$\begin{aligned}
\#\mathcal{N}_0(x) &\geq \binom{\lfloor \alpha s \rfloor}{s} \binom{\lfloor \alpha s \rfloor - s}{s} 2^{\lfloor \alpha s \rfloor - 2s} \\
&= \frac{\big(\lfloor \alpha s \rfloor\big)!}{s!^2 \big(\lfloor \alpha s \rfloor - 2s\big)!} \, 2^{\lfloor \alpha s \rfloor - 2s} \\
&= \exp\left(s\left(\alpha \log \alpha - (\alpha - 2) \log\left(\frac{(\alpha-2)}{2} \right) + o(1) \right) \right) .
\end{aligned}$$

Since $t = \dfrac{\log x}{\log 10}\big(1 + o(1)\big)$, we get that $s = \dfrac{\log x}{(\alpha+1)\log 10}\big(1+o(1)\big)$. Thus, as $x \to \infty$, we get that

$$\#\mathcal{N}_0(x) \geq x^{c(\alpha)+o(1)} ,$$

where

$$c(\alpha) = \frac{\alpha \log \alpha - (\alpha - 2)\log\big((\alpha-2)/2\big)}{(\alpha+1)\log 10} .$$

To find the optimal α, we solve the equation $\dfrac{dc(\alpha)}{d\alpha} = 0$, getting $\alpha = 5.5385\ldots$, which in turn leads to $c(\alpha) = 0.495599\ldots$, thus establishing the lower bound of Theorem 1 (i).

The Case $\mathcal{N}$. For a positive integer k we write

$$\mathcal{A}_k = \left\{ \overline{a_{k-1}\ldots a_0} \equiv 0 \pmod{2^k} : \; a_i \in \{1, 2, 4\}, \; i = 0, \ldots, k-1 \right\} .$$

We begin with the following preliminary result.

Lemma 2. *The following estimate holds for all $k \ge 1$:*

$$\min\Big\{\mathcal{P}(a):\ a \in \mathcal{A}_k\Big\} \ \le\ 2^{3\lfloor k/4\rfloor+6}\ . \tag{3}$$

Proof: We shall construct a smaller subset $\mathcal{A}'_k$ of $\mathcal{A}_k$ formed by all numbers $a = \overline{a_{k-1}\dots a_0}$ such that $a_0 = 2$ and $\overline{a_{m-1}\dots a_0} \in \mathcal{A}_m$ holds for all $1 \le m \le k$. Thus, $\mathcal{A}'_1 = \{2\}$ and $\mathcal{A}'_2 = \{12\}$. Assume that $m < k$ and that $\mathbf{a}_m = \overline{a_{m-1}\dots a_0} \in \mathcal{A}'_m$. To *extend* this number to $\mathcal{A}'_k$ for $k > m$, we do the following:

(i) if $2^m \,||\, \mathbf{a}_m$, we then note that $10^m + \mathbf{a}_m \equiv 0 \pmod{2^{m+1}}$. Hence, we may set $a_{m+1} = 1$ and $\mathbf{a}_{m+1} = \overline{1\,a_m\dots a_0}$ and note that $\mathbf{a}_{m+1} \in \mathcal{A}'_{m+1}$.

(ii) if $2^{m+2} \,|\, \mathbf{a}_m$, we then set $a_{m+2} = 2$, $a_{m+1} = 1$ and note that $2\cdot 10^m + \mathbf{a}_m \equiv 0 \pmod{2^{m+1}}$ and also that $10^{m+1} + 2\cdot 10^m + \mathbf{a}_m \equiv 0 \pmod{2^{m+2}}$, in which case $\mathbf{a}_{m+2} = \overline{12\,a_m\dots a_0} \in \mathcal{A}'_{m+2}$.

(iii) if $2^{m+1} \,||\, \mathbf{a}_m$ we can do one of the following:

- **(1)** set $a_{m+1} = 4$, $a_{m+2} = 1$. In this case, $4\cdot 10^m + \mathbf{a}_m \equiv 0 \pmod{2^{m+1}}$ as well as $10^{m+1} + 4\cdot 10^m + \mathbf{a}_m \equiv 0 \pmod{2^{m+2}}$, so that $\mathbf{a}_{m+2} = \overline{14\,a_m\dots a_0} \in \mathcal{A}'_{m+2}$.
- **(2)** set $a_{m+1} = 2$, $a_{m+2} = 2$. In this case, $2\cdot 10^m + \mathbf{a}_m \equiv 0 \pmod{2^{m+1}}$ as well as $2\cdot 10^{m+1} + 2\cdot 10^m + \mathbf{a}_m \equiv 0 \pmod{2^{m+2}}$, so that $\mathbf{a}_{m+2} = \overline{22\,a_m\dots a_0} \in \mathcal{A}'_{m+2}$.

Assume that $\mathbf{a}_k$ is a number constructed by the above process. We shall show that there exists a choice at (iii) of doing either (1) or (2) in such a way that the resulting number has the property that the product of its digits fulfills the desired inequality (3). Note that the last two digits of $\mathbf{a}_k$ are 12. As we move from $\mathbf{a}_m$ for some $m < k$ up towards k, we encounter one of the three situations (i), (ii) or (iii) above. Every time we encounter situation (i), we save a factor of 2 as we used one digit of 1. Every time we encounter situation (ii), we save one factor of 2 from 2 digits, since we have a group 12. If we encounter (iii) however, we save nothing as both 14 and 22 have products $2^2 = 2^{\text{number of digits}}$. If after encountering (iii), we encounter either (i) or (ii), then we save a factor of 2 from 4 possible locations. Thus, let us assume that (iii) appears twice, consecutively, regardless of whether (1) or (2) is being performed. We claim that this is impossible. Indeed, for if it were not so, we would conclude that 2^{m+4} divides both $\overline{1414\,a_m\dots a_0}$ and $\overline{2222\,a_m\dots a_0}$; hence, their difference $(2222-1414)\,10^m$. We thus get that $2^4 \,|\, 808$, which is false. The above argument shows that as we move from 2 towards k, we create blocks of consecutive digits of length $\ell \in \{1,\dots,4\}$ in $\mathbf{a}_k$ such that the product of the digits in any block is $2^{\ell-1}$. This immediately implies inequality (3). ∎

We are now ready to handle a lower bound for $\mathcal{N}(x)$. We let again $\alpha > 1/4$ to be determined later. We let s be a large positive integer, and $\mathbf{a}_s$ be a number in $\mathcal{A}_s$ satisfying inequality (3). We let $t = \lfloor \alpha s \rfloor + s$. We also let n be such that $n_i = a_i$ for $i = 0, \ldots, s-1$ and $n_j \in \{1, 2\}$ for $j \geq s$. Since $n \equiv \mathbf{a}_s \pmod{2^s}$, we get that $2^s \mid n$. Since $\mathcal{P}(\mathbf{a}_s) \leq 2^{3\lfloor s/4 \rfloor + 6}$, it follows that we can take $\lfloor s/4 \rfloor - 6$ of the digits n_j of n for $j \geq s$ to be 2, and the remaining ones being 1, and the resulting number belongs to $\mathcal{N}(x)$, where $x = 2(10^t - 1)/9$. Thus, the number of such numbers is, by Lemma 1 with $\ell = 2$, $v_1 = 1/4$, $v_2 = \alpha - 1/4$,

$$\begin{aligned} \#\mathcal{N}(x) &\geq \binom{\lfloor \alpha s \rfloor}{\lfloor s/4 \rfloor - 6} \\ &= \exp\Big(s\Big(\alpha \log \alpha - 1/4 \log(1/4) - (\alpha - 1/4) \log(\alpha - 1/4) + o(1) \Big) \Big) . \end{aligned}$$

Since $t = \big(1 + o(1)\big) \dfrac{\log x}{\log 10}$, we get that $s = \big(1 + o(1)\big) \dfrac{\log x}{(\alpha + 1) \log 10}$. Thus,

$$\#\mathcal{N}(x) \geq x^{c(\alpha) + o(1)} ,$$

where

$$c(\alpha) = \frac{\alpha \log \alpha - 1/4 \log(1/4) - (\alpha - 1/4) \log(\alpha - 1/4)}{(\alpha + 1) \log 10} .$$

As usual, to encounter the best value for α, we take the derivative of $c(\alpha)$ and equal it to zero getting $\alpha = 1.01989\ldots$, which leads to $c(\alpha) = 0.122123\ldots$, thus establishing the lower bound in Theorem 1 (ii).

2.3. Upper bounds

The Case $\mathcal{N}_0$. We let β and γ be constants in $(0, 1)$ to be determined later. We let $\mathcal{N}_{0,1}(x) = \{n \leq x^{1-\beta}\}$. Clearly,

$$\#\mathcal{N}_{0,1}(x) \leq x^{1-\beta} . \tag{4}$$

We now put $y = x^\gamma$ and set

$$\mathcal{N}_{0,2}(x) = \Big\{ n \leq x : \ d \mid n \text{ for some } d > y \text{ with } P(d) \leq 7 \Big\} .$$

Fix $n \in \mathcal{N}_{0,2}(x)$. Then there exists $d > y$ with $P(d) \leq 7$ such that $d \mid n$. For a fixed d, there are at most x/d such values for n. Since $\mathcal{A}(t) = \Psi(t, 7) = \{n \leq t :$

$P(n) \le 7\}$ satisfies $\#\mathcal{A}(t) \ll (\log t)^4$, we get, by summing over all the acceptable values of d, that

$$\#\mathcal{N}_{0,2}(x) \le \sum_{\substack{d>y \\ P(d)\le 7}} \frac{x}{d} \le x\int_y^x \frac{dA(t)}{t} \ll \frac{A(y)}{y} + \int_y^x \frac{(\log t)^4}{t^2}\,dt$$
$$\ll \frac{x\,(\log x)^4}{y} = x^{1-\beta+o(1)} .$$

From now on, we assume that $n \in \mathcal{N}_{0,3}(x) = \mathcal{N}_0(x)\setminus\big(\mathcal{N}_{0,1}(x)\cup\mathcal{N}_{0,2}(x)\big)$. Then $\mathcal{P}_0(n) \le x^\gamma = 2^{\gamma\log x/\log 2}$. Thus, n can have at most $\gamma\log x/\log 2$ digits distinct from 0 or 1. Since $n > x^{1-\beta}$, it follows that n has at least

$$\left\lfloor (1-\beta)\,\frac{\log x}{\log 10}\right\rfloor + 1 - \frac{\gamma\log x}{\log 2} > \delta\log x$$

digits equal to either zero or 1, where

$$\delta = \frac{1-\beta}{\log 10} - \frac{\gamma}{\log 2} .$$

Since the totality of digits of n does not exceed $\lfloor\log x/\log 10\rfloor + 1$, we get, by Lemma 1 with $\ell = 2$, $v_1 = \delta$, $v_2 = 1/\log 10 - \delta$, that

$$\#\mathcal{N}_{0,3}(x) \le \binom{\lfloor\log x/\log 10\rfloor + 1}{\lfloor\delta\log x\rfloor} 2^{\lfloor\delta\log x\rfloor}\,10^{\lfloor\log x/\log 10\rfloor - \lfloor\delta\log x\rfloor + 1}$$

$$(5)\qquad = \exp\Bigg(\log x\bigg(\frac{1}{\log 10}\log\Big(\frac{1}{\log 10}\Big) - \delta\log\delta - \Big(\frac{1}{\log 10} - \delta\Big)\log\Big(\frac{1}{\log 10} - \delta\Big) + \delta\log 2 + 1 - \delta\log 10 + o(1)\bigg)\Bigg).$$

Comparing (4) with (5), we see that we should take $\beta = \gamma$, therefore $\delta = 1 - \beta(1/\log 10 + 1/\log 2)$. Imposing that also $\#\mathcal{N}_{0,3}(x) \le x^{1-\beta+o(1)}$, we are led to solve the equation

$$1-\beta = \frac{1}{\log 10}\log\Big(\frac{1}{\log 10}\Big) - \delta\log\delta - \Big(\frac{1}{\log 10} - \delta\Big)\log\Big(\frac{1}{\log 10} - \delta\Big)$$
$$+ \delta\log 2 + 1 - \delta\log 10 ,$$

with $\delta = 1 - \beta(1/\log 10 + 1/\log 2)$, which gives $\beta = 0.3467\ldots$, and therefore leads to $\#\mathcal{N}_0(x) < x^{.6533}$ if x is sufficiently large, which yields the upper bound in Theorem 1 (i).

The Case $\mathcal{N}(x)$. We follow the same procedure as before. We let $\mathcal{N}_1(x) = \mathcal{N}_{0,1}(x) \cap \mathcal{N}(x)$ and $\mathcal{N}_2(x) = \mathcal{N}_{0,2}(x) \cap \mathcal{N}(x)$. Certainly,

$$\#\mathcal{N}_1(x) \le x^{1-\beta} \qquad \text{and} \qquad \#\mathcal{N}_2(x) \le x^{1-\gamma+o(1)} .$$

If we now put $\mathcal{N}_3(x) = \mathcal{N}(x) \setminus \big(\mathcal{N}_1(x) \cup \mathcal{N}_2(x)\big)$, we get that any $n \in \mathcal{N}_3(x)$ has at least $\delta \log x$ digits equal to 1. The same argument as before now shows that

$$\begin{aligned}\#\mathcal{N}_3(x) \;&\le\; \binom{\lfloor \log x/\log 10\rfloor + 1}{\lfloor \delta \log x\rfloor} \, 9^{\lfloor \log x/\log 10\rfloor - \lfloor \delta \log x\rfloor + 1} \\ &=\; \exp\Bigg(\log x \left(\frac{1}{\log 10} \log\left(\frac{1}{\log 10}\right) - \delta \log \delta \right. \\ &\quad \left. - \left(\frac{1}{\log 10} - \delta\right) \log\left(\frac{1}{\log 10} - \delta\right) + \log 9/\log 10 - \delta \log 9 + o(1)\right)\Bigg) .\end{aligned}$$

Taking again $\beta = \gamma$, we are led to solve the equation

$$\begin{aligned}1 - \beta \;=\; &\frac{1}{\log 10} \log\left(\frac{1}{\log 10}\right) - \delta \log \delta - \left(\frac{1}{\log 10} - \delta\right) \log\left(\frac{1}{\log 10} - \delta\right) \\ &+ \log 9/\log 10 - \delta \log 9 ,\end{aligned}$$

again with $\delta = 1 - \beta(1/\log 10 + 1/\log 2)$, which gives $\beta = .38276\ldots$, and therefore leads to $\#\mathcal{N}(x) < x^{.618}$ if x is sufficiently large, which yields the upper bound in Theorem 1 (ii).

3 – The Proofs of the Propositions

Proof of Proposition 1: Let $d = 2^a\, 3^b\, 5^c\, 7^d$. We put $f = \max\{a, c\}$ and $n = m\, 10^f$, where m is coprime to 10. It is clear that $\mathcal{P}_0(n) = \mathcal{P}_0(m)$ and n is a multiple of $2^a\, 5^c$. Thus, it remains to show that we can choose m to be a multiple of $M = 3^b\, 7^d$ such that it has a digits 2, b of 3, c of 5, d of 7, and the remaining 0 and 1. We search for m of the form

$$m \;=\; \sum_{\ell=0}^{t-1} a_\ell \; 10^{\ell \phi(M)} ,$$

where ϕ is Euler's totient function and $a_\ell \in \{1,2,3,5,7\}$ are such that a of them are 2, b of them are 3, c of them are 5, and d of them are 7. Assume that $s = t-a-b-c-d$ of them are 1. Then, by Euler's Theorem,

$$m \equiv 2a+3b+5c+7d+s \pmod{M} .$$

Thus, it suffices to choose $s>0$ such that $2a+3b+5c+7d+s$ is a multiple of M. The condition $s>0$ insures that we may choose $a_0 = 1$ so m is coprime to 10. It is now clear that if m is chosen as shown above, then $n = m\,10^f \in \mathcal{N}_0$. ∎

Proof of Proposition 2: (**i**) No number of the form $\overline{\ldots a4}$ or $\overline{\ldots a8}$, where a is an odd digit, can belong to $\mathcal{N}_0$, as such numbers are not divisible by 4 but the product of their digits is. This shows that any string of consecutive integers in $\mathcal{N}$ is either contained in an interval of length 15 starting with $m = \overline{\ldots a9}$, where a is odd, or in an interval of length 3 starting with $m = \overline{\ldots a5}$, where a is odd. We analyze only the first situation. If the string of consecutive integers in $\mathcal{N}_0$ starts with m, then it cannot contain $m+10$, because both these numbers end in 9 but not both of them can be multiples of 9. So, any string of consecutive integers in $\mathcal{N}_0$ containing m can have length at most 10. A similar argument can be used to deduce that any string of consecutive integers in $\mathcal{N}_0$ ending in $m+14 = \overline{\ldots 3}$, cannot contain $m+4$ (which ends also with 3), therefore it can have length at most 10. Hence, either the string has length at most 10, or is contained in $\{m+1,\ldots,m+13\}$. It remains to show that there are infinitely many m such that this last string has all its members in $\mathcal{N}_0$. We choose

$$n = \overline{11\ldots1000} ,$$

where the string of 1's is of length $\ell = 18\,k$ for some $k \geq 1$. It is clear that $n,\ n+1,\ n+2,\ n+4,\ n+5,\ n+8,\ n+10,\ n+11,\ n+12$ are all in $\mathcal{N}_0$. Since the number of 1's is a multiple of 9, it is also clear that $n+3,\ n+6,\ n+9$ are also in $\mathcal{N}_0$. Finally, since the number of 1's is a multiple of 7, one checks using Euler's Theorem that

$$n+7 = 10^3\left(\frac{10^\ell-1}{9}\right)+7 \equiv 0 \pmod{7} ,$$

which shows that $n+7$ is also in $\mathcal{N}_0$.

(**ii**) It is clear that a string of consecutive integers in $\mathcal{N}$ cannot contain a multiple of 10. Assume that $10|n$ and that the string of consecutive integers is contained in $\{n+1,\ldots,n+9\}$. Write n as shown in (1). Hence, $n_0 = 0$.

If n_1 is odd, then we saw that $n+4$ and $n+8$ cannot be members of $\mathcal{N}_0$. Thus, the length of the string is at most 3 in this case. If n_1 is even, then neither one of $n+1$, $n+3$, $n+5$, $n+7$ or $n+9$ can be part of the string because they are odd but the products of their digits is even. Hence, there is no string of consecutive integers in $\mathcal{N}$ of length ≥ 4. Taking

$$n = \overline{1\ldots 1}\ ,$$

with a number ℓ of 1's which is congruent to 1 modulo 3, we see that n, $n+1$ and $n+2$ are all in $\mathcal{N}$. ∎

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