PORTUGALIAE MATHEMATICA
Vol. 64 Fasc. 1 – 2007
Nova Série

ON THE CAUCHY PROBLEM FOR A ONE-DIMENSIONAL COMPRESSIBLE VISCOUS POLYTROPIC IDEAL GAS

Yuming Qin, Yumei Wu * and Fagui Liu

Recommended by Hugo Beirão da Veiga

Abstract: In this paper, we first prove the regularity and continuous dependence on initial data for H^i-solutions $(i=1,2,4)$ for large initial data and then show the large-time behavior of H^i $(i=2,4)$-global solutions for small initial data to the Cauchy problem for the compressible Navier–Stokes equations of a one-dimensional viscous polytropic ideal gas. Moreover, we also obtain the large-time behavior of "small" classical solutions *in the norm of classical solutions* for this model.

1 – Introduction

In this paper we study the regularity, continuous dependence on initial data and large-time behavior of H^i $(i=1,2,4)$ solutions to the Cauchy problem for the compressible Navier–Stokes equations of a one-dimensional viscous polytropic ideal gas in Lagrangian coordinates (see [21–27, 32–35, 39, 41–42, 49–50]):

$$u_t = v_x \ , \tag{1.1}$$

$$v_t = \sigma_x \ , \quad \left(\sigma := \mu \frac{v_x}{u} - R\frac{\theta}{u}\right) \tag{1.2}$$

$$C_V\,\theta_t = \left[\lambda \frac{\theta_x}{u}\right]_x + \sigma v_x \tag{1.3}$$

Received: March 23, 2006; *Revised*: November 14, 2006.
AMS Subject Classification: 76N10, 35B40, 35M10, 35Q30.
Keywords: large-time behavior; viscous polytropic ideal gas; Cauchy problem; generalized solutions.

* Post-doctoral Researcher of CNPq, from Institute of Mathematics, Federal University of Rio de Janeiro, Brazil.

subject to the following initial conditions

$$\Big(u(x,0),\, v(x,0),\, \theta(x,0)\Big) = \Big(u_0(x),\, v_0(x),\, \theta_0(x)\Big)\,, \quad \forall\, x\in\mathbb{R}\,. \tag{1.4}$$

The equations (1.1)–(1.3) describe the motion of a one-dimensional viscous polytropic ideal gas, where u, v, θ are the specific volume, velocity, and absolute temperature, respectively; σ is the stress, μ, C_V and λ are positive constants.

We introduce the following definition of H^i-solutions ($i=2,4$).

Definition. For a fixed constant $T>0$ and some positive constants $\bar{u}$ and $\bar{\theta}$, we call $\big(u(t), v(t), \theta(t)\big)$ to be an H^2-generalized solution to the Cauchy problem (1.1)–(1.4) if it satisfies the following conditions

$$u-\bar{u},\ v,\ \theta-\bar{\theta}\ \in\ L^\infty\big([0,T], H^2(\mathbb{R})\big)\,, \tag{1.5}$$

$$u_t\ \in\ L^\infty\big((0,T), H^1(\mathbb{R})\big)\cap L^2\big((0,T), H^2(\mathbb{R})\big)\,, \tag{1.6}$$

$$v_t,\ \theta_t\ \in\ L^\infty\big((0,T), L^2(\mathbb{R})\big)\cap L^2\big((0,T), H^1(\mathbb{R})\big)\,, \tag{1.7}$$

$$u_x\in L^2\big((0,T), H^1(\mathbb{R})\big)\,, \qquad v_x,\ \theta_x\in L^2\big((0,T), H^2(\mathbb{R})\big)\,. \tag{1.8}$$

Furthermore, in addition to (1.5)–(1.8), if the following conditions hold,

$$u-\bar{u},\ v,\ \theta-\bar{\theta}\ \in\ L^\infty\big([0,T], H^4(\mathbb{R})\big)\,, \tag{1.9}$$

$$u_t\ \in\ L^\infty\big((0,T), H^3(\mathbb{R})\big)\cap L^2\big((0,T), H^2(\mathbb{R})\big)\,, \tag{1.10}$$

$$v_t,\ \theta_t\ \in\ L^\infty\big((0,T), H^2(\mathbb{R})\big)\cap L^2\big((0,T), H^3(\mathbb{R})\big)\,, \tag{1.11}$$

$$u_{tt}\ \in\ L^\infty\big((0,T), H^1(\mathbb{R})\big)\cap L^2\big((0,T), H^2(\mathbb{R})\big)\,, \tag{1.12}$$

$$v_{tt},\ \theta_{tt}\ \in\ L^\infty\big((0,T), L^2(\mathbb{R})\big)\cap L^2\big((0,T), H^1(\mathbb{R})\big)\,, \tag{1.13}$$

$$u_x\in L^2\big((0,T), H^3(\mathbb{R})\big)\,, \tag{1.14}$$

$$v_x,\ \theta_x\in L^2\big((0,T), H^4(\mathbb{R})\big)\,, \qquad u_{ttt}\in L^2\big((0,T), L^2(\mathbb{R})\big)\,, \tag{1.15}$$

then we call $\big(u(t), v(t), \theta(t)\big)$ to be an H^4-solution to the Cauchy problem (1.1)–(1.4). □

Now let us recall some related results for the equations (1.1)–(1.3) in the literature. For the one-dimensional Cauchy problem (1.1)–(1.4), Kanel [23] obtained the global existence and large-time behavior (only for v,θ) of H^1-solutions (see the definition below) with *small initial data*; Kazhikhov and Shelukhin [26, 27] proved that if $u_0-\bar{u}, v_0, \theta_0-\bar{\theta}\in H^1(\mathbb{R})$ with some positive constants $\bar{u}, \bar{\theta}$ and $u_0(x), \theta_0(x)>0$ on $\mathbb{R}$, then there exists a unique global (*large*) solution

$\big(u(t), v(t), \theta(t)\big)$ with positive $u(x,t)$ and $\theta(x,t)$ to the Cauchy problem (1.1)–(1.4) on $\mathbb{R}\times[0,+\infty)$ such that for any $T>0$,

$$u-\bar{u},\, v,\, \theta-\bar{\theta} \in L^\infty\big([0,T], H^1(\mathbb{R})\big)\,, \qquad u_t \in L^\infty\big((0,T), L^2(\mathbb{R})\big)\,, \tag{1.16}$$

$$v_t,\, u_x,\, \theta_t,\, u_{xt},\, v_{xx},\, \theta_{xx} \in L^2\big((0,T), L^2(\mathbb{R})\big)\,. \tag{1.17}$$

Now we call $\big(u(t), v(t), \theta(t)\big)$ verifying (1.9)–(1.10) to be an H^1-generalized solution to the Cauchy problem (1.1)–(1.4). It is noteworthy that there is *no any* result on asymptotic behavior given in [26, 27]. In this case, Okada and Kawashima [39] established the global existence and large-time behavior of classical (or H^1-) solution with *small* initial data and Jiang [21] proved the large-time behavior of H^1-solution with weighted small initial data. For one-dimensional initial boundary value problems, we refer to the works [1–3, 11, 13, 22, 24–25, 27–28, 33–36, 39, 41–42, 47, 50]. For two or three dimensional Cauchy problems or initial boundary value problems, the global existence and large-time behavior of *smooth* solutions have been investigated for general domains only in case of "small initial data" (see [1, 4, 12, 14, 18–21, 29–32, 40, 46–47, 49, 51]). We also note the recent works of Feireisl, Petzeltova, Novotny and Straskraba ([5–10, 37–38, 48]) on the large-time behavior of weak solutions to multi-dimensional compressible fluids. For related general real gases, we refer to [41–45].

It is well-known that continuous dependence of solutions on initial data is very important (especially when we study infinite-dimensional dynamics, which is equivalent to that the associated semigroup is continuous with respect to initial data or this semigroup, as an operator, is continuous for any but fixed time t). For example, we refer to [15–17]. In [15], Hoff established the continuous dependence on initial data in $L^2(\mathbb{R})$ for the Cauchy problem of the Navier–Stokes equations of one-dimensional compressible flow with discontinuous initial data. In this paper, we prove both continuous dependence on initial data in $H^i(\mathbb{R})$ $(i=1,2,4)$ and global existence and large-time behavior in $H^i(\mathbb{R})$ $(i=2,4)$. Note that the large-time behavior of global solutions in $H^4(\mathbb{R})$ implies that of solutions in $C^{3+1/2}(\mathbb{R})$ in which the classical solution exists globally. This is a new ingredient of this paper.

It is worthy to point out here that since the domain is unbounded, the Poincaré inequality can not be applied to this domain, and further the large-time behavior of *large* initial data and the decay rate can not be anticipated. This is why we only establish the large-time behavior of solutions with "small initial data" and *no* decay rate is given in our results.

The aim of this paper is to prove the global existence and continuous dependence on initial data of $H^i(\mathbb{R})$ $(i=1,2,4)$ (global) solutions for large initial data and then to show the large-time behavior of these $H^i(\mathbb{R})$ $(i=2,4)$ solutions for "small initial data".

The notation in this paper will be as follows:

L^p, $1\le p\le +\infty$, $W^{m,p}$, $m\in\mathbb{N}$, $H^1=W^{1,2}$, $H_0^1=W_0^{1,2}$ denote the usual (Sobolev) spaces on $\mathbb{R}$. In addition, $\|\cdot\|_B$ denotes the norm in the space B; we also put $\|\cdot\|=\|\cdot\|_{L^2(\mathbb{R})}$. We denote by $C^k(I,B)$, $k\in\mathbb{N}_0$, the space of k-times continuously differentiable functions from $J\subseteq\mathbb{R}$ into a Banach space B, and likewise by $L^p(J,B)$, $1\le p\le+\infty$, the corresponding Lebesgue spaces. $C^\beta([0,T],B)$ denotes the Hölder space of B-valued continuous functions with exponent $\beta\in(0,1]$ in variable t. We use C_i $(i=1,2,3,4)$ to denote the universal constant depending only on $\min\limits_{x\in\mathbb{R}} u_0(x)$, $\min\limits_{x\in\mathbb{R}} \theta_0(x)$, the $H^i(\mathbb{R})$ $(i=1,2,3,4)$ norm of $(u_0-\bar{u}, v_0, \theta_0-\bar{\theta})$ (for some positive constants $\bar{u},\bar{\theta}$) and e_0 or E_0, E_1 (see Theorem 1.3), but independent of any length of time $T>0$.

We are now in a position to state our main theorems.

Theorem 1.1. *Assume that $u_0-\bar{u}, v_0, \theta_0-\bar{\theta}\in H^2(\mathbb{R})$ with some positive constants $\bar{u}$, $\bar{\theta}$ and $u_0(x)>0$, $\theta_0(x)>0$ on $\mathbb{R}$ and the compatibility conditions hold. Then for any but fixed constant $T>0$, the Cauchy problem (1.1)–(1.4) admits a unique H^2-generalized global solution $\big(u(t),v(t),\theta(t)\big)$ on Q_T verifying (1.5)–(1.8) and the following estimates hold for any $t\in[0,T]$,*

$$(1.18)\qquad 0<C_1^{-1}(T)\le\theta(x,t)\le C_1(T)\quad \text{on }\ \mathbb{R}\times[0,T]\ ,$$

$$(1.19)\qquad 0<C_1^{-1}(T)\le u(x,t)\le C_1(T)\quad \text{on }\ \mathbb{R}\times[0,T]\ ,$$

$$(1.20)\qquad \begin{aligned}&\|u(t)-\bar{u}\|_{H^2}^2+\|u(t)-\bar{u}\|_{W^{1,\infty}}^2+\|u_t(t)\|_{H^1}^2+\|v(t)\|_{H^2}^2+\|v(t)\|_{W^{1,\infty}}^2\\&+\|v_t(t)\|^2+\|\theta(t)-\bar{\theta}\|_{H^2}^2+\|\theta(t)-\bar{\theta}\|_{W^{1,\infty}}^2+\|\theta_t(t)\|^2\\&+\int_0^t\Big[\|u_x\|_{H^1}^2+\|u_x\|_{L^\infty}^2+\|u_t\|_{H^2}^2+\|v_x\|_{H^2}^2+\|v_x\|_{W^{1,\infty}}^2\\&+\|v_t\|_{H^1}^2+\|\theta_x\|_{H^2}^2+\|\theta_x\|_{W^{1,\infty}}^2+\|\theta_t\|_{H^1}^2\Big](\tau)\,d\tau\ \le\ C_2(T)\ .\end{aligned}$$

Moreover, the H^i-generalized global solutions $(i=1,2)$ are continuously dependent on initial data in the sense that

$$(1.21)\qquad \begin{aligned}&\Big\|\Big(u_1(t)-u_2(t),\, v_1(t)-v_2(t),\, \theta_1(t)-\theta_2(t)\Big)\Big\|_{H^i}\le\\&\le C_i(T)\Big\|\Big(u_{01}-u_{02},\, v_{01}-v_{02},\, \theta_{01}-\theta_{02}\Big)\Big\|_{H^i}\ ,\qquad i=1,2\,,\end{aligned}$$

where $\big(u_j(t), v_j(t), \theta_j(t)\big)$ $(j=1,2)$ *is the* H^i*-generalized global solution* $(i=1,2)$ *to the Cauchy problem (1.1)–(1.4) with the initial datum* $(u_{0j}, v_{0j}, \theta_{0j}) \in H^i(\mathbb{R})\times H^i(\mathbb{R})\times H^i(\mathbb{R})$ *satisfying* $u_{0j}-\bar{u}, v_{0j}, \theta_{0j}-\bar{\theta} \in H^i(\mathbb{R})$, $u_{0j}(x)>0$, $\theta_{0j}(x)>0$ *on* $\mathbb{R}$ *and the compatibility conditions* $(j=1,2)$. *This property implies the uniqueness of* H^i*-generalized global solution* $(i=1,2)$.

Theorem 1.2. *Assume that* $u_0-\bar{u}, v_0, \theta_0-\bar{\theta} \in H^4(\mathbb{R})$ *with some positive constants* $\bar{u}$, $\bar{\theta}$ *and* $u_0(x)>0$, $\theta_0(x)>0$ *on* $\mathbb{R}$ *and the compatibility conditions hold. Then for any but fixed constant* $T>0$, *the Cauchy problem (1.1)–(1.4) admits a unique* H^4*-global solution* $\big(u(t), v(t), \theta(t)\big)$ *on* Q_T *verifying (1.9)–(1.15) and (1.18)–(1.19), and the following estimates hold for any* $t\in[0,T]$,

$$\tag{1.22} \begin{aligned} &\|u(t)-\bar{u}\|_{H^4}^2 + \|u(t)-\bar{u}\|_{W^{3,\infty}}^2 + \|u_t(t)\|_{H^3}^2 + \|u_{tt}(t)\|_{H^1}^2 \\ &\quad + \|v(t)\|_{H^4}^2 + \|v(t)\|_{W^{3,\infty}}^2 + \|v_t(t)\|_{H^2}^2 + \|v_{tt}(t)\|^2 \\ &\quad + \|\theta(t)-\bar{\theta}\|_{H^4}^2 + \|\theta(t)-\bar{\theta}\|_{W^{3,\infty}}^2 + \|\theta_t(t)\|_{H^2}^2 + \|\theta_{tt}(t)\|^2 \;\le\; C_4(T)\;, \end{aligned}$$

$$\tag{1.23} \begin{aligned} &\int_0^t \Big(\|u_x\|_{H^3}^2 + \|u_t\|_{H^4}^2 + \|u_{tt}\|_{H^2}^2 + \|u_{ttt}\|^2 + \|u_x\|_{W^{2,\infty}}^2 \\ &\quad + \|v_x\|_{H^4}^2 + \|v_t\|_{H^3}^2 + \|v_{tt}\|_{H^1}^2 + \|v_x\|_{W^{3,\infty}}^2 \\ &\quad + \|\theta_x\|_{H^4}^2 + \|\theta_t\|_{H^3}^2 + \|\theta_{tt}\|_{H^1}^2 + \|\theta_x\|_{W^{3,\infty}}^2\Big)(\tau)\,d\tau \;\le\; C_4(T)\;. \end{aligned}$$

Moreover, the H^4*-global solutions is continuously dependent on initial data in the sense of (1.21) with* $i=4$.

Remark 1.1. We know that H^2-generalized global solution $\big(u(t), v(t), \theta(t)\big)$ obtained in Theorem 1.1 is not classical one. By the embedding theorem (the Morrey theorem), we have $u_0-\bar{u}$, v_0, $\theta_0-\bar{\theta} \in C^{1+\frac{1}{2}}(\mathbb{R})$. If we impose on the higher regularities of v_0, $\theta_0-\bar{\theta}\in C^{2+\gamma}(\mathbb{R})$, $\gamma\in(0,1)$, then the global existence of classical solutions was obtained in [27]. □

Remark 1.2. From Remark 1.1 we know that the H^2-generalized global solution $\big(u(t), v(t), \theta(t)\big)$ obtained in Theorem 1.1 can be understood as a generalized (global) solution between the classical (global) solution and the H^1-generalized (global) solution. □

Remark 1.3. The similar results in Theorems 1.1–1.2 with $\bar{\theta}=0$ hold for the initial boundary value problem (1.1)–(1.3) with the boundary conditions $v|_{x=0,1}=\theta|_{x=0,1}=0$. □

Theorem 1.3. *Assume that* $u_0-\bar{u}$, v_0, $\theta_0-\bar{\theta}\in H^i(\mathbb{R})$ $(i=2,4)$ *with some positive constants* $\bar{u},\bar{\theta}$ *and* $u_0(x)>0$, $\theta_0(x)>0$ *on* $\mathbb{R}$ *and the compatibility conditions hold. Define*

$$e_0 \;:=\; \|u_0-\bar{u}\|^2_{L^\infty}+\int_{\mathbb{R}}(1+x^2)^\alpha\left[\big(u_0(x)-\bar{u}\big)^2+v_0^2(x)+\big(\theta_0(x)-\bar{\theta}\big)^2+v_0^4(x)\right]dx$$

with $\alpha>\frac{1}{2}$ *being an arbitrary but fixed constant, and*

$$E_l \;=\; \left\|\Big(\log(\rho_0/\bar{\rho}),\log(v_0),\log(\theta_0/\bar{\theta})\Big)\right\|_{H^l}\,,\qquad (l=0,1),\quad \rho_0=1/u_0,\quad \bar{\rho}=1/\bar{u}\;.$$

Then there exists a constant $\epsilon_0\in(0,1]$ *such that if* $e_0\le\epsilon_0$ *or* $E_0E_1\le\epsilon_0$, *then the* H^i*-global solution* $\big(u(t),v(t),\theta(t)\big)$ $(i=2,4)$ *obtained in Theorems 1.1–1.2 to the Cauchy problem (1.1)–(1.4) verifies*

$$0<C_1^{-1}\le\theta(x,t)\le C_1\quad \text{on}\ \ \mathbb{R}\times[0,+\infty)\;, \tag{1.24}$$

$$0<C_1^{-1}\le u(x,t)\le C_1\quad \text{on}\ \ \mathbb{R}\times[0,+\infty) \tag{1.25}$$

and for $i=2$, *estimates (1.5)–(1.8) with* $T=+\infty$ *and the following inequality hold*

$$\begin{aligned}&\|u(t)-\bar{u}\|^2_{H^2}+\|u(t)-\bar{u}\|^2_{W^{1,\infty}}+\|u_t(t)\|^2_{H^1}+\|v(t)\|^2_{H^2}\\&+\|v(t)\|^2_{W^{1,\infty}}+\|v_t(t)\|^2+\|\theta(t)-\bar{\theta}\|^2_{H^2}+\|\theta(t)-\bar{\theta}\|^2_{W^{1,\infty}}+\|\theta_t(t)\|^2\\&+\int_0^t\Big[\|u_x\|^2_{H^1}+\|u_x\|^2_{L^\infty}+\|u_t\|^2_{H^2}+\|v_x\|^2_{H^2}+\|v_x\|^2_{W^{1,\infty}}+\|v_t\|^2_{H^1}\\&+\|\theta_x\|^2_{H^2}+\|\theta_x\|^2_{W^{1,\infty}}+\|\theta_t\|^2_{H^1}\Big](\tau)\,d\tau\;\le\;C_2\;,\qquad\forall\,t>0\;,\end{aligned} \tag{1.26}$$

and for $i=4$, *estimates (1.23)–(1.25) and (1.9)–(1.15) with* $T=+\infty$ *and the following inequalities hold*

$$\begin{aligned}&\|u(t)-\bar{u}\|^2_{H^4}+\|u(t)-\bar{u}\|^2_{W^{3,\infty}}+\|u_t(t)\|^2_{H^3}+\|u_{tt}(t)\|^2_{H^1}\\&+\|v(t)\|^2_{H^4}+\|v_t(t)\|^2_{H^2}+\|v_{tt}(t)\|^2+\|v_x(t)\|^2_{W^{3,\infty}}\\&+\|\theta(t)-\bar{\theta}\|^2_{H^4}+\|\theta(t)-\bar{\theta}\|^2_{W^{3,\infty}}+\|\theta_t(t)\|^2_{H^2}+\|\theta_{tt}(t)\|^2\;\le\;C_4\,,\quad\forall\,t>0\;,\end{aligned} \tag{1.27}$$

$$\begin{aligned}&\int_0^t\Big(\|u_x\|^2_{H^3}+\|u_t\|^2_{H^4}+\|u_{tt}\|^2_{H^2}+\|u_{ttt}\|^2+\|u_x\|^2_{W^{2,\infty}}\\&+\|v_x\|^2_{H^4}+\|v_t\|^2_{H^3}+\|v_{tt}\|^2_{H^1}+\|v_x\|^2_{W^{3,\infty}}\\&+\|\theta_x\|^2_{H^4}+\|\theta_t\|^2_{H^3}+\|\theta_{tt}\|^2_{H^1}+\|\theta_x\|^2_{W^{3,\infty}}\Big)(\tau)\,d\tau\;\le\;C_4\,,\quad\forall\,t>0\;.\end{aligned} \tag{1.28}$$

Moreover, the H^i-(generalized) global solutions $(i = 1,2,4)$ are continuously dependent on initial data in the sense that

$$\Big\|\Big(u_1(t)-u_2(t),\, v_1(t)-v_2(t),\, \theta_1(t)-\theta_2(t)\Big)\Big\|_{H^i} \le \\ \le C_i \Big\|\Big(u_{01}-u_{02},\, v_{01}-v_{02},\, \theta_{01}-\theta_{02}\Big)\Big\|_{H^i}\,, \qquad i=1,2\,, \tag{1.29}$$

where $\big(u_j(t), v_j(t), \theta_j(t)\big)$ $(j=1,2)$ has the same sense as in (1.21).

Finally, for the H^2-global solution $\big(u(t), v(t), \theta(t)\big)$, as $t \to +\infty$,

$$\|u_t(t)\|_{H^1} + \|u_t(t)\|_{L^\infty} + \|v_t(t)\| + \|\theta_t(t)\| \;\to\; 0\,, \tag{1.30}$$

$$\big\|\big(u(t), v(t), \theta(t)\big) - (\bar u, 0, \bar\theta)\big\|_{W^{1,\infty}} + \big\|\big(u_x(t), v_x(t), \theta_x(t)\big)\big\|_{H^1} \;\to\; 0 \tag{1.31}$$

and for the H^4-global solution $\big(u(t), v(t), \theta(t)\big)$, as $t \to +\infty$,

$$\big\|\big(u_x(t), v_x(t), \theta_x(t)\big)\big\|_{H^3} + \|u_t(t)\|_{H^3} + \|u_t(t)\|_{W^{2,\infty}} \\ + \|v_t(t)\|_{H^2} + \|v_t(t)\|_{W^{1,\infty}} + \|\theta_t(t)\|_{H^2} + \|\theta_t(t)\|_{W^{1,\infty}} \;\to\; 0\,, \tag{1.32}$$

$$\|u_{tt}(t)\|_{H^1} + \|v_{tt}(t)\| + \|\theta_{tt}(t)\| + \big\|\big(u_x(t), v_x(t), \theta_x(t)\big)\big\|_{W^{2,\infty}} \;\to\; 0\,. \tag{1.33}$$

Corollary 1.1. *The H^4-global solution $\big(u(t), v(t), \theta(t)\big)$ obtained in Theorem 1.2 is a classical one. Moreover, under assumptions in Theorem 1.3, we have the following large-time behavior of classical solution $\big(u(t), v(t), \theta(t)\big)$: as $t \to +\infty$,*

$$\big\|\big(u_x(t), v_x(t), \theta_x(t)\big)\big\|_{C^{2+1/2}} + \|u_t(t)\|_{C^{2+1/2}} \\ + \big\|\big(v_t(t), \theta_t(t)\big)\big\|_{C^{1+1/2}} + \|u_{tt}(t)\|_{C^{1/2}} \;\to\; 0\,. \tag{1.34}$$

2 – Global Existence in $H^2(\mathbb{R})$

In this section we complete the proof of Theorem 1.1. We begin with the following lemma on the estimates in $H^1(\mathbb{R})$.

Lemma 2.1. *If the assumptions of Theorem 1.1 are valid, then (1.16)–(1.17) hold and the H^1-generalized global solution $\big(u(t), v(t), \theta(t)\big)$ to the Cauchy prob-*

lem (1.1)–(1.4) verifies (1.18)–(1.19) and for any $t \in [0,T]$,

$$\|u(t)-\bar{u}\|_{H^1}^2 + \|v(t)\|_{H^1}^2 + \|\theta(t)-\bar{\theta}\|_{H^1}^2 + \|u_t(t)\|^2 + \int_0^t \Big(\|v_x\|_{H^1}^2 + \|\theta_x\|_{H^1}^2 + \|u_x\|^2 + \|v_t\|^2 + \|\theta_t\|^2\Big)(\tau)\, d\tau \;\le\; C_1(T)\ , \tag{2.1}$$

$$\|u(t)-\bar{u}\|_{L^\infty}^2 + \|v(t)\|_{L^\infty}^2 + \|\theta(t)-\bar{\theta}\|_{L^\infty}^2 + \int_0^t \Big(\|u_t\|_{H^1}^2 + \|v_x\|_{L^\infty}^2 + \|\theta_x\|_{L^\infty}^2\Big)(\tau)\, d\tau \;\le\; C_1(T)\ . \tag{2.2}$$

Proof: Estimates (1.18)–(1.19) and (2.1) were obtained in [26, 27]. By the interpolation inequality, we infer that

$$\|u(t)-\bar{u}\|_{L^\infty} \le C\,\|u(t)-\bar{u}\|^{1/2}\,\|u_x(t)\|^{1/2} \le C\,\|u(t)-\bar{u}\|_{H^1} \tag{2.3}$$

here and hereafter $C>0$ stands for a generic absolute positive constant independent of $T>0$, any length of time.

Similarly,

$$\|v(t)\|_{L^\infty} \le C\,\|v(t)\|_{H^1}\ , \qquad \|\theta(t)-\bar{\theta}\|_{L^\infty} \le C\,\|\theta(t)-\bar{\theta}\|_{H^1}\ , \tag{2.4}$$

$$\|v_x(t)\|_{L^\infty} \le C\,\|v_x(t)\|_{H^1}\ , \qquad \|\theta_x(t)\|_{L^\infty} \le C\,\|\theta_x(t)\|_{H^1}\ . \tag{2.5}$$

By (1.1), we get

$$\|u_t(t)\|_{H^1} = \|v_x(t)\|_{H^1}\ . \tag{2.6}$$

Thus estimate (2.2) follows from (2.1) and (2.3)–(2.6). The proof is complete. ∎

Lemma 2.2. *Under the assumptions in Theorem 1.1, the following estimates hold for any* $t \in [0,T]$,

$$\|\theta_t(t)\|^2 + \|v_t(t)\|^2 + \int_0^t \Big(\|v_{xt}\|^2 + \|\theta_{xt}\|^2\Big)(\tau)\, d\tau \;\le\; C_2(T)\ , \tag{2.7}$$

$$\|v_x(t)\|_{L^\infty}^2 + \|v_{xx}(t)\|^2 + \|\theta_x(t)\|_{L^\infty}^2 + \|\theta_{xx}(t)\|^2 \;\le\; C_2(T)\ , \tag{2.8}$$

$$\|u(t)-\bar{u}\|_{H^2}^2 + \|v(t)\|_{H^2}^2 + \|\theta(t)-\bar{\theta}\|_{H^2}^2 + \|u_t(t)\|_{H^1}^2 \;\le\; C_2(T)\ . \tag{2.9}$$

Proof: Differentiating (1.2) with respect to t, then multiplying the resulting equation by v_t in $L^2(\mathbb{R})$, and using Lemma 2.1, we get

$$\begin{aligned}
&\frac{d}{dt}\|v_t(t)\|^2 + C_1^{-1}(T)\,\|v_{xt}(t)\|^2 \;\le \\
(2.10)\qquad &\le\; \frac{1}{2\,C_1(T)}\,\|v_{xt}(t)\|^2 + C_2(T)\,\Big(\|v_x(t)\|^3\,\|v_{xx}(t)\| + \|\theta_t(t)\|^2 + \|v_x(t)\|^2\Big) \\
&\le\; \frac{1}{2\,C_1(T)}\,\|v_{xt}(t)\|^2 + C_2(T)\,\Big(\|v_x(t)\|^2 + \|\theta_t(t)\|^2 + \|v_{xx}(t)\|^2\Big)
\end{aligned}$$

which, together with Lemma 2.1, yields

$$\begin{aligned}
\|v_t(t)\|^2 + \int_0^t \|v_{xt}\|^2(\tau)\,d\tau \;&\le\; C_2(T) + C_1(T)\int_0^t\Big(\|v_x\|^2 + \|\theta_t\|^2 + \|v_{xx}\|^2\Big)(\tau)\,d\tau \\
(2.11)\qquad &\le\; C_2(T)\;.
\end{aligned}$$

Hence, by (1.2), Lemma 2.1, the embedding theorem and Young's inequality, we have

$$\begin{aligned}
\|v_{xx}(t)\| \;&\le\; C_1(T)\,\Big(\|v_t(t)\| + \|v_x(t)\| + \|u_x(t)\| + \|v_x(t)\|^{1/2}\,\|v_{xx}(t)\|\Big) \\
&\le\; \frac{1}{2}\,\|v_{xx}(t)\| + C_1(T)\,\Big(\|v_t(t)\| + \|v_x(t)\| + \|u_x(t)\|\Big)
\end{aligned}$$

which, combined with (2.11) and (2.1)–(2.2), leads to

$$(2.12)\qquad \|v_{xx}(t)\| \le C_1(T)\,\Big(\|v_t(t)\| + \|v_x(t)\| + \|u_x(t)\|\Big) \le C_2(T)\;,\quad \forall\, t\in[0,T]\;,$$

$$(2.13)\quad \|v_x(t)\|^2_{L^\infty} \le C_1(T)\,\|v_x(t)\|\,\|v_{xx}(t)\| \le C_2(T)\;,\qquad \forall\, t\in[0,T]\;.$$

Similarly, by (1.3) and (2.13), we deduce

$$\begin{aligned}
(2.14)\quad &\frac{d}{dt}\|\theta_t(t)\|^2 + C_1^{-1}(T)\,\|\theta_{xt}(t)\|^2 \;\le \\
&\le\; \frac{1}{2\,C_1(T)}\,\|\theta_{xt}(t)\|^2 + C_2(T)\,\Big(\|\theta_x(t)\|^2 + \|v_x(t)\|^2 + \|\theta_t(t)\|^2 + \|v_{tx}(t)\|^2\Big)
\end{aligned}$$

which, combined with Lemma 2.1, gives

$$(2.15)\qquad \|\theta_t(t)\|^2 + \int_0^t \|\theta_{xt}\|^2(\tau)\,d\tau \;\le\; C_2(T)\;,\qquad \forall\, t\in[0,T]\;.$$

Similarly to (2.12), by equation (1.3), Lemma 2.1, (2.15) and the interpolation inequality, we obtain

$$\begin{aligned}
\|\theta_{xx}(t)\| \;&\le\; C_1(T)\,\Big(\|\theta_t(t)\| + \|\theta_x(t)\|^{1/2}\,\|\theta_{xx}(t)\|^{1/2}\,\|u_x(t)\| \\
&\qquad + \|v_x(t)\|^{3/2}\,\|v_{xx}(t)\|^{1/2} + \|v_x(t)\|\Big) \\
&\le\; C_1(T)\,\Big(\|\theta_t(t)\| + \|\theta_x(t)\| + \|v_x(t)\| + \|v_{xx}(t)\|\Big) + \frac{1}{2}\,\|\theta_{xx}(t)\|
\end{aligned}$$

whence

$$\|\theta_{xx}(t)\| \le C_1(T)\Big(\|\theta_t(t)\| + \|\theta_x(t)\| + \|v_x(t)\| + \|v_{xx}(t)\|\Big) \le C_2(T)\,, \tag{2.16}$$

$$\|\theta_x(t)\|^2_{L^\infty} \le C_1(T)\,\|\theta_x(t)\|\,\|\theta_{xx}(t)\| \le C_2(T)\,. \tag{2.17}$$

Thus estimates (2.7)–(2.9) follow from (1.1), (2.11)–(2.13) and (2.15)–(2.17) and Lemma 2.1. The proof is complete. ∎

Lemma 2.3. *Under the assumptions in Theorem 1.1, the following estimates hold for any* $t \in [0,T]$,

$$\|u_{xx}(t)\|^2 + \|u_x(t)\|^2_{L^\infty} + \int_0^t \Big(\|u_{xx}\|^2 + \|u_x\|^2_{L^\infty}\Big)(\tau)\,d\tau \le C_2(T)\,, \tag{2.18}$$

$$\int_0^t \Big(\|v_{xxx}\|^2 + \|\theta_{xxx}\|^2\Big)(\tau)\,d\tau \le C_2(T)\,. \tag{2.19}$$

Proof: Differentiating (1.2) with respect to x, and using equation (1.1), we get

$$\begin{aligned}\mu\,\frac{\partial}{\partial t}\Big(\frac{u_{xx}}{u}\Big) + \frac{R\theta\,u_{xx}}{u^2} &= \\ &= v_{tx} + \frac{R\theta_{xx}}{u} + \frac{2\mu v_{xx}u_x - 2R\theta_x u_x}{u^2} + \frac{2R\theta u_x^2 - 2\mu v_x u_x^2}{u^3}\,.\end{aligned} \tag{2.20}$$

Multiplying (2.20) by u_{xx}/u in $L^2(\mathbb{R})$, and using Lemmas 2.1–2.2, we deduce that

$$\begin{aligned}&\frac{d}{dt}\Big\|\frac{u_{xx}}{u}(t)\Big\|^2 + C_1^{-1}(T)\,\|u_{xx}(t)\|^2 \le \\ &\le \frac{1}{2C_1(T)}\,\|u_{xx}(t)\|^2 + C_2(T)\Big(\|\theta_x(t)\|^2 + \|u_x(t)\|^2 + \|v_{xx}(t)\|^2 + \|\theta_{xx}(t)\|^2 + \|v_{tx}(t)\|^2\Big)\end{aligned} \tag{2.21}$$

which, together with Lemma 2.2, implies that for any $t \in [0,T]$,

$$\|u_{xx}(t)\|^2 + \int_0^t \|u_{xx}\|^2(\tau)\,d\tau \le C_2(T)\,, \tag{2.22}$$

$$\|u_x(t)\|^2_{L^\infty} \le C\,\|u_x(t)\|\,\|u_{xx}(t)\| \le C_2(T)\,, \tag{2.23}$$

$$\int_0^t \|u_x(t)\|^2_{L^\infty}(\tau)\,d\tau \le C\int_0^t \Big(\|u_x(t)\|^2 + \|u_{xx}(t)\|^2\Big)(\tau)\,d\tau \le C_2(T)\,. \tag{2.24}$$

Differentiating (1.2) and (1.3) with respect to x respectively, using Lemmas 2.1–2.2 and (2.23), we deduce that for any $t \in [0,T]$,

$$\begin{aligned}(2.25)\quad \|v_{xxx}(t)\| \le\ & C_2(T)\Big(\|v_t(t)\| + \|v_{tx}(t)\| + \|v_{xx}(t)\| + \|u_{xx}(t)\| + \|v_x(t)\| \\ & + \|\theta_{xx}(t)\| + \|\theta_x(t)\| + \|u_x(t)\|\Big)\,,\end{aligned}$$

$$\begin{aligned}(2.26)\quad \|\theta_{xxx}(t)\| \le\ & C_2(T)\Big(\|\theta_t(t)\| + \|\theta_{tx}(t)\| + \|\theta_{xx}(t)\| + \|u_{xx}(t)\| + \|v_{xx}(t)\| \\ & + \|\theta_x(t)\|\Big)\,.\end{aligned}$$

Thus estimates (2.18)–(2.19) follow from (2.22)–(2.26) and Lemmas 2.1–2.2. The proof is complete. ∎

Lemma 2.4. *Under the assumptions in Theorem 1.1, the Cauchy problem (1.1)–(1.4) admits a unique H^2-generalized global solution $\big(u(t), v(t), \theta(t)\big)$ satisfying that for any $t \in [0,T]$,*

$$(2.27)\qquad \Big\|\Big(u(t)-\bar u,\ v(t),\ \theta(t)-\bar\theta\Big)\Big\|_{H^2} \le C_2(T)\,.$$

Moreover, H^i-generalized global solutions ($i=1,2$) are continuously dependent on initial data in the sense of (1.21).

Proof: Obviously we infer estimate (2.27) from Lemmas 2.1–2.3. Thus global existence of H^2-generalized solutions follows. Now we prove estimate (1.21). For $i=1$ in (1.21), we assume that $u_{0j}-\bar u,\ v_{0j},\ \theta_{0j}-\bar\theta \in H^1(\mathbb{R}),\ u_{0j}(x)>0$, $\theta_{0j}(x)>0$ on $\mathbb{R}$ and the compatibility conditions hold ($j=1,2$). We denote by $u=u_1-u_2,\ v=v_1-v_2,\ \theta=\theta_1-\theta_2$ and $u_0=u_{01}-u_{02},\ v_0=v_{01}-v_{02}$, $\theta_0=\theta_{01}-\theta_{02}$. Subtracting the corresponding equations (1.1)–(1.3) satisfied by (u_1,v_1,θ_1) and (u_2,v_2,θ_2), we obtain

$$(2.28)\qquad u_t = v_x\,,$$

$$(2.29)\qquad v_t = \mu\left(\frac{v_x}{u_1}-\frac{v_{2x}\,u}{u_1u_2}\right)_x + R\left(\frac{\theta_2\,u-\theta\,u_2}{u_1u_2}\right)_x\,,$$

$$(2.30)\ C_V\,\theta_t = \lambda\left[\frac{\theta_x}{u_1}-\frac{\theta_{2x}\,u}{u_1u_2}\right]_x + \frac{1}{u_1}\big[\mu v_x - R\theta\big]\,v_{1x} + \big[\mu v_{2x}-R\theta_2\big]\,\frac{u_2\,v_x - v_{2x}\,u}{u_1u_2}\,,$$

$$t=0\colon\quad u=u_0\,,\quad v=v_0\,,\quad \theta=\theta_0\,.$$

By Lemma 2.1, we know that for any $t \in [0,T]$,

$$\Big\|\Big(u_j(t)-\bar u,\, v_j(t),\, \theta_j(t)-\bar\theta\Big)\Big\|^2_{H^1} + \tag{2.31}$$
$$+\int_0^t \Big(\|u_{jx}\|^2+\|v_{jx}\|^2_{H^1}+\|\theta_{jx}\|^2_{H^1}+\|v_{jt}\|^2+\|\theta_{jt}\|^2\Big)(\tau)\,d\tau \;\le\; C_1(T)\,, \quad j=1,2$$

where $C_1(T)>0$ denotes the universal constant depending only on the H^1 norm of $(u_{0j}-\bar u,\, v_{0j},\, \theta_{0j}-\bar\theta)$ and $\min_{x\in\mathbb{R}} u_{0j}(x)$, $\min_{x\in\mathbb{R}} \theta_{0j}(x)$ $(j=1,2)$ and $T>0$.

Multiplying (2.28), (2.29) and (2.30) by u, v and θ respectively, adding the results up and integrating the results over $\mathbb{R}$, and using Lemmas 2.1–2.3 and (2.31), we deduce that for any small $\epsilon>0$,

$$\frac12\,\frac{d}{dt}\Big(\|u(t)\|^2+\|v(t)\|^2+C_V\|\theta(t)\|^2\Big)+\int_{\mathbb{R}}\frac{\mu v_x^2+\lambda\theta_x^2}{u_1}\,dx \;\le$$
$$\le\; \epsilon\Big(\|v_x(t)\|^2+\|\theta_x(t)\|^2\Big)+C_1(T)\,H_1(t)\,\Big(\|u(t)\|^2+\|v(t)\|^2+\|\theta(t)\|^2\Big)$$

where $H_1(t)=\|v_{1xx}(t)\|^2+\|v_{2xx}(t)\|^2+\|\theta_{2xx}(t)\|^2+1$ satisfies $\int_0^T H_1(\tau)\,d\tau\le C_1(T)$.

This, by taking ϵ small enough, implies

$$\frac{d}{dt}\Big(\|u(t)\|^2+\|v(t)\|^2+C_V\|\theta(t)\|^2\Big)+C_1^{-1}(T)\,\Big(\|v_x(t)\|^2+\|\theta_x(t)\|^2\Big) \;\le \tag{2.32}$$
$$\le\; C_1(T)\,H_1(t)\,\Big(\|u(t)\|^2+\|v(t)\|^2+\|\theta(t)\|^2\Big)\,.$$

By Lemmas 2.1–2.3 and the interpolation inequality, we get

$$\begin{aligned}\|v_{xx}(t)\|^2 \;&\le\; C_1(T)\,\Big[\|v_t(t)\|^2+\|v_x(t)\|^2_{L^\infty}+\|\theta(t)\|^2_{H^1}+\|v_{2xx}(t)\|^2\,\|u(t)\|^2_{H^1}\Big]\\ &\le\; \frac12\,\|v_{xx}(t)\|^2+C_1(T)\,\Big(\|v_t(t)\|^2+\|\theta(t)\|^2_{H^1}+\|v_x(t)\|^2\Big)\\ &\quad+C_1(T)\,\|v_{2xx}(t)\|^2\,\|u(t)\|^2_{H^1}\end{aligned}$$

implying

$$\|v_{xx}(t)\|^2 \;\le\; C_1(T)\,\|v_t(t)\|^2+C_1(T)\,H_1(t)\,\Big(\|v_x(t)\|^2+\|u(t)\|^2_{H^1}+\|\theta(t)\|^2_{H^1}\Big)\,. \tag{2.33}$$

Differentiating (2.28) with respect to x, multiplying the result by u_x in $L^2(\mathbb{R})$ and using (2.35), we obtain that for any $\delta>0$,

$$\frac{d}{dt}\|u_x(t)\|^2 \;\le\; \delta\,\|v_{xx}(t)\|^2+\frac1\delta\,\|u_x(t)\|^2$$
$$\le\; C_1(T)\,\delta\,\|v_t(t)\|^2+C_1(T)\,\delta^{-1}H_1(t)\,\Big(\|v_x(t)\|^2+\|u(t)\|^2+\|\theta(t)\|^2\Big)\,. \tag{2.34}$$

Multiplying (2.29) by v_t in $L^2(\mathbb{R})$, and using Lemmas 2.1–2.3 and (2.32), we obtain

$$\frac{d}{dt}\left\|\frac{v_x}{\sqrt{u_1}}(t)\right\|^2 + C_1^{-1}(T)\,\|v_t(t)\|^2 \;\le\; \le\; C_1(T)\,H_1(t)\Big(\|v_x(t)\|^2 + \|u(t)\|^2_{H^1} + \|\theta(t)\|^2_{H^1}\Big)\,. \tag{2.35}$$

Similarly, multiplying (2.30) by θ_t in $L^2(\mathbb{R})$, we obtain

$$\frac{d}{dt}\left\|\frac{\theta_x}{\sqrt{u_1}}(t)\right\|^2 + C_1^{-1}(T)\,\|\theta_t(t)\|^2 \;\le\; \le\; C_1(T)\,H_1(t)\Big(\|v_x(t)\|^2 + \|u(t)\|^2_{H^1} + \|\theta(t)\|^2_{H^1}\Big)\,. \tag{2.36}$$

Adding up (2.32) and (2.34)–(2.36), and then taking δ small enough, we finally conclude

$$\begin{aligned}\frac{d}{dt}G_1(t) \;&\le\; C_1(T)\,H_1(t)\Big(\|v_x(t)\|^2 + \|u(t)\|^2_{H^1} + \|\theta(t)\|^2_{H^1}\Big)\\ &\le\; C_1(T)\,H_1(t)\,G_1(t)\end{aligned} \tag{2.37}$$

where

$$G_1(t) \;=\; \|u(t)\|^2 + \|u_x(t)\|^2 + \|v(t)\|^2 + \left\|\frac{v_x}{\sqrt{u_1}}(t)\right\|^2 + C_V\,\|\theta(t)\|^2 + \left\|\frac{\theta_x}{\sqrt{u_1}}(t)\right\|^2$$

satisfies

$$C_1^{-1}(T)\Big(\|u(t)\|^2_{H^1} + \|v(t)\|^2_{H^1} + \|\theta(t)\|^2_{H^1}\Big) \;\le\; \le\; G_1(t) \;\le\; C_1(T)\Big(\|u(t)\|^2_{H^1} + \|v(t)\|^2_{H^1} + \|\theta(t)\|^2_{H^1}\Big)\,. \tag{2.38}$$

Thus Gronwall's inequality and (2.37)–(2.38) yield that for any $t \in [0,T]$,

$$\begin{aligned}\|u(t)\|^2_{H^1} + \|v(t)\|^2_{H^1} + \|\theta(t)\|^2_{H^1} \;&\le\; C_1(T)\,G_1(0)\,\exp\!\left(C_1(T)\int_0^T H_1(\tau)\,d\tau\right)\\ &\le\; C_1(T)\Big(\|u_0\|^2_{H^1} + \|v_0\|^2_{H^1} + \|\theta_0\|^2_{H^1}\Big)\end{aligned}$$

which is estimate (1.21) with $i = 1$.

For $i = 2$ in (1.21), we further assume that $u_{0j} - \bar{u}$, v_{0j}, $\theta_{0j} - \bar{\theta} \in H^2(\mathbb{R})$ with $u_{0j}(x) > 0$, $\theta_{0j}(x) > 0$ on $\mathbb{R}$, $(j = 1, 2)$.

Similarly to (2.33), by Lemmas 2.1–2.3, we have

$$\|\theta_{xx}(t)\|^2 \;\le\; C_1(T)\Big(\|\theta_t(t)\|^2 + H_1(t)\,G_1(t)\Big) \;\le\; C_2(T)\Big(\|\theta_t(t)\|^2 + G_1(t)\Big) \tag{2.39}$$

where $C_2(T) > 0$ denotes the universal constant depending only on the H^2 norm of $(u_{0j} - \bar{u},\ v_{0j},\ \theta_{0j} - \bar{\theta})$ and $\min_{x\in\mathbb{R}} u_{0j}(x)$, $\min_{x\in\mathbb{R}} \theta_{0j}(x)$ $(j=1,2)$ and T.

Differentiating (2.29) with respect to x, we see that

$$\frac{\mu v_{xxx}}{u_1} + \frac{R\theta_2 u_{xx}}{u_1 u_2} = v_{tx} + \frac{2\mu v_{xx} u_{1x}}{u_1^2} + \mathcal{R}(x,t) \tag{2.40}$$

where

$$\begin{aligned}\mathcal{R}(x,t) = {} & \frac{\mu(uv_{2x})_{xx} - R(2\theta_{2x}u_x + \theta_{2xx}u) + R(u_2\theta)_{xx}}{u_1u_2} \\ & + 2\Big[\mu(uv_{2x})_x - R(\theta_2 u - u_2\theta)_x\Big]\left(\frac{1}{u_1u_2}\right)_x - \mu v_x\left(\frac{1}{u_1}\right)_x \\ & + \Big[\mu u v_{2x} - R(\theta_2 u - u_2\theta)\Big]\left(\frac{1}{u_1u_2}\right)_{xx}.\end{aligned}$$

By Lemmas 2.1–2.3 and the embedding theorem, we easily obtain

$$\|\mathcal{R}(t)\|^2 \le C_2(T)\left(1 + \|v_{2xxx}(t)\|^2\right)\left(\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + \|v_{xx}(t)\|^2\right). \tag{2.41}$$

On the other hand, we conclude from (2.40)–(2.41) and the interpolation inequality that

$$\begin{aligned}\|v_{xxx}(t)\|^2 &\le C_1(T)\,\|v_{tx}(t)\|^2 + C_2(T)\left(\|u_{xx}(t)\|^2 + \|v_{xx}(t)\|_{L^\infty}^2 + \|\mathcal{R}(t)\|^2\right) \\ &\le \frac{1}{2}\,\|v_{xxx}(t)\|^2 + C_1(T)\,\|v_{tx}(t)\|^2 \\ &\quad + C_2(T)\left(1 + \|v_{2xxx}(t)\|^2\right)\left(\|v_{xx}(t)\|^2 + \|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2\right)\end{aligned}$$

whence

$$\begin{aligned}\|v_{xxx}(t)\|^2 \le {} & C_1(T)\,\|v_{tx}(t)\|^2 \\ & + C_2(T)\left(1 + \|v_{2xxx}(t)\|^2\right)\left(\|v_{xx}(t)\|^2 + \|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2\right).\end{aligned} \tag{2.42}$$

Using (2.28), (2.40) and Lemmas 2.1–2.3, noting that $u_{txx} = v_{xxx}$, $v_{xxx}/u_1 = (u_{xx}/u_1)_t + u_{xx}v_{1x}/u_1^2$, multiplying (2.40) by u_{xx}/u_1 in $L^2(\mathbb{R})$, we see that

$$\frac{d}{dt}\left\|\frac{u_{xx}}{u_1}(t)\right\|^2 + C_1^{-1}(T)\,\|u_{xx}(t)\|^2 \le \tag{2.43}$$

$$\begin{aligned}&\le C_1(T)\,\|v_{tx}(t)\|^2 + C_2(T)\,H_2(t)\left(\|u(t)\|_{H^2}^2 + \|v_{xx}(t)\|^2 + \|\theta(t)\|_{H^2}^2\right) \\ &\le C_1(T)\,\|v_{tx}(t)\|^2 + C_2(T)\,H_2(t)\left(\|u(t)\|_{H^2}^2 + \|v(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + \|v_t(t)\|^2\right)\end{aligned}$$

where $H_2(t) = 1 + \|v_{2xxx}(t)\|^2 + \|v_{2tx}(t)\|^2$ satisfies $\int_0^T H_2(s)\,ds \le C_2(T)$.

Similarly, differentiating (2.29) and (2.30) with respect to t, multiplying the results by v_t and θ_t in $L^2(\mathbb{R})$ respectively, and using Lemmas 2.1–2.3, we finally deduce that

$$\frac{d}{dt}\|v_t(t)\|^2 + C_1^{-1}(T)\,\|v_{tx}(t)\|^2 \;\le\; \le\; C_2(T)\,H_2(t)\Big(\|v_t(t)\|^2 + \|\theta_t(t)\|^2 + \|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 + \|\theta(t)\|_{H^1}^2\Big)\,, \tag{2.44}$$

$$\frac{d}{dt}\|\theta_t(t)\|^2 + C_1^{-1}(T)\,\|\theta_{tx}(t)\|^2 \;\le\; \le\; C_1(T)\,\|v_{tx}(t)\|^2 + C_2(T)\,H_2(t)\Big(\|v_t(t)\|^2 + \|\theta_t(t)\|^2 + \|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 + \|\theta(t)\|_{H^1}^2\Big)\,. \tag{2.45}$$

Now multiplying (2.44) by $2C_1^2(T)$, then adding up the result to (2.43) and (2.45), we arrive at

$$\begin{aligned}\frac{d}{dt}G_2(t) &\le C_2(T)\,H_2(t)\Big(\|v_t(t)\|^2 + \|\theta_t(t)\|^2 + \|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2\Big)\\ &\le C_2(T)\,H_2(t)\Big(G_1(t) + G_2(t)\Big)\end{aligned} \tag{2.46}$$

where $G_2(t) = \big\|\frac{u_{xx}}{u_1}(t)\big\|^2 + 2C_1^2(T)\,\|v_t(t)\|^2 + \|\theta_t(t)\|^2$.

Thus adding (2.46) to (2.37) gives

$$\frac{d}{dt}\hat{G}(t) \;\le\; C_2(T)\,H_2(t)\,\hat{G}(t) \tag{2.47}$$

where $\hat{G}(t) = G_1(t) + G_2(t)$.

Similarly to (2.33) and (2.39), we infer from (2.30)–(2.31)

$$\|v_t(t)\|^2 + \|\theta_t(t)\|^2 \;\le\; C_2(T)\Big(\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2\Big)$$

which with (2.39) implies

$$\hat{G}(t) \;\le\; C_2(T)\Big(\|u(t)\|_{H^2}^2 + \|v(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2\Big)\,. \tag{2.48}$$

On the other hand, we deduce from (2.33) and (2.38)–(2.39) that

$$\begin{aligned}\hat{G}(t) &\ge C_1^{-1}(T)\Big(\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 + \|\theta(t)\|_{H^1}^2\Big)\\ &\quad + C_2^{-1}(T)\Big(\|u_{xx}(t)\|^2 + \|v_t(t)\|^2 + \|\theta_t(t)\|^2\Big)\\ &\ge C_1^{-1}(T)\Big(\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 + \|\theta(t)\|_{H^1}^2\Big)\\ &\quad + C_2^{-1}(T)\Big[\|u_{xx}(t)\|^2 + \Big(\|v_t(t)\|^2 + G_1(t)\Big) + \Big(\|\theta_t(t)\|^2 + G_1(t)\Big)\Big]\\ &\ge C_2^{-1}(T)\Big(\|u(t)\|_{H^2}^2 + \|v(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2\Big)\end{aligned}$$

which together with (2.48) gives

$$C_2^{-1}(T)\Big(\|u(t)\|^2_{H^2}+\|v(t)\|^2_{H^2}+\|\theta(t)\|^2_{H^2}\Big)\ \le\ \le\ \hat{G}(t)\ \le\ C_2(T)\Big(\|u(t)\|^2_{H^2}+\|v(t)\|^2_{H^2}+\|\theta(t)\|^2_{H^2}\Big)\ . \tag{2.49}$$

Thus it follows from Gronwall's inequality, (2.38) and (2.49) that

$$\begin{aligned}\|u(t)\|^2_{H^2}+\|v(t)\|^2_{H^2}+\|\theta(t)\|^2_{H^2} &\le C_2(T)\,\hat{G}(t)\\ &\le C_2(T)\,\hat{G}(0)\,\exp\bigg(C_2(T)\int_0^T H_2(\tau)\,d\tau\bigg)\\ &\le C_2(T)\Big(\|u_0\|^2_{H^2}+\|v_0\|^2_{H^2}+\|\theta_0\|^2_{H^2}\Big),\quad \forall\, t\in[0,T]\,,\end{aligned}$$

which is estimate (1.21) with $i=2$ and implies the uniqueness of H^2-generalized global solutions. Thus the proof is complete. ∎

Till now we have completed the proof of Theorem 1.1. ∎

3 – Global Existence in $H^4(\mathbb{R})$

In this section we derive estimates in $H^4(\mathbb{R})$ and complete the proof of Theorem 1.2. The following several lemmas concern with the estimates in $H^4(\mathbb{R})$.

Lemma 3.1. *Under the assumptions of Theorem 1.2, the following estimates hold for any $t\in[0,T]$,*

$$\|v_{tx}(x,0)\|+\|\theta_{tx}(x,0)\|\ \le\ C_3(T)\ , \tag{3.1}$$

$$\|v_{tt}(x,0)\|+\|\theta_{tt}(x,0)\|+\|v_{txx}(x,0)\|+\|\theta_{txx}(x,0)\|\ \le\ C_4(T)\ , \tag{3.2}$$

$$\|v_{tt}(t)\|^2+\int_0^t\|v_{ttx}\|^2(\tau)\,d\tau\ \le\ C_4(T)+C_4(T)\int_0^t\|\theta_{txx}\|^2(\tau)\,d\tau\ , \tag{3.3}$$

$$\|\theta_{tt}(t)\|^2+\int_0^t\|\theta_{ttx}\|^2(\tau)\,d\tau\ \le\ C_4(T)+C_4(T)\int_0^t\Big(\|\theta_{txx}\|^2+\|v_{txx}\|^2\Big)(\tau)\,d\tau\ . \tag{3.4}$$

Proof: We easily infer from (1.2) and Lemmas 2.1–2.4 that

$$\|v_t(t)\|\ \le\ C_2(T)\Big(\|v_x(t)\|_{H^1}+\|u_x(t)\|+\|\theta_x(t)\|\Big)\ . \tag{3.5}$$

Differentiating (1.2) with respect to x and exploiting Lemmas 2.1–2.4, we have

$$(3.6)\qquad \|v_{tx}(t)\| \le C_2(T)\Big(\|v_x(t)\| + \|v_{xxx}(t)\| + \|\theta_x(t)\|_{H^1} + \|u_x(t)\|_{H^1}\Big)$$

or

$$(3.7)\qquad \|v_{xxx}(t)\| \le C_2(T)\Big(\|v_x(t)\| + \|u_x(t)\|_{H^1} + \|\theta_x(t)\|_{H^1} + \|v_{tx}(t)\|\Big)\ .$$

Differentiating (1.2) with respect to x twice, using Lemmas 2.1–2.4 and the embedding theorem, we have

$$(3.8)\qquad \|v_{txx}(t)\| \le C_2(T)\Big(\|u_x(t)\|_{H^2} + \|v_x(t)\|_{H^3} + \|\theta_x(t)\|_{H^2}\Big)$$

or

$$(3.9)\qquad \|v_{xxxx}(t)\| \le C_2(T)\Big(\|u_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|v_{txx}(t)\|\Big)\ .$$

In the same manner, we deduce from (1.3) that

$$(3.10)\qquad \|\theta_t(t)\| \le C_2(T)\Big(\|\theta_x(t)\|_{H^1} + \|v_x(t)\| + \|u_x(t)\|\Big)\ ,$$

$$(3.11)\qquad \|\theta_{tx}(t)\| \le C_2(T)\Big(\|\theta_x(t)\|_{H^2} + \|v_x(t)\|_{H^1} + \|u_{xx}(t)\|\Big)$$

or

$$(3.12)\qquad \|\theta_{xxx}(t)\| \le C_2(T)\Big(\|\theta_x(t)\|_{H^1} + \|v_x(t)\|_{H^1} + \|u_{xx}(t)\| + \|\theta_{tx}(t)\|\Big)$$

and

$$(3.13)\qquad \|\theta_{txx}(t)\| \le C_2(T)\Big(\|u_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^3}\Big)$$

or

$$(3.14)\qquad \|\theta_{xxxx}(t)\| \le C_2(T)\Big(\|u_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|\theta_{txx}(t)\|\Big)\ .$$

Differentiating (1.2) with respect to t, and using Lemmas 2.1–2.4 and (1.1), we deduce that

$$(3.15)\qquad \|v_{tt}(t)\| \le C_2(T)\Big(\|\theta_x(t)\| + \|u_x(t)\| + \|v_{xx}(t)\| + \|v_{tx}(t)\|_{H^1} + \|\theta_{xt}(t)\| + \|\theta_t(t)\|\Big)$$

which together with (3.6), (3.8) and (3.11) implies

$$(3.16)\qquad \|v_{tt}(t)\| \le C_2(T)\Big(\|\theta_x(t)\|_{H^2} + \|v_x(t)\|_{H^3} + \|u_x(t)\|_{H^2}\Big)\ .$$

Analogously, we derive from (1.3) and Lemmas 2.1–2.4 that

$$(3.17)\qquad \|\theta_{tt}(t)\| \le C_2(T)\Big(\|\theta_t(t)\| + \|\theta_x(t)\| + \|\theta_{tx}(t)\|_{H^1} + \|\theta_{txx}(t)\| + \|v_x(t)\| + \|v_{xt}(t)\|\Big)$$

which combined with (3.10)–(3.11), (3.13) and (3.6) gives

$$\|\theta_{tt}(t)\| \le C_2(T)\left(\|\theta_x(t)\|_{H^3} + \|v_x(t)\|_{H^2} + \|u_x(t)\|_{H^2}\right). \tag{3.18}$$

Thus estimates (3.1)–(3.2) follow from (3.6), (3.8), (3.11), (3.13), (3.16) and (3.18).

Now differentiating (1.2) with respect to t twice, multiplying the resulting equation by v_{tt} in $L^2(\mathbb{R})$, and using (1.1) and Lemmas 2.1–2.4, we deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v_{tt}(t)\|^2 &= -\int_{\mathbb{R}}\sigma_{tt}v_{ttx}\,dx - \mu\int_{\mathbb{R}}\frac{v_{ttx}^2}{u}\,dx \\ &\quad + C_2(T)\,\|v_{ttx}(t)\|\left(\|\theta_{tt}(t)\| + \|v_{tx}(t)\| + \|\theta_t(t)\| + \|v_x(t)\|\right) \\ &\le -\left(2\,C_1(T)\right)^{-1}\|v_{ttx}(t)\|^2 \\ &\quad + C_2(T)\left(\|\theta_{tt}(t)\|^2 + \|v_{tx}(t)\|^2 + \|\theta_t(t)\|^2 + \|v_x(t)\|^2\right)\end{aligned}$$

which with (3.17) implies

$$\begin{aligned}&\frac{d}{dt}\|v_{tt}(t)\|^2 + C_1^{-1}(T)\,\|v_{ttx}(t)\|^2 \le \\ &\le C_2(T)\left(\|\theta_{txx}(t)\|^2+\|\theta_{xx}(t)\|^2+\|\theta_{tx}(t)\|^2+\|v_x(t)\|_{H^1}^2+\|v_{tx}(t)\|^2+\|\theta_t(t)\|^2+\|u_x(t)\|^2\right).\end{aligned} \tag{3.19}$$

Thus estimate (3.3) follows from Lemmas 2.1–2.3, (3.2) and (3.19).

Analogously, we obtain from (1.3) that

$$\begin{aligned}\frac{C_V}{2}\frac{d}{dt}\|\theta_{tt}(t)\|^2 &\le -\lambda\int_{\mathbb{R}}\frac{\theta_{ttx}^2}{u}\,dx \\ &\quad + C_2(T)\,\|\theta_{ttx}(t)\|\left(\|\theta_{tx}(t)\| + \|v_{tx}(t)\| + \|v_x(t)\|\right) \\ &\quad + C_2(T)\,\|\theta_{tt}(t)\|\left(\|\sigma_{tt}(t)\| + \|\sigma_t(t)\|\,\|v_{tx}(t)\|_{L^\infty} + \|v_{ttx}(t)\|\right).\end{aligned} \tag{3.20}$$

By Lemmas 2.1–2.3, and the interpolation inequality, we get

$$\|\sigma_t(t)\| \le C_2(T)\left(\|v_{tx}(t)\| + \|\theta_t(t)\| + \|v_x(t)\|\right), \tag{3.21}$$

$$\|\sigma_{tt}(t)\| \le C_2(T)\left(\|v_{ttx}(t)\| + \|\theta_{tt}(t)\| + \|v_{tx}(t)\| + \|\theta_t(t)\| + \|v_x(t)\|\right) \tag{3.22}$$

and

$$\|v_{tx}(t)\|_{L^\infty}^2 \le C\,\|v_{tx}(t)\|\,\|v_{txx}(t)\|. \tag{3.23}$$

By virtue of (3.21)–(3.33), we infer from (3.20)

$$\frac{d}{dt}\|\theta_{tt}(t)\|^2 + C_1^{-1}(T)\,\|\theta_{ttx}(t)\|^2 \;\le \tag{3.24}$$
$$\le\; C_2(T)\Big(\|\theta_{tx}(t)\|^2 + \|v_x(t)\|^2 + \|v_{tx}(t)\|^2 + \|\theta_t(t)\|^2 + \|v_{ttx}(t)\|^2 + \|\theta_{tt}(t)\|^2\Big)$$
$$+\, C_2(T)\,\|\theta_{tt}(t)\|\,\Big(\|v_{tx}(t)\| + \|\theta_t(t)\| + \|v_x(t)\|)(\|v_{tx}(t)\| + \|v_{txx}\|\Big)$$

which together with (3.2)–(3.3), (3.17) and Lemmas 2.1–2.3 yields

$$\|\theta_{tt}(t)\|^2 + C_1^{-1}(T)\int_0^t \|\theta_{ttx}(t)\|^2(\tau)\,d\tau \;\le \tag{3.25}$$
$$\begin{aligned}
&\le\; C_4(T) + C_4(T)\int_0^t\Big(\|\theta_{tt}\|^2 + \|v_{ttx}\|^2\Big)(\tau)\,d\tau\\
&\quad+\, C_2(T)\left[\int_0^t\Big(\|\theta_{tt}\|^2\left(\|v_{tx}\|^2 + \|\theta_t\|^2 + \|v_x\|^2\right)\Big)(\tau)\,d\tau\right]^{1/2}\\
&\quad\cdot\left[\int_0^t\Big(\|v_{tx}\|^2 + \|v_{txx}\|^2\Big)(\tau)\,d\tau\right]^{1/2}\\
&\le\; C_4(T) + C_4(T)\int_0^t\|\theta_{txx}\|^2(\tau)\,d\tau\\
&\quad+\, C_2(T)\sup_{0\le\tau\le t}\|\theta_{tt}(\tau)\|\left[1 + \left(\int_0^t\|v_{txx}\|^2(\tau)\,d\tau\right)^{1/2}\right]\\
&\le\; \frac{1}{2}\sup_{0\le\tau\le t}\|\theta_{tt}(\tau)\|^2 + C_4(T) + C_4(T)\int_0^t\Big(\|v_{txx}\|^2 + \|\theta_{txx}\|^2\Big)(\tau)\,d\tau\;.
\end{aligned}$$

Hence taking the supremum on the right-hand side of (3.25) gives required estimate (3.4). The proof is complete. ∎

Lemma 3.2. *Under the assumptions of Theorem 1.2, the following estimates hold for any $t \in [0,T]$,*

$$\|v_{tx}(t)\|^2 + \int_0^t\|v_{txx}\|^2(\tau)\,d\tau \;\le\; C_3(T)\;, \tag{3.26}$$

$$\|\theta_{tx}(t)\|^2 + \int_0^t\|\theta_{txx}\|^2(\tau)\,d\tau \;\le\; C_3(T)\;, \tag{3.27}$$

$$\|\theta_{tt}(t)\|^2 + \|v_{tt}(t)\|^2 + \int_0^t\Big(\|v_{ttx}\|^2 + \|\theta_{ttx}\|^2\Big)(\tau)\,d\tau \;\le\; C_4(T)\;. \tag{3.28}$$

Proof: Differentiating (1.2) with respect to x and t, multiplying the resulting equation by v_{tx} in $L^2(\mathbb{R})$, and integrating by parts, we deduce that

$$\begin{aligned}(3.29)\qquad & \frac{1}{2}\frac{d}{dt}\|v_{tx}(t)\|^2 \le \\ &\le -\mu\int_{\mathbb{R}}\frac{v_{txx}^2}{u}\,dx \\ &\quad + C_2(T)\,\|v_{txx}(t)\|\Big(\|\theta_{tx}(t)\|+\|v_{tx}(t)\|+\|\theta_t(t)\|+\|v_{xx}(t)\|+\|\theta_x(t)\|+\|u_x(t)\|\Big) \\ &\le -\big(2\,C_1(T)\big)^{-1}\|v_{txx}(t)\|^2 \\ &\quad + C_2(T)\Big(\|\theta_{tx}(t)\|^2+\|v_{tx}(t)\|^2+\|\theta_t(t)\|^2+\|v_{xx}(t)\|^2+\|\theta_x(t)\|^2+\|u_x(t)\|^2\Big)\end{aligned}$$

which combined with Lemmas 2.1–2.3 and (3.2) gives estimate (3.26).

Analogously, we infer from (1.3),

$$\begin{aligned}(3.30)\qquad & \frac{C_V}{2}\frac{d}{dt}\|\theta_{tx}(t)\|^2 \le \\ &\le -\lambda\int_{\mathbb{R}}\frac{\theta_{txx}^2}{u}\,dx + C_2(T)\,\|\theta_{txx}(t)\|\Big(\|\theta_{tx}(t)\|+\|\theta_{xx}(t)\|+\|u_x(t)\|+\|v_{xx}(t)\|\Big) \\ &\le -\big(2\,C_1(T)\big)^{-1}\|\theta_{txx}(t)\|^2 + C_2(T)\Big(\|\theta_{tx}(t)\|^2+\|\theta_{xx}(t)\|^2+\|v_{xx}(t)\|^2+\|u_x(t)\|^2\Big)\end{aligned}$$

which combined with Lemmas 2.1–2.3 implies estimate (3.27). Inserting (3.26)–(3.27) into (3.3)–(3.4) yields estimate (3.28). The proof is now complete. ∎

Lemma 3.3. *Under the assumptions of Theorem 1.2, the following estimates hold for any* $t\in[0,T]$,

$$(3.31)\qquad \|u_{xxx}(t)\|_{H^1}^2+\|u_{xx}(t)\|_{W^{1,\infty}}^2+\int_0^t\Big(\|u_{xxx}\|_{H^1}^2+\|u_{xx}\|_{W^{1,\infty}}^2\Big)(\tau)\,d\tau \le C_4(T)\,,$$

$$\begin{aligned}(3.32)\qquad & \|v_{xxx}(t)\|_{H^1}^2+\|v_{xx}(t)\|_{W^{1,\infty}}^2+\|\theta_{xxx}(t)\|_{H^1}^2+\|\theta_{xx}(t)\|_{W^{1,\infty}}^2+\|u_{txxx}(t)\|^2+ \\ &+\|v_{txx}(t)\|^2+\|\theta_{txx}(t)\|^2+\int_0^t\Big(\|v_{tt}\|^2+\|\theta_{tt}\|^2+\|v_{xx}\|_{W^{2,\infty}}^2+\|\theta_{xx}\|_{W^{2,\infty}}^2 \\ &+\|\theta_{txx}\|_{H^1}^2+\|v_{txx}\|_{H^1}^2+\|\theta_{tx}\|_{W^{1,\infty}}^2+\|v_{tx}\|_{W^{1,\infty}}^2+\|u_{txxx}\|_{H^1}^2\Big)(\tau)\,d\tau \le C_4(T)\,,\end{aligned}$$

$$(3.33)\qquad \int_0^t\Big(\|v_{xxxx}\|_{H^1}^2+\|\theta_{xxxx}\|_{H^1}^2\Big)(\tau)\,d\tau \le C_4(T)\,.$$

Proof: Differentiating (2.22) with respect to x, and using (1.1), we arrive at

$$(3.34)\qquad \mu\frac{\partial}{\partial t}\Big(\frac{u_{xxx}}{u}\Big)+\frac{R\theta\,u_{xxx}}{u^2} = E_1(x,t)$$

with

$$\begin{aligned} E_1(x,t) \;&=\; \mu\left[\frac{v_{xxx}\,u_x + u_{xx}\,v_{xx}}{u^2} - \frac{2\,u_x\,u_{xx}\,v_x}{u^3}\right] - \frac{\theta_x\,u_{xx}}{u^2} \\ &\quad + \frac{2\,R\theta\,u_x\,u_{xx}}{u^3} + v_{txx} + E_x(x,t)\;, \\ E(x,t) \;&=\; \frac{R\theta_{xx}}{u} + \frac{2\,\mu\,v_{xx}\,u_x - 2\,R\theta_x\,u_x}{u^2} + \frac{2\,R\theta\,u_x^2 - 2\,\mu\,v_x\,u_x^2}{u^3}\;. \end{aligned}$$

An easy calculation with Lemmas 2.1–2.4 and Lemmas 3.1–3.2 gives

$$\|E_1(t)\| \;\le\; C_2(T)\,\Big(\|u_x(t)\|_{H^1} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|v_{tx}(t)\|_{H^1}\Big) \tag{3.35}$$

and

$$\int_0^T \|E_1\|^2(\tau)\,d\tau \;\le\; C_4(T)\;. \tag{3.36}$$

Now multiplying (3.34) by $\frac{u_{xxx}}{u}$ in $L^2(\mathbb{R})$, we obtain

$$\frac{d}{dt}\Big\|\frac{u_{xxx}}{u}(t)\Big\|^2 + C_1^{-1}\Big\|\frac{u_{xxx}}{u}(t)\Big\|^2 \;\le\; C_1(T)\,\|E_1(t)\|^2 \tag{3.37}$$

which combined with (3.36) and Lemmas 2.1–2.3 and Lemmas 3.1–3.2 yields

$$\|u_{xxx}(t)\|^2 + \int_0^t \|u_{xxx}\|^2(\tau)\,d\tau \;\le\; C_4(T)\;, \qquad \forall\, t\in[0,T]\;. \tag{3.38}$$

In view of (3.7), (3.9), (3.12), (3.14) and Lemmas 2.1–2.3 and Lemmas 3.1–3.2, we get that for any $t\in[0,T]$,

$$\|v_{xxx}(t)\|^2 + \|\theta_{xxx}(t)\|^2 + \int_0^t \Big(\|v_{xxx}\|_{H^1}^2 + \|\theta_{xxx}\|_{H^1}^2\Big)(\tau)\,d\tau \;\le\; C_4(T)\,, \tag{3.39}$$

$$\|v_{xx}(t)\|_{L^\infty}^2 + \|\theta_{xx}(t)\|_{L^\infty}^2 + \int_0^t \Big(\|v_{xx}\|_{W^{1,\infty}}^2 + \|\theta_{xx}\|_{W^{1,\infty}}^2\Big)(\tau)\,d\tau \;\le\; C_4(T)\,. \tag{3.40}$$

Differentiating (1.2) with respect to t, we infer that for any $t\in[0,T]$,

$$\begin{aligned} \|v_{txx}(t)\| \;&\le\; C_1(T)\,\|v_{tt}(t)\| \\ &\qquad + C_2\Big(\|u_x(t)\| + \|v_{xx}(t)\| + \|v_{tx}(t)\| + \|\theta_x(t)\| + \|\theta_t(t)\| + \|\theta_{tx}(t)\|\Big) \\ &\le\; C_4(T) \end{aligned} \tag{3.41}$$

which with (3.9) gives,

$$\|v_{xxxx}(t)\|^2 + \int_0^t \Big(\|v_{txx}\|^2 + \|v_{xxxx}\|^2\Big)(\tau)\,d\tau \;\le\; C_4(T)\;. \tag{3.42}$$

Similarly, we can infer from (3.13)–(3.14) and (3.39)–(3.40) that

$$\|\theta_{txx}(t)\|^2+\|\theta_{xxxx}(t)\|^2+\int_0^t\Big(\|\theta_{txx}\|^2+\|\theta_{xxxx}\|^2\Big)(\tau)\,d\tau \;\le\; C_4(T)\,, \quad \forall\, t\in[0,T]\,. \tag{3.43}$$

which combined with (3.39) and (3.42)–(3.43) implies

$$\|v_{xxx}(t)\|_{L^\infty}^2+\|\theta_{xxx}(t)\|_{L^\infty}^2+\int_0^t\Big(\|v_{xxx}\|_{L^\infty}^2+\|\theta_{xxx}\|_{L^\infty}^2\Big)(\tau)\,d\tau \;\le\; C_4(T)\,,\ \forall\, t\in[0,T]\,. \tag{3.44}$$

Differentiating (3.44) with respect to x, we see that

$$\mu\,\frac{\partial}{\partial t}\left(\frac{u_{xxxx}}{u}\right)+\frac{R\theta\,u_{xxxx}}{u^2} \;=\; E_2(x,t) \tag{3.45}$$

with

$$E_2(x,t) \;=\; \mu\left[\frac{v_{xx}\,u_{xxx}+u_x\,v_{xxxx}}{u^2}-\frac{2\,u_x\,v_x\,u_{xxx}}{u^3}\right]+\frac{2\,R\theta\,u_x\,u_{xxx}}{u^3}-\frac{R\theta_x\,u_{xxx}}{u^2} + E_{1x}(x,t)\ .$$

Using Lemmas 2.1–2.3 and Lemmas 3.1–3.2, we can deduce that

$$\|E_{xx}(t)\| \;\le\; C_4(T)\,\Big(\|\theta_x(t)\|_{H^3}+\|u_x(t)\|_{H^2}+\|v_x(t)\|_{H^3}\Big)\ , \tag{3.46}$$

$$\|E_{1x}(t)\| \;\le\; C_4(T)\,\Big(\|v_x(t)\|_{H^3}+\|u_x(t)\|_{H^2}+\|v_{tx}(t)\|_{H^2}+\|\theta_x(t)\|_{H^3}\Big)\ , \tag{3.47}$$

$$\|E_2(t)\| \;\le\; C_4(T)\,\Big(\|v_x(t)\|_{H^3}+\|u_x(t)\|_{H^2}+\|v_{tx}(t)\|_{H^2}+\|\theta_x(t)\|_{H^3}\Big)\ . \tag{3.48}$$

On the other hand, differentiating (1.2) with respect to t and x, we infer that

$$\begin{aligned}\|v_{txxx}(t)\| \;\le\; C_1(T)\,\|v_{ttx}(t)\|+C_2(T)\,\Big(&\|v_{xx}\|_{H^1}+\|\theta_x(t)\|_{H^1}+\|u_x(t)\|_{H^1}\\ &+\|\theta_{tx}(t)\|_{H^1}+\|\theta_t(t)\|+\|v_{tx}(t)\|_{H^1}\Big)\ .\end{aligned} \tag{3.49}$$

Similarly, we have

$$\begin{aligned}\|\theta_{txxx}(t)\| \;\le\; C_1(T)\,\|\theta_{ttx}(t)\|+C_2(T)\,\Big(&\|u_x(t)\|+\|v_{xx}\|_{H^1}+\|\theta_x(t)\|_{H^2}\\ &+\|\theta_{tx}(t)\|_{H^1}+\|\theta_t(t)\|+\|v_{tx}(t)\|_{H^1}\Big)\ .\end{aligned} \tag{3.50}$$

Thus it follows from Lemmas 2.1–2.3, Lemmas 3.1–3.2 and (3.49)–(3.50) that

$$\int_0^t\Big(\|v_{txxx}\|^2+\|\theta_{txxx}\|^2\Big)(\tau)\,d\tau \;\le\; C_4(T)\ , \qquad \forall\, t\in[0,T]\ . \tag{3.51}$$

By virtue of (3.38), (3.42)–(3.43), (3.48)–(3.49), Lemmas 2.1–2.3 and Lemmas 3.1–3.2, we have

$$\int_0^t \|E_2\|^2(\tau)\,d\tau \;\le\; C_4(T)\;, \qquad \forall\, t\in[0,T]\;. \tag{3.52}$$

Multiplying (3.45) by $\frac{u_{xxxx}}{u}$ in $L^2(\mathbb{R})$, we get

$$\frac{d}{dt}\Big\|\frac{u_{xxxx}}{u}(t)\Big\|^2 + C_1^{-1}\Big\|\frac{u_{xxxx}}{u}(t)\Big\|^2 \le C_1(T)\,\|E_2(t)\|^2 \tag{3.53}$$

which combined with (3.52) implies

$$\|u_{xxxx}(t)\|^2 + \int_0^t \|u_{xxxx}\|^2(\tau)\,d\tau \;\le\; C_4(T)\;, \qquad \forall\, t\in[0,T]\;. \tag{3.54}$$

Exploiting (3.15)–(3.18), Lemmas 2.1–2.3, Lemmas 3.1–3.2 and (3.38)–(3.44), we derive

$$\int_0^t \Big(\|v_{tt}\|^2 + \|\theta_{tt}\|^2\Big)(\tau)\,d\tau \;\le\; C_4(T)\;, \qquad \forall\, t\in[0,T]\;. \tag{3.55}$$

Differentiating (1.2) with respect to x three times, and using the following estimates

$$\|\sigma_x(t)\| \;\le\; C_2(T)\,\Big(\|v_{xx}(t)\| + \|\theta_x(t)\| + \|u_x(t)\|\Big)\;,$$

$$\|\sigma_{xx}(t)\| \;\le\; C_2(T)\,\Big(\|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^1} + \|u_x(t)\|_{H^1}\Big)\;,$$

$$\|\sigma_{xxx}(t)\| \;\le\; C_2(T)\,\Big(\|v_x(t)\|_{H^3} + \|\theta_x(t)\|_{H^2} + \|u_x(t)\|_{H^2}\Big)\;,$$

we deduce that

$$\|v_{xxxxx}(t)\| \;\le\; C_1(T)\,\|v_{txxx}(t)\| + C_2(T)\,\Big(\|u_x(t)\|_{H^3} + \|v_x(t)\|_{H^3} + \|\theta_x(t)\|_{H^3}\Big)\,. \tag{3.56}$$

Thus we conclude from (1.1), (3.42)–(3.43), (3.51), (3.54), (3.56), and Lemmas 2.1–2.2 and Lemmas 3.1–3.2 that

$$\int_0^t \Big(\|v_{xxxxx}\|^2 + \|u_{txxx}\|^2_{H^1}\Big)(\tau)\,d\tau \;\le\; C_4(T)\;, \qquad \forall\, t\in[0,T]\;. \tag{3.57}$$

Similarly, we can deduce that for any $t\in[0,T]$,

$$\int_0^t \|\theta_{xxxxx}\|^2(\tau)\,d\tau \;\le\; C_4(T)\;, \tag{3.58}$$

$$\int_0^t \Big(\|v_{xx}\|^2_{W^{2,\infty}} + \|\theta_{xx}\|^2_{W^{2,\infty}}\Big)(\tau)\,d\tau \;\le\; C_4(T)\;. \tag{3.59}$$

Thus exploiting (1.1), (3.38)–(3.44), (3.51), (3.54)–(3.55), (3.57)–(3.58) and the interpolation inequality, we can derive the desired estimates (3.31)–(3.33). The proof is complete. ∎

Lemma 3.4. *Under the assumptions of Theorem 1.2, the following estimates hold for any* $t \in [0,T]$,

$$\begin{aligned}(3.60)\quad &\|u(t)-\bar u\|^2_{H^4} + \|u_t(t)\|^2_{H^3} + \|u_{tt}(t)\|^2_{H^1} + \|v(t)\|^2_{H^4} + \|v_t(t)\|^2_{H^2} + \|v_{tt}(t)\|^2 + \\ &+ \|\theta(t)-\bar\theta\|^2_{H^4} + \|\theta_t(t)\|^2_{H^2} + \|\theta_{tt}(t)\|^2 + \int_0^t \Big(\|u_x\|^2_{H^3} + \|v_x\|^2_{H^4} + \|v_t\|^2_{H^3} \\ &+ \|v_{tt}\|^2_{H^1} + \|\theta_x\|^2_{H^4} + \|\theta_t\|^2_{H^3} + \|\theta_{tt}\|^2_{H^1}\Big)(\tau)\,d\tau \;\le\; C_4(T)\;,\end{aligned}$$

$$(3.61)\qquad \int_0^t \Big(\|u_t\|^2_{H^4} + \|u_{tt}\|^2_{H^2} + \|u_{ttt}\|^2\Big)(\tau)\,d\tau \;\le\; C_4(T)\;.$$

Proof: Using (1.1), Lemmas 2.1–2.3 and Lemmas 3.1–3.3, we can derive estimates (3.60)–(3.61). The proof is complete. ∎

Proof of Theorem 1.2: By Lemmas 3.1–3.4, we have proved the global existence of H^4-solution to problem (1.1)–(1.4) and the uniqueness follows from that of the H^1-global solution or the H^2-global solution. To complete the proof, we need only prove that (1.21) holds for $i=4$, which will be done in the next lemma.

Lemma 3.5. *Under the assumptions of Theorem 1.2, the* H^4*-global solution to problem (1.1)–(1.4) is continuously dependent on initial data in the sense of (1.21) for* $i=4$.

Proof: Similarly to the proof of Lemma 2.4, we have equations (2.28)–(2.30), but now we assume that $u_{0j}-\bar u$, v_{0j}, $\theta_{0j}-\bar\theta \in H^4(\mathbb{R})$, $u_{0j}(x)>0$, $\theta_{0j}(x)>0$ on $\mathbb{R}$ and the corresponding compatibility conditions hold, and u, v and θ are the same sense as in Lemma 2.4.

By Lemma 3.4, we get that for any $t \in [0,T]$,

$$\begin{aligned}(3.62)\quad &\Big\|\Big(u_j(t)-\bar u,\, v_j(t),\, \theta_j(t)-\bar\theta\Big)\Big\|^2_{H^4} + \|u_{jt}(t)\|^2_{H^3} + \|u_{jtt}(t)\|^2_{H^1} + \|v_{jt}(t)\|^2_{H^2} + \\ &+ \|v_{jtt}(t)\|^2 + \|\theta_{jt}(t)\|^2_{H^2} + \|\theta_{jtt}(t)\|^2 + \int_0^t \Big(\|u_{jx}\|^2_{H^3} + \|v_{jx}\|^2_{H^4} + \|\theta_{jx}\|^2_{H^4} + \|v_{jt}\|^2_{H^3} \\ &+ \|v_{jtt}\|^2_{H^1} + \|\theta_{jt}\|^2_{H^3} + \|\theta_{jtt}\|^2_{H^1} + \|u_{jt}\|^2_{H^4} + \|u_{jtt}\|^2_{H^2} + \|u_{jttt}\|^2\Big)(\tau)\,d\tau \;\le\; C_4(T)\,.\end{aligned}$$

Inserting the relation $v_{xxx}/u_1 = (u_{xx}/u_1)_t - v_{1x}u_{xx}/u_1^2$ into (2.40), we arrive at

$$\mu\left(\frac{u_{xx}}{u_1}\right)_t + \frac{R\theta_2 u_{xx}}{u_1u_2} = \mathcal{R}_1 \tag{3.63}$$

where

$$\mathcal{R}_1(x,t) = (2\mu v_{xx}u_{1x} + \mu v_{1x}u_{xx})/u_1^2 + \mathcal{R}(x,t) + v_{tx} .$$

Differentiating (3.63) with respect to x, we arrive at

$$\mu\left(\frac{u_{xxx}}{u_1}\right)_t + \frac{R\theta_2 u_{xxx}}{u_1u_2} = \mathcal{R}_2 \tag{3.64}$$

with

$$\mathcal{R}_2(x,t) = \mathcal{R}_{1x} + \mu\left(\frac{u_{1x}u_{xx}}{u_1^2}\right)_t + \frac{R\theta_2(u_1u_2)_x u_{xx}}{u_1^2u_2^2} - \frac{R\theta_{2x}u_{xx}}{u_1u_2} .$$

By virtue of Lemmas 2.1–2.2 and Lemmas 3.1–3.4, we can infer that

$$\|\mathcal{R}(t)\|^2 \le C_4(T)\left(\|u(t)\|_{H^2}^2 + \|v(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2\right) , \tag{3.65}$$

$$\|\mathcal{R}_x(t)\|^2 \le C_4(T)\left(\|u(t)\|_{H^3}^2 + \|\theta(t)\|_{H^3}^2 + \|v(t)\|_{H^3}^2\right) , \tag{3.66}$$

$$\|\mathcal{R}_{xx}(t)\|^2 \le C_4(T)\left(\|u(t)\|_{H^4}^2 + \|v(t)\|_{H^4}^2 + \|\theta(t)\|_{H^4}^2\right) , \tag{3.67}$$

$$\|\mathcal{R}_{1x}(t)\|^2 \le C_1(T)\,\|v_{txx}(t)\|^2 + C_4(T)\left(\|u(t)\|_{H^3}^2 + \|\theta(t)\|_{H^3}^2 + \|v(t)\|_{H^3}^2\right) \tag{3.68}$$

and

$$\begin{aligned}\|\mathcal{R}_{1xx}(t)\|^2 \le\ & C_4(T)\left(\|v_{txx}(t)\|^2 + \|v_{txxx}(t)\|^2\right)\\ & + C_4(T)\left(1 + \|v_{2xxxxx}(t)\|^2\right)\left(\|u(t)\|_{H^4}^2 + \|\theta(t)\|_{H^4}^2 + \|v(t)\|_{H^4}^2\right) .\end{aligned} \tag{3.69}$$

Hence, with the help of (3.65)–(3.69), we derive that

$$\begin{aligned}\|\mathcal{R}_2(t)\|^2 \le\ & C_1(T)\,\|v_{txx}(t)\|^2\\ & + C_4(T)\left(\|u(t)\|_{H^3}^2 + \|v(t)\|_{H^3}^2 + \|\theta(t)\|_{H^3}^2 + \|v_{tx}(t)\|^2\right)\end{aligned} \tag{3.70}$$

and

$$\begin{aligned}\|\mathcal{R}_{2x}(t)\|^2 \le\ & C_4(T)\left(\|v_{txx}(t)\|^2 + \|v_{txxx}(t)\|^2\right)\\ & + C_4(T)\left(1 + \|v_{2x}(t)\|_{H^4}^2\right)\left(\|u(t)\|_{H^4}^2 + \|v(t)\|_{H^4}^2 + \|\theta(t)\|_{H^4}^2\right) .\end{aligned} \tag{3.71}$$

Differentiating (3.64) with respect to x, we find that

$$\mu\left(\frac{u_{xxxx}}{u_1}\right)_t + \frac{R\theta_2 u_{xxxx}}{u_1u_2} = \mathcal{R}_3(x,t) \tag{3.72}$$

where

$$\mathcal{R}_3(x,t) = \mu\left(\frac{u_{1x}u_{xxx}}{u_1^2}\right)_t - \frac{R\theta_{2x}u_{xxx}}{u_1u_2} + \frac{R(u_1u_2)_x\theta_2 u_{xxx}}{u_1^2u_2^2} + \mathcal{R}_{2x}(x,t)\ .$$

Multiplying (3.64) and (3.72) by $\frac{u_{xxx}}{u_1}$ and $\frac{u_{xxxx}}{u_1}$ in $L^2(\mathbb{R})$ respectively, we have

$$\frac{d}{dt}\left\|\frac{u_{xxx}}{u_1}(t)\right\|^2 + C_1^{-1}(T)\left\|\frac{u_{xxx}}{u_1}(t)\right\|^2 \le C_1(T)\,\|\mathcal{R}_2(t)\|^2\ , \tag{3.73}$$

$$\frac{d}{dt}\left\|\frac{u_{xxxx}}{u_1}(t)\right\|^2 + C_1^{-1}(T)\left\|\frac{u_{xxxx}}{u_1}(t)\right\|^2 \le C_1(T)\,\|\mathcal{R}_3(t)\|^2\ . \tag{3.74}$$

Differentiating (2.29) with respect to t and x, we can derive

$$\begin{aligned}\|v_{txxx}(t)\| \le\ & C_4(T)\Big(\|v_{ttx}(t)\| + \|v_{txx}(t)\|\Big) + C_4(T)\Big(1 + \|v_{2t}(t)\|_{H^3}\Big)\\ & \times\Big(\|u(t)\|_{H^2} + \|v(t)\|_{H^2} + \|\theta(t)\|_{H^2} + \|\theta_t(t)\| + \|v_{tx}(t)\|\Big)\end{aligned}$$

which with (3.71) gives

$$\begin{aligned}\|\mathcal{R}_3(t)\|^2 \le\ & C_4(T)\Big(\|v_{ttx}(t)\|^2 + \|v_{txx}(t)\|^2\Big) + C_4(T)\Big(1 + \|v_{2x}(t)\|_{H^4}^2 + \|v_{2t}(t)\|_{H^3}^2\Big)\\ & \times\Big(\|u(t)\|_{H^4}^2 + \|v(t)\|_{H^4}^2 + \|\theta(t)\|_{H^4}^2 + \|\theta_t(t)\|^2 + \|v_t(t)\|^2 + \|v_{tx}(t)\|^2\Big)\ .\end{aligned} \tag{3.75}$$

On the other hand, we deduce from (2.30) that

$$\|\theta_t(t)\| \le C_4(T)\Big(\|\theta(t)\|_{H^2} + \|u(t)\|_{H^1} + \|v(t)\|_{H^1}\Big)\ , \tag{3.76}$$

or

$$\|\theta_{xx}(t)\| \le C_4(T)\Big(\|\theta_t(t)\| + \|\theta(t)\|_{H^1} + \|u(t)\|_{H^1} + \|v(t)\|_{H^1}\Big) \tag{3.77}$$

and

$$\|\theta_{tx}(t)\| \le C_4(T)\Big(\|\theta(t)\|_{H^3} + \|u(t)\|_{H^2} + \|v(t)\|_{H^2}\Big) \tag{3.78}$$

or

$$\begin{aligned}(3.79)\qquad \|\theta_{xxx}(t)\| &\le C_4(T)\Big(\|\theta_{tx}(t)\| + \|u(t)\|_{H^2} + \|v(t)\|_{H^1} + \|v_t(t)\| \\ &\quad + \|\theta(t)\|_{H^1} + \|\theta_t(t)\|\Big)\,,\end{aligned}$$

$$\begin{aligned}(3.80)\qquad \|\theta_{xxxx}(t)\| &\le C_4(T)\Big(\|u(t)\|_{H^3} + \|v(t)\|_{H^1} + \|\theta(t)\|_{H^1} + \|v_t(t)\| \\ &\quad + \|\theta_t(t)\| + \|v_{tx}(t)\| + \|\theta_{tx}(t)\| + \|\theta_{tt}(t)\|\Big)\,,\end{aligned}$$

$$(3.81)\qquad \|\theta_{tt}(t)\| \le C_4(T)\Big(\|u(t)\|_{H^3} + \|v(t)\|_{H^3} + \|\theta(t)\|_{H^4}\Big)\,,$$

$$\begin{aligned}(3.82)\qquad \|\theta_{txx}(t)\| &\le C_1(T)\,\|\theta_{tt}(t)\| + C_4(T)\Big(\|u(t)\|_{H^1} + \|v(t)\|_{H^1} + \|\theta(t)\|_{H^1} \\ &\quad + \|v_t(t)\| + \|v_{tx}(t)\| + \|\theta_t(t)\| + \|\theta_{tx}(t)\|\Big) \\ &\le C_4(T)\Big(\|u(t)\|_{H^3} + \|v(t)\|_{H^3} + \|\theta(t)\|_{H^4}\Big)\,.\end{aligned}$$

In the same manner, we infer from (2.29) that

$$(3.83)\qquad \|v_t(t)\| \le C_4(T)\Big(\|u(t)\|_{H^1} + \|v(t)\|_{H^2} + \|\theta(t)\|_{H^1}\Big)\,,$$

$$(3.84)\qquad \|v_{xx}(t)\| \le C_4(T)\Big(\|u(t)\|_{H^1} + \|v(t)\|_{H^1} + \|v_t(t)\| + \|\theta(t)\|_{H^1}\Big)\,,$$

$$(3.85)\qquad \|v_{tx}(t)\| \le C_4(T)\Big(\|\theta(t)\|_{H^2} + \|u(t)\|_{H^2} + \|v(t)\|_{H^3}\Big)\,,$$

$$\begin{aligned}(3.86)\qquad \|v_{xxx}(t)\| &\le C_4(T)\Big(\|u(t)\|_{H^2} + \|v(t)\|_{H^1} + \|\theta(t)\|_{H^1} + \|v_t(t)\| \\ &\quad + \|\theta_t(t)\| + \|v_{tx}(t)\|\Big)\,,\end{aligned}$$

$$\begin{aligned}(3.87)\qquad \|v_{xxxx}(t)\| &\le C_4(T)\Big(\|u(t)\|_{H^3} + \|v(t)\|_{H^1} + \|\theta(t)\|_{H^1} + \|v_t(t)\| \\ &\quad + \|v_{tx}(t)\| + \|v_{tt}(t)\| + \|\theta_t(t)\| + \|\theta_{tx}(t)\|\Big)\,,\end{aligned}$$

$$(3.88)\qquad \|v_{tt}(t)\| \le C_4(T)\Big(\|u(t)\|_{H^3} + \|v(t)\|_{H^4} + \|\theta(t)\|_{H^3}\Big)\,,$$

$$(3.89)\qquad \|v_{txx}(t)\| \le C_4(T)\Big(\|u(t)\|_{H^3} + \|v(t)\|_{H^4} + \|\theta(t)\|_{H^3}\Big)\,.$$

Differentiating (2.29) with respect to t twice, multiplying the resulting equations by v_{tt} in $L^2(\mathbb{R})$, using Lemmas 2.1–2.4, Lemmas 3.1–3.4, and (3.76)–(3.89), we deduce that

$$\begin{aligned}(3.90)\qquad &\frac{1}{2}\frac{d}{dt}\|v_{tt}(t)\|^2 + C_1^{-1}(T)\,\|v_{ttx}(t)\|^2 \le \\ &\le C_4(T)\Big(1 + \|v_{2ttx}\|^2\Big)\Big(\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 + \|\theta(t)\|_{H^1}^2 \\ &\quad + \|v_t(t)\|^2 + \|v_{tx}(t)\|^2 + \|v_{tt}\|^2 + \|\theta_t(t)\|^2 + \|\theta_{tx}(t)\|^2 + \|\theta_{tt}(t)\|^2\Big)\,.\end{aligned}$$

Analogously, we can derive from (2.29)–(2.30) that for any $\delta > 0$,

$$\begin{aligned}(3.91)\qquad & \frac{C_V}{2}\frac{d}{dt}\|\theta_{tt}(t)\|^2 + C_1^{-1}(T)\,\|\theta_{ttx}(t)\|^2 \ \le \\ &\le\ \delta\,\|v_{ttx}(t)\|^2 + C_4(T,\delta)\left(1+\|v_{1ttx}(t)\|^2+\|v_{2ttx}(t)\|^2+\|\theta_{2ttx}(t)\|^2\right)\\ &\quad\times\Big(\|u(t)\|_{H^1}^2+\|v(t)\|_{H^1}^2+\|\theta(t)\|_{H^1}^2+\|v_{tx}(t)\|^2+\|v_{tt}(t)\|^2\\ &\quad+\|\theta_t(t)\|^2+\|\theta_{tx}(t)\|^2+\|\theta_{tt}(t)\|^2\Big)\ ,\end{aligned}$$

$$\begin{aligned}(3.92)\qquad & \frac{1}{2}\frac{d}{dt}\|v_{tx}(t)\|^2 + C_1^{-1}(T)\,\|v_{txx}(t)\|^2 \ \le \\ &\le\ \delta\left(\|v_{txx}(t)\|^2+\|v_{ttx}(t)\|^2+\|\theta_{txx}(t)\|^2\right)+C_4(T,\delta)\left(1+\|v_{2txxx}(t)\|^2\right)\\ &\times\left(\|u(t)\|_{H^2}^2+\|v(t)\|_{H^1}^2+\|\theta(t)\|_{H^1}^2+\|v_t(t)\|^2+\|v_{tx}(t)\|^2+\|\theta_t(t)\|^2+\|\theta_{tx}(t)\|^2\right),\end{aligned}$$

$$\begin{aligned}(3.93)\qquad & \frac{C_V}{2}\frac{d}{dt}\|\theta_{tx}(t)\|^2 + C_1^{-1}(T)\,\|\theta_{txx}(t)\|^2 \ \le \\ &\le\ \delta\left(\|\theta_{txx}(t)\|^2+\|\theta_{ttx}(t)\|^2\right)+C_4(T,\delta)\left(1+\|\theta_{2txxx}(t)\|^2\right)\\ &\quad\times\Big(\|u(t)\|_{H^2}^2+\|v(t)\|_{H^1}^2+\|\theta(t)\|_{H^1}^2+\|v_t(t)\|^2+\|v_{tx}(t)\|^2\\ &\quad+\|\theta_t(t)\|^2+\|\theta_{tx}(t)\|^2+\|v_{tt}(t)\|^2+\|\theta_{tt}(t)\|^2\Big)\ .\end{aligned}$$

Let

$$\begin{aligned}G_3(t)\ &=\ \frac{1}{2}\left(\|v_{tt}(t)\|^2+\|v_{tx}(t)\|^2\right)+\frac{C_V}{2}\left(\|\theta_{tt}(t)\|^2+\|\theta_{tx}(t)\|^2\right)\\ &\quad+\delta\left(\left\|\frac{u_{xxx}}{u_1}(t)\right\|^2+\left\|\frac{u_{xxxx}}{u_1}(t)\right\|^2\right)\ .\end{aligned}$$

Now multiplying (3.73)–(3.74) by δ respectively, adding up the resulting equations and (3.90)–(3.93), and picking $\delta > 0$ small enough, we get

$$\begin{aligned}(3.94)\qquad \frac{d}{dt}G_3(t)+C_4^{-1}(T)\Big(&\|v_{ttx}(t)\|^2+\|v_{txx}(t)\|^2+\|\theta_{ttx}(t)\|^2+\|\theta_{txx}(t)\|^2\\ &+\|u_{xxx}(t)\|^2+\|u_{xxxx}(t)\|^2\Big)\ \le\ C_4(T)\,H_3(t)\,M(t)\end{aligned}$$

where

$$\begin{aligned}M(t)\ &=\ \|u(t)\|_{H^4}^2+\|v(t)\|_{H^4}^2+\|\theta(t)|_{H^4}^2+\|v_t(t)\|^2+\|\theta_t(t)\|^2\\ &\quad+\|v_{tx}(t)\|^2+\|\theta_{tx}(t)\|^2+\|v_{tt}(t)\|^2+\|\theta_{tt}(t)\|^2\end{aligned}$$

and

$$H_3(t) \;=\; 1 + \|v_{1ttx}(t)\|^2 + \|v_{2ttx}(t)\|^2 + \|\theta_{2ttx}(t)\|^2 \\ + \|\theta_{2t}(t)\|^2_{H^3} + \|v_{2t}(t)\|^2_{H^3} + \|v_{2x}(t)\|^2_{H^4}$$

verifies, by Lemmas 2.1–2.3 and Lemmas 3.1–3.4,

$$\int_0^t H_3(\tau)\,d\tau \;\le\; C_4(T)\,(1{+}t) \;\le\; C_4(T)\;, \qquad \forall\, t\in[0,T]\;. \tag{3.95}$$

Obviously, it follows from (3.76), (3.78), (3.81), (3.83), (3.85), (3.88) and the definition of $M(t)$ that

$$\|u(t)\|^2_{H^4} + \|v(t)\|^2_{H^4} + \|\theta(t)\|^2_{H^4} \;\le\; M(t) \\ \le C_4(T)\left(\|u(t)\|^2_{H^4} + \|v(t)\|^2_{H^4} + \|\theta(t)\|^2_{H^4}\right). \tag{3.96}$$

Let

$$G(t) \;=\; G_1(t) + G_2(t) + G_3(t) \;=\; \hat{G}(t) + G_3(t)\;.$$

Then we can infer from (3.77), (3.79)–(3.80), (3.84) and (3.86)–(3.87) that

$$M(t) \;\le\; C_4(T)\Big(\|u(t)\|^2_{H^4} + \|v(t)\|^2_{H^1} + \|\theta(t)\|^2_{H^1} + \|v_t(t)\|^2 + \|\theta_t(t)\|^2 \\ + \|v_{tx}(t)\|^2 + \|\theta_{tx}(t)\|^2 + \|v_{tt}(t)\|^2 + \|\theta_{tt}(t)\|^2\Big) \\ \le\; C_4(T)\,G(t)\;. \tag{3.97}$$

Moreover, we find from the definition of $G(t)$ that

$$G(t) \;\le\; C_4(T)\Big(\|u(t)\|^2_{H^4} + \|v(t)\|^2_{H^1} + \|\theta(t)\|^2_{H^1} + \|v_t(t)\|^2 + \|\theta_t(t)\|^2 \\ + \|v_{tx}(t)\|^2 + \|\theta_{tx}(t)\|^2 + \|v_{tt}(t)\|^2 + \|\theta_{tt}(t)\|^2\Big) \\ \le\; C_4(T)\,M(t)$$

which with (3.95)–(3.96) implies

$$C_4^{-1}(T)\left(\|u(t)\|^2_{H^4} + \|v(t)\|^2_{H^4} + \|\theta(t)\|^2_{H^4}\right) \;\le\; G(t) \;\le \\ \le\; C_4(T)\left(\|u(t)\|^2_{H^4} + \|v(t)\|^2_{H^4} + \|\theta(t)\|^2_{H^4}\right). \tag{3.98}$$

Adding (2.41) to (3.94) yields

$$\frac{d}{dt}G(t) \;\le\; C_4(T)\,H_3(t)\,G(t)\;. \tag{3.99}$$

Thus using (3.97) and Gronwall's inequality, we deduce from (3.99),

$$\begin{aligned}\|u(t)\|^2_{H^4} + \|v(t)\|^2_{H^4} + \|\theta(t)\|^2_{H^4} &\le C_4(T)\, G(t)\\ &\le C_4(T)\, G(0) \exp\Big(C_4(T)\int_0^t H_3(\tau)\, d\tau\Big)\\ &\le C_4(T)\,\Big(\|u_0\|^2_{H^4} + \|v_0\|^2_{H^4} + \|\theta_0\|^2_{H^4}\Big)\end{aligned}$$

which implies (1.21) with $i=4$. The proof is complete. ∎

Till we have finished the proof of Theorem 1.2. ∎

4 – Proof of Theorem 1.3

In this section, we finish the proof of Theorem 1.3. In order to study the large-time behavior of the H^i-global solutions ($i = 2, 4$), obviously all the estimates established in Section 2 and Section 3 will *no longer* work because those estimates depend heavily on $T > 0$, any given length of time. Thus we have to derive the uniform estimates in $H^i(\mathbb{R})$ ($i = 1, 2, 4$) in which all the constants depend *only* on $\min_{x\in\mathbb{R}} u_0(x)$, $\min_{x\in\mathbb{R}} \theta_0(x)$, the $H^i(\mathbb{R})$ ($i = 1, 2, 4$) norm of $(u_0 - \bar{u}, v_0, \theta_0 - \bar{\theta})$ (and e_0 or E_0, E_1 (see Theorem 1.3)), *but* independent of any length of time $T > 0$. Since for any unbounded domain, the Poincaré inequality will not be valid and hence, unlike the corresponding initial boundary value problems of (1.1)–(1.3) in bounded domains (see e.g. [1–3, 11–13, 21, 24, 27–28, 31–36, 39, 41–46, 50–51]), the exponential decay of solutions will not be anticipated (see e.g. [1, 4, 14, 19, 21–23, 25–26, 29–32, 39–40, 49]). Note that H^1-solutions do not possess enough regularity and summability to allow all operations performed in Sections 2 and 3. Now we first use some H^1-estimates given in [21, 23, 26, 27, 39] to establish *uniform* H^1-estimates similar to (2.1)–(2.4) in the following lemma.

Lemma 4.1. *Assume that $u_0 - \bar{u}$, v_0, $\theta_0 - \bar{\theta} \in H^1(\mathbb{R})$ with some constants $\bar{u} > 0$, $\bar{\theta} > 0$ and $u_0(x) > 0$, $\theta_0(x) > 0$ on $\mathbb{R}$, and the compatibility conditions hold. Then there exists a constant $\epsilon_0 \in (0, 1]$ such that*

- **(I)** *if $E_0 E_1 \le \epsilon_0$, then, estimates (1.16)–(1.17) with $T = +\infty$ hold and the H^1-generalized global solution $\big(u(t), v(t), \theta(t)\big)$ to the Cauchy problem (1.1)–(1.4) satisfies that for any $(x, t) \in \mathbb{R}\times[0, +\infty)$,*

$$0 < C_1^{-1} \le \theta(x,t) \le C_1\,, \tag{4.1}$$

$$0 < C_1^{-1} \le u(x,t) \le C_1 \tag{4.2}$$

and for any $t > 0$,

$$\|u(t)-\bar{u}\|_{H^1}^2 + \|v(t)\|_{H^1}^2 + \|\theta(t)-\bar{\theta}\|_{H^1}^2 + \int_0^t \Big(\|v_x\|_{H^1}^2 + \|\theta_x\|_{H^1}^2 + \|u_x\|^2 + \|v_t\|^2 + \|\theta_t\|^2\Big)(\tau)\,d\tau \;\le\; C_1\;, \tag{4.3}$$

$$\|u(t)-\bar{u}\|_{L^\infty}^2 + \|v(t)\|_{L^\infty}^2 + \|\theta(t)-\bar{\theta}\|_{L^\infty}^2 + \int_0^t \Big(\|u_t\|_{H^1}^2 + \|v_x\|_{L^\infty}^2 + \|\theta_x\|_{L^\infty}^2\Big)(\tau)\,d\tau \;\le\; C_1 \tag{4.4}$$

and as $t \to +\infty$,

$$\Big\|\Big(u(t)-\bar{u},\, v(t),\, \theta(t)-\bar{\theta}\Big)\Big\|_{L^\infty} + \Big\|\Big(u_x(t),\, v_x(t),\, \theta_x(t)\Big)\Big\| \;\to\; 0 \tag{4.5}$$

or

(II) *if $e_0 \le \epsilon_0$, then estimates (1.16)–(1.17) with $T = +\infty$ and (4.1)–(4.5) hold and the H^1-generalized global solution $\big(u(t), v(t), \theta(t)\big)$ satisfies that for any $(x,t) \in \mathbb{R}\times[0,+\infty)$,*

$$\big|u(x,t)-\bar{u}\big| + \phi(t)\,\big|\theta(x,t)-\bar{\theta}\big| \;<\; \frac{1}{3}\min(\bar{u},\bar{\theta}) \tag{4.6}$$

where $\phi(t) = \min(1,t)$.

Proof: Case I: From [39] (see e.g. Theorem 2.1) it follows that there exists a constant $\epsilon_1 \in (0,1]$ such that if $E_0 E_1 \le \epsilon_1$, then H^1-generalized global solution $\big(u(t), v(t), \theta(t)\big)$ to the Cauchy problem (1.1)–(1.4) satisfies estimates (4.1)–(4.3) and (4.5). Using the interpolation inequality: $\|f\|_{L^\infty} \le C\,\|f\|^{1/2}\,\|f_x\|^{1/2}$ for any $f \in H^1(\mathbb{R})$ where $C > 0$ is a positive constant independent of any length of time, we easily deduce (4.4) from (4.3).

Case II: We know from [21] (see e.g. Theorem 1.1 (ii) or [22]) there is a constant $\epsilon_2 \in (0,1]$ such that if $e_0 \le \epsilon_2$, then estimates (4.5)–(4.6) and

$$\|u(t)-\bar{u}\|^2 + \|v(t)\|^2 + \|\theta(t)-\bar{\theta}\|^2 + \int_0^t \Big(\|v_x\|^2 + \|\theta_x\|^2\Big)(\tau)\,d\tau \;\le\; C_1, \quad \forall t>0 \tag{4.7}$$

hold. Clearly, (4.2) is the direct result of (4.6). By (4.6) we get that for any $t \ge 1$,

$$0 \;<\; C_1^{-1} \le \theta(x,t) \le C_1\;, \qquad \forall x \in \mathbb{R}\;. \tag{4.8}$$

Moreover, we find from the proofs in [26, 27] that

$$C_1^{-1} e^{-C_1 t} \le\; \theta(x,t) \;\le\; C_1 e^{C_1 t}\;, \qquad \forall (x,t) \in \mathbb{R}\times[0,+\infty)$$

which together with (4.8) yields estimate (4.1). In view of (1.1), we can write (1.2) in the form

$$\mu\Big(\frac{u_x}{u}\Big)_t = v_t + R\Big(\frac{\theta}{u}\Big)_x . \tag{4.9}$$

Multiplying (4.9) by u_x/u in $L^2(\mathbb{R})$, using (4.1)–(4.2) and (4.7), integrating by parts, and noting that $(u_x/u)_t = (u_t/u)_x = (v_x/u)_x$, we deduce that

$$\begin{aligned}
&\frac{\mu}{2}\int_{\mathbb{R}}\Big(\frac{u_x}{u}\Big)^2 dx + R\int_0^t\int_{\mathbb{R}}\frac{\theta u_x^2}{u^3}\,dx\,d\tau \le \\
&\quad\le C_1 + \int_{\mathbb{R}} v\frac{u_x}{u}\Big|_0^t dx + \int_0^t\int_{\mathbb{R}}\frac{v_x^2}{u}\,dx\,d\tau + R\int_0^t\int_{\mathbb{R}}\frac{\theta_x u_x}{u^2}\,dx\,d\tau \\
&\quad\le C_1 + \frac{R}{2}\int_0^t\int_{\mathbb{R}}\frac{\theta u_x^2}{u^3}\,dx\,d\tau + \frac{\mu}{4}\int_{\mathbb{R}}\Big(\frac{u_x}{u}\Big)^2 dx
\end{aligned}$$

which, together with (4.1)–(4.2), gives

$$\|u_x(t)\|^2 + \int_0^t \|u_x\|^2(\tau)\,d\tau \le C_1 , \qquad \forall t>0 . \tag{4.10}$$

Multiplying (1.2) by v_{xx} in $L^2(\mathbb{R})$, using (4.1)–(4.2), (4.7), (4.10), the interpolation inequality and integrating by parts, we have

$$\begin{aligned}
\|v_x(t)\|^2 + \int_0^t \|v_{xx}\|^2(\tau)\,d\tau &\le C_1 + C_1\int_0^t \Big(\|v_x\|\,\|v_{xx}\|\,\|u_x\|^2 + \|\theta_x\|^2 + \|u_x\|^2\Big)(\tau)\,d\tau \\
&\le C_1 + \frac{1}{2}\int_0^t \|v_{xx}\|^2(\tau)\,d\tau
\end{aligned}$$

whence

$$\|v_x(t)\|^2 + \int_0^t \|v_{xx}\|^2(\tau)\,d\tau \le C_1 , \qquad \forall t>0 . \tag{4.11}$$

Analogously, from (1.3) we get

$$\begin{aligned}
\|\theta_x(t)\|^2 + \int_0^t \|\theta_{xx}\|^2(\tau)\,d\tau &\le C_1 + C_1\int_0^t \Big(\|\theta_x\|\,\|\theta_{xx}\|\,\|u_x\|^2 + \|v_x\|^3\,\|v_{xx}\| + \|v_x\|^2\Big)(\tau)\,d\tau \\
&\le C_1 + \frac{1}{2}\int_0^t \|\theta_{xx}\|^2(\tau)\,d\tau
\end{aligned}$$

implying

$$\|\theta_x(t)\|^2 + \int_0^t \|\theta_{xx}\|^2(\tau)\,d\tau \le C_1 , \qquad \forall t>0 . \tag{4.12}$$

By (1.2)–(1.3), (4.1)–(4.2), (4.7) and (4.10)–(4.12), using the interpolation inequality, we derive

$$
\begin{aligned}
\|v_t(t)\| &\le C_1\Big(\|v_{xx}(t)\| + \|v_x(t)\|^{1/2}\,\|v_{xx}(t)\|^{1/2}\,\|u_x\| + \|\theta_x(t)\| + \|u_x(t)\|\Big)\\
&\le C_1\Big(\|v_{xx}(t)\| + \|v_x(t)\| + \|u_x(t)\| + \|\theta_x(t)\|\Big)\ , \qquad (4.13)\\
\|\theta_t(t)\| &\le C_1\Big(\|\theta_{xx}(t)\| + \|\theta_x(t)\|^{1/2}\,\|\theta_{xx}(t)\|^{1/2}\,\|u_x(t)\|\\
&\quad + \|v_x(t)\|^{3/2}\,\|v_{xx}(t)\|^{1/2} + \|v_x(t)\|\Big)\\
&\le C_1\Big(\|\theta_{xx}(t)\| + \|v_x(t)\| + \|\theta_x(t)\| + \|v_{xx}(t)\|\Big)
\end{aligned}
$$

which, combined with (4.7) and (4.10)–(4.13) implies estimate (4.3). Taking $\epsilon_0 = \min[\epsilon_1, \epsilon_2]$ ends the proof. ∎

Since we have established in Lemma 4.1 *uniform* H^1-estimates similar to (2.1)–(2.4) in Lemma 2.1, we only need to repeat the same argumentations as in Lemmas 2.2–2.4 and Lemmas 3.1–3.4 to be able to reach estimates (1.24)–(1.29) in Theorem 1.3. Now all constants in these estimates will *no longer* depend on $T>0$, any length of time, i.e., $C_i(+\infty) = C_i$ $(i=1,2,4)$. In order to finish the proof of Theorem 1.3, it suffices to prove the results on the large-time behavior of the $H^i\,(i=2,4)$-global solutions in Theorem 1.3. To this end, we need the following lemma.

Lemma 4.2. *Suppose y and h are nonnegative functions on $[0,+\infty)$, y' is locally integrable, and y,h satisfy*

$$
\begin{aligned}
&\forall\, t>0: \quad y'(t) \le A_1 y^2(t) + A_2 + h(t)\ ,\\
&\forall\, T>0: \quad \int_0^T y(s)\,ds \le A_3\,, \quad \int_0^T h(s)\,ds \le A_4\ ,
\end{aligned}
$$

with A_1, A_2, A_3, A_4 being positive constants independent of t and T. Then for any $r>0$

$$
\forall\, t\ge 0: \quad y(t+r) \le \left(\frac{A_3}{r} + A_2 r + A_4\right) e^{A_1 A_2}\ .
$$

Moreover,

$$
\lim_{t\to+\infty} y(t) = 0\ .
$$

Proof: See, e.g. [52]. ∎

The next two lemmas concern the large-time behavior of H^2 and H^4 global solutions respectively.

Lemma 4.3. *Under the assumptions in Theorem 1.3 with $i=2$, if $e_0 \le \epsilon_0$ or $E_0 E_1 \le \epsilon_0$, then the H^2-generalized global solution $\big(u(t), v(t), \theta(t)\big)$ obtained in Theorem 1.1 to the Cauchy problem (1.1)–(1.4) satisfies (1.30)–(1.31).*

Proof: We start from Lemma 4.1, repeat the same reasoning as derivation of (2.10), (2.12)–(2.14), (2.16)–(2.17), (2.21) and (2.23)–(2.24) in Lemmas 2.2–2.4 and keep in mind that at this time all constants $C_i(T)$ $(i=1,2,3,4)$ in Lemmas 2.2–2.4 will not depend on $T>0$ to obtain

$$\frac{d}{dt}\|v_t(t)\|^2 + (2C_1)^{-1}\|v_{tx}(t)\|^2 \le C_2\Big(\|v_x(t)\|^2 + \|v_{xx}(t)\|^2 + \|\theta_t(t)\|^2\Big) , \tag{4.14}$$

$$\frac{d}{dt}\|\theta_t(t)\|^2 + (2C_1)^{-1}\|\theta_{tx}(t)\|^2 \le C_2\Big(\|v_x(t)\|^2 + \|\theta_x(t)\|^2 + \|\theta_t(t)\|^2 + \|v_{tx}(t)\|^2\Big) , \tag{4.15}$$

$$\frac{d}{dt}\Big\|\frac{u_{xx}}{u}(t)\Big\|^2 + (2C_1)^{-1}\|u_{xx}(t)\|^2 \le C_2\Big(\|\theta_x(t)\|^2 + \|u_x(t)\|^2 + \|v_{xx}(t)\|^2 + \|\theta_{xx}(t)\|^2 + \|v_{tx}(t)\|^2\Big) , \tag{4.16}$$

$$\|v_{xx}(t)\| \le C_1\Big(\|v_t(t)\| + \|v_x(t)\| + \|u_x(t)\|\Big) \le C_2 , \tag{4.17}$$

$$\|\theta_{xx}(t)\| \le C_1\Big(\|\theta_t(t)\| + \|\theta_x(t)\| + \|v_x(t)\| + \|v_{xx}(t)\|\Big) \le C_2 , \tag{4.18}$$

$$\begin{aligned} &\|v_x(t)\|^2_{L^\infty} \le C\,\|v_x(t)\|\,\|v_{xx}(t)\| \le C_2 , \\ &\|\theta_x(t)\|^2_{L^\infty} \le C\,\|\theta_x(t)\|\,\|\theta_{xx}(t)\| \le C_2 , \end{aligned} \tag{4.19}$$

$$\|u_x(t)\|^2_{L^\infty} \le C\,\|u_x(t)\|\,\|u_{xx}(t)\| \le C_2 . \tag{4.20}$$

Applying Lemma 4.2 to (4.14)–(4.16) and using estimate (1.26), we obtain that as $t \to +\infty$,

$$\|v_t(t)\| \to 0 , \quad \|\theta_t(t)\| \to 0 , \quad \|u_{xx}(t)\| \to 0 \tag{4.21}$$

which with (1.1), (4.5) and (4.17)–(4.20) implies that as $t \to +\infty$,

$$\|v_{xx}(t)\| + \|\theta_{xx}(t)\| + \|u_t(t)\|_{H^1} \to 0 , \tag{4.22}$$

$$\|u_t(t)\|_{L^\infty} + \big\|\big(u_x(t), v_x(t), \theta_x(t)\big)\big\|_{L^\infty} \to 0. \tag{4.23}$$

Thus (1.30)–(1.31) follows from (4.5) and (4.21)–(4.23). The proof is complete. ∎

Lemma 4.4. *Under the assumptions in Theorem 1.3 with $i=4$, if $e_0 \le \epsilon_0$ or $E_0 E_1 \le \epsilon_0$, then the H^4-global solution $\big(u(t), v(t), \theta(t)\big)$ obtained in Theorem 1.2 to the Cauchy problem (1.1)–(1.4) satisfies (1.32)–(1.33).*

Proof: Similarly to (3.19), (3.34), (3.39)–(3.40), (3.47), (3.63) and using (1.28), we derive

$$\tag{4.24} \begin{aligned} &\frac{d}{dt}\|v_{tt}(t)\|^2 + (2C_1)^{-1}\|v_{ttx}(t)\|^2 \le \\ &\quad\le C_2\Big(\|\theta_{xx}(t)\|^2 + \|\theta_{tx}(t)\|^2_{H^1} + \|v_x(t)\|^2_{H^1} + \|v_{tx}(t)\|^2 + \|\theta_t(t)\|^2 + \|u_x(t)\|^2\Big)\,, \end{aligned}$$

$$\tag{4.25} \begin{aligned} &\frac{d}{dt}\|\theta_{tt}(t)\|^2 + C_1^{-1}\|\theta_{ttx}(t)\|^2 \le \\ &\quad\le C_4\Big(\|\theta_{tx}(t)\|^2 + \|v_{tx}(t)\|^2_{H^1} + \|v_x(t)\|^2 + \|\theta_t(t)\|^2 + \|v_{ttx}(t)\|^2 + \|\theta_{tt}(t)\|^2\Big)\,, \end{aligned}$$

$$\tag{4.26} \begin{aligned} &\frac{d}{dt}\|v_{tx}(t)\|^2 + C_1^{-1}\|v_{txx}(t)\|^2 \le \\ &\quad\le C_2\Big(\|\theta_{tx}(t)\|^2 + \|v_{tx}(t)\|^2 + \|\theta_t(t)\|^2 + \|v_{xx}(t)\|^2 + \|\theta_x(t)\|^2 + \|u_x(t)\|^2\Big)\,, \end{aligned}$$

$$\tag{4.27} \begin{aligned} &\frac{d}{dt}\|\theta_{tx}(t)\|^2 + C_1^{-1}\|\theta_{txx}(t)\|^2 \le \\ &\qquad\qquad\le C_2\Big(\|\theta_{tx}(t)\|^2 + \|\theta_{xx}(t)\|^2 + \|v_{xx}(t)\|^2 + \|u_x(t)\|^2\Big)\,, \end{aligned}$$

$$\tag{4.28} \frac{d}{dt}\Big\|\frac{u_{xxx}}{u}(t)\Big\|^2 + C_1^{-1}\Big\|\frac{u_{xxx}}{u}(t)\Big\|^2 \le C_1\|E_1(t)\|^2\,,$$

$$\tag{4.29} \frac{d}{dt}\Big\|\frac{u_{xxxx}}{u}(t)\Big\|^2 + C_1^{-1}\Big\|\frac{u_{xxxx}}{u}(t)\Big\|^2 \le C_1\|E_2(t)\|^2$$

where, by (1.28), (3.36) and (3.52),

$$\tag{4.30} \int_0^t \Big(\|E_1\|^2 + \|E_2\|^2\Big)(\tau)\,d\tau \le C_4\,, \qquad \forall\, t>0\,.$$

Applying Lemma 4.2 to (4.24)–(4.29) and using estimates (1.28) and (4.30), we infer that as $t\to+\infty$,

$$\tag{4.31} \|v_{tt}(t)\|\to 0\,, \quad \|\theta_{tt}(t)\|\to 0\,, \quad \|v_{tx}(t)\|\to 0\,,$$

$$\tag{4.32} \|\theta_{tx}(t)\|\to 0\,, \quad \|u_{xxx}(t)\|\to 0\,, \quad \|u_{xxxx}(t)\|\to 0\,.$$

In the same manner as (3.7), (3.9), (3.41) and using the interpolation inequality, we deduce that

$$\|v_{xxx}(t)\| \le C_2\Big(\|v_x(t)\|+\|u_x(t)\|_{H^1}+\|\theta_x(t)\|_{H^1}+\|v_{tx}(t)\|\Big)\,, \tag{4.33}$$

$$\begin{aligned}\|v_{txx}(t)\| \le\; & C_1\,\|v_{tt}(t)\| + C_2\Big(\|v_{xx}(t)\|+\|u_x(t)\|+\|v_{tx}(t)\| \\ & +\|\theta_x(t)\|+\|\theta_t(t)\|+\|\theta_{tx}\|\Big)\,,\end{aligned} \tag{4.34}$$

$$\|v_{xxxx}(t)\| \le C_2\Big(\|v_x(t)\|_{H^2}+\|u_x(t)\|_{H^2}+\|\theta_x(t)\|_{H^2}+\|v_{txx}(t)\|\Big)\,, \tag{4.35}$$

$$\begin{aligned}\|v_{tx}(t)\|^2_{L^\infty} &\le C\,\|v_{tx}(t)\|\,\|v_{txx}(t)\|\,,\\ \|v_t(t)\|^2_{L^\infty} &\le C\,\|v_t(t)\|\,\|v_{tx}(t)\|\;,\end{aligned} \tag{4.36}$$

$$\begin{aligned}\|v_{xx}(t)\|^2_{L^\infty} &\le C\,\|v_{xx}(t)\|\,\|v_{xxx}(t)\|\,,\\ \|v_{xxx}(t)\|^2_{L^\infty} &\le C\,\|v_{xxx}(t)\|\,\|v_{xxxx}(t)\|\;,\end{aligned} \tag{4.37}$$

$$\begin{aligned}\|u_{xx}(t)\|^2_{L^\infty} &\le C\,\|u_{xx}(t)\|\,\|u_{xxx}(t)\|\,,\\ \|u_{xxx}(t)\|^2_{L^\infty} &\le C\,\|u_{xxx}(t)\|\,\|u_{xxxx}(t)\|\;.\end{aligned} \tag{4.38}$$

Thus it follows from (1.1), (4.31)–(4.38) and Lemma 4.3 that as $t\to+\infty$,

$$\begin{aligned}&\big\|\big(u_x(t),v_x(t)\big)\big\|_{H^3}+\|v_t(t)\|_{H^2}+\|u_t(t)\|_{H^3}+\|u_t(t)\|_{W^{2,\infty}}\\ &\qquad +\|u_{tt}(t)\|_{H^1}+\|(u_x(t),v_x(t))\|_{W^{2,\infty}} \;\to\; 0\;.\end{aligned} \tag{4.39}$$

Analogously, we can derive that as $t\to+\infty$,

$$\|\theta_x(t)\|_{H^3}+\|\theta_t(t)\|_{H^2}+\|\theta_t(t)\|_{W^{1,\infty}}+\|\theta_x(t)\|_{W^{2,\infty}} \;\to\; 0$$

which together with Lemma 4.3 and (4.39) implies estimates (1.32)–(1.33). The proof is complete. ∎

Till now we have finished the proof of Theorem 1.3. ∎

Proof of Corollary 1.1: Applying the embedding theorem, we readily get estimate (1.34) and complete the proof from Theorem 1.2. ∎

ACKNOWLEDGEMENTS – Y. Qin is supported in part by the Jie Chu Qing Nian grant (contract number 10225102) of China, and by the grants of the NNSF of China (contract number 10571024), the Prominent Youth (No. 0412000100) from Henan Province of China. Y. Wu is supported in part by a postdoctoral grant (proc. 200877/01-1) from CNPq in Brazil.

REFERENCES

[**1**] Antontsev, S.N.; Kazhikhov, A.V. and Monakhov, V.N. – *Boundary Value Problems in Mechanics of Nonhomogeneous Fluids*, Amsterdam, New York, 1990.

[**2**] Chen, G. – Global solutions to the compressible Navier–Stokes equations for a reacting mixture, *SIAM J. Math. Anal.,* 3 (1992), 609–634.

[**3**] Chen, G.; Hoff, D. and Trivisa, K. – Global solutions of the compressible Navier–Stokes equations with large discontinuous initial data, *Comm. PDE,* 25 (2000), 2233–2257.

[**4**] Deckelnick, K. – L^2 decay for the compressible Navier–Stokes equations in unbounded domains, *Comm. PDE,* 18 (1993), 1445–1476.

[**5**] Feireisl, E. – Compressible Navier–Stokes equations with a non-monotone pressure law, *J. Differential Equations,* 184 (2002), 97–108.

[**6**] Feireisl, E. – On compactness of solutions to the compressible isentropic Navier–Stokes equations when the density is not square integrable, *Comment. Math. Univ. Carolin.,* 42 (2001), 83–98.

[**7**] Feireisl, E. and Petzeltova, H. – Bounded absorbing sets for the Navier–Stokes equations of compressible fluid, *Comm. Partial Differential Equations,* 26 (2001), 1133–1144.

[**8**] Feireisl, E.; Novotny, A. and Petzeltova, H. – On the existence of globally defined weak solutions to the Navier–Stokes equations, *J. Math. Fluid. Mech.,* 3 (2001), 358–392.

[**9**] Feireisl, E. and Petzeltova, H. – Asymptotic compactness of global trajectories generaalized by the Navier–Stokes equations of a compressible fluid, *J. Differential Equations,* 173 (2001), 390–409.

[**10**] Feireisl, E. and Petzeltova, H. – The zero-velocity limit solutions of the Navier–Stokes equations of compressible fluid revisited. Navier–Stokes equations and related nonlinear problems, *Ann. Univ. Ferrara Sez. VII (N.S.),* 46 (2002), 209–218.

[**11**] Fujita-Yashima, H. and Benabidallah, R. – Unicité de la solution de l'équation monodimensionnelle oua' symétrie sphérique d'un gaz visqueux et calorifére, *Rendi. del Circolo Mat. di Palermo, Ser. II,* XLII (1993), 195–218.

[**12**] Fujita-Yashima, H. and Benabidallah, R. – Equation á symétrie sphérique d'un gaz visqueux et calorifére avec la surface libre, *Annali Mat. Pura ed Applicata,* CLXVIII (1995), 75–117.

[**13**] Fujita-Yashima, H.; Padula, M. and Novotny, A. – Équation monodimensionnelle dúmgaz vizqueux et calorifére avec des conditions initialmoins restrictives, *Ricerche Mat.,* 42 (1993), 199–248.

[**14**] Hoff, D. – Global solutions of the Navier–Stokes equations for muldimensional compressible flow with discontinuous initial data, *J. Diff. Eqs.,* 120 (1995), 215–254.

[**15**] Hoff, D. – Continuous dependence on initial data for discontinuous solutions of the Navire-Stokes equations for one-dimensional, compressible flow, *SIAM J. Math. Anal.,* 27 (1996), 1193–1211.

[**16**] Hoff, D. and Serre, D. – The failure of continuous dependence on initial data for the Navier–Stokes equations of compressible flow, *SIAM J. Appl. Math.,* 51 (1991), 887–898.

[**17**] Hoff, D. and Zarnowski, R. – Continuous dependence in L^2 for discontinuous solutions of viscous p-system, *Ann. Inst. H. Poincaré Analyse Nonlinéaire,* 11 (1994), 159–187.

[**18**] Hoff, D. – Discontinuous solutions of the Navier–Stokes equations for multidimensional heat-conducting flows, *Arch. Rational Mech. Anal.,* 139 (1997), 303–354.

[**19**] Jiang, S. – Global spherically symmetric solutions to the equations of a viscous polytropic ideal gas in an exterior domain, *Commun. Math. Phys.,* 178 (1996), 339–374.

[**20**] Jiang, S. – Global solutions of the Cauchy problem for a viscous polytropic ideal gas, *Ann. Scuola Norm Sup, Pisa Cl. Sci (4),* XXVI (1998), 47–74.

[**21**] Jiang, S. – Large-time behavior of solutions to the equations of a viscous polytropic ideal gas, *Ann. Mat. Pura Appl.,* CLXXV (1998), 253–275.

[**22**] Jiang, S. – Large-time behavior of solutions to the equations of a one-dimensional viscous polytropic ideal gas in unbounded domains, *Comm. Math. Phys.,* 200 (1999), 181–193.

[**23**] Kanel, Y.I. – Cauchy problem for the equations of gasdynamics with viscosity, *Siberian Math. J.,* 20 (1979), 208–218.

[**24**] Kawashima, S. – Large-time behavior of solutions to hyperbolic-parabolic system of conservation laws and applications, *Proc. Roy. Soc. Edinburgh,* 106 A (1987), 169–194.

[**25**] Kawashima, S. and Nishida, T. – Global solutions to the initial value problem for the equations of one-dimensional motion of viscous polytropic gases, *J. Math. Kyoto Univ.,* 21 (1981), 825–837.

[**26**] Kazhikhov, A.V. – Cauchy problem for viscous gas equations, *Siberian Math. J.,* 23 (1982), 44–49.

[**27**] Kazhikhov, A.V. and Shelukhin, V.V. – Unique global solution with respect to time of initial boundary value problems for one-dimensional equations of a viscous gas, *J. Appl. Math. Mech.,* 41 (1977), 273–282.

[**28**] Liu, T.P. and Zeng, Y. – Large-time behavior of solutions of general quasilinear hyperbolic-parabolic systems of conservation laws, *Memoirs of the AMS,* 599 (1997).

[**29**] Matsumura, A. and Nishida, T. – The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids, *Proc. Japan Acad. Ser. A Math. Sci.,* 55 (1979), 337–342.

[**30**] Matsumura, A. and Nishida, T. – The initial value problem for the equations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.,* 20 (1980), 67–104.

[**31**] Matsumura, A. and Nishida, T. – *Initial boundary value problems for the equations of motion of general fluids*, In: "Computing Meth. in Appl. Sci. and Engin. V." (R. Glowinski and J.L. Lions, Eds.), NorthHolland, Amsterdam, 1982, pp. 389–406.

[**32**] Matsumura, A. and Nishida, T. – Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids, *Comm. Math. Phys.,* 89 (1983), 445–464.

[**33**] Nagasawa, T. – On the outer pressure problem of the one-dimensional polytropic ideal gas, *Japan J. Appl. Math.,* 5 (1988), 53–85.

[**34**] Nagasawa, T. – On the asymptotic behaviour of the one-dimensional motion of the polytropic ideal gas with stress-free condition, *Quart. Appl. Math.,* 46 (1988), 665–679.

[**35**] Nagasawa, T. – On the one-dimensional motion of the polytropic ideal gas non-fixed on the boundary, *J. Diff. Eqs.,* 65 (1986), 49–67.

[**36**] Nikolaev, V.B. – On the solvability of mixed problem for one-dimensional axisymmetrical viscous gas flow, *Dinamicheskie Zadachi Mekhaniki Sploshnoj Sredy,* 63 Sibirsk. Otd. Acad. Nauk SSSR, Inst. Gidrodinamiki, 1983 (in Russian).

[**37**] Novotny, A. and Straskraba, I. – Stabilization of weak solutions to compressible Navier–Stokes equations, *J. Math. Kyoto Univ.,* 40 (2000), 217–245.

[**38**] Novotny, A. and Straskraba, I. – Convergence to equilibria for compressible Navier–Stokes equations with large data, *Ann. Mat. Pura. Appl.,* 179 (2001), 263–287.

[**39**] Okada, M. and Kawashima, S. – On the equations of one-dimensional motion of compressible viscous fluids, *J. Math. Kyoto Univ.,* 23 (1983), 55–71.

[**40**] Padula, M. – Stability properties of regular flows of heat-conducting compressible fluids, *J. Math. Kyoto Univ.,* 32 (1992), 401–442.

[**41**] Qin, Y. – Global existence and asymptotic behaviour of solutions to a system of equations for a nonlinear one-dimensional viscous heat-conductive real gas, *Chin. Ann. Math.,* 20A (1999), 343–354.

[**42**] Qin, Y. – *Global existence and asymptotic behaviour of solutions to nonlinear hyperbolic-parabolic coupled systems with arbitrary initial data*, Ph. D. Thesis, Fudan University, 1998.

[**43**] Qin, Y. – Global existence and asymptotic behaviour for the solutions to nonlinear viscous, heat-conductive, one-dimensional real gas, *Adv. Math. Sci. Appl.,* 10 (2000), 119–148.

[**44**] Qin, Y. – Exponential stability for a nonlinear one-dimensional heat-conductive viscous real gas, *J. Math. Anal. Appl.,* 272 (2002), 507–535.

[**45**] Qin, Y. and Rivera, J.E.M. – Universal attractors for a nonlinear one-dimensional heat-conductive viscous real gas, *Proc. Roy. Soc. Edinburgh,* 132 A (2002), 685–709.

[**46**] Qin, Y. – *Exponential stability for the compressible Navier–Stokes equations*, Preprint.

[**47**] Serrin, J. – *Mathematical principles of classical fluids mechanics*, In "Hanbuch der Physik" VIII/1, Springer-Verlag, Berlin, Heidelberg, New York, 1972, pp. 125–262.

[**48**] Stravskraba, I. – *Large time behaviour of solutions to compressible Navier–Stokes equations: theory and numerical methods* (Varenna, 1997), pp. 125–138, Pitman Res. Notes Math. Ser., 338, Longman, Harlow, 1998.

[**49**] Valli, A. and Zajaczkowski, W.M. – Navier–Stokes equation for compressible fluids: global existence and qualitative properties of the solutions in the general case, *Comm. Math. Phys.,* 103 (1986), 259–296.

[**50**] Zheng, S. and Qin, Y. – Maximal attractor for the system of one-dimensional polytropic viscous ideal gas, *Quart. Appl. Math.,* 3 (2001), 579–599.

[**51**] Zheng, S. and Qin, Y. – Universal attractors for the Navier–Stokes equations of compressible and heat conductive fluids in bounded annular domains in $\mathbb{R}^n$, *Arch. Rational Mech. Anal.,* 160 (2001), 153–179.

[**52**] Shen, W. and Zheng, S. – On the coupled Cahn-Hilliard equations, *Comm. PDE,* 18 (1993), 701–727.

Yuming Qin,
Department of Applied Mathematics, Donghua University,
Shanghai 200051 – P. R. CHINA
E-mail: yuming@dhu.edu.cn

and

Yumei Wu,
Department of Physics, Baylor University,
PO Box 97316, Waco, TX – USA
E-mail: Yumei_Wu@baylor.edu

and

Fagui Liu,
North China Institute of Water Conservancy and Hydroelectric Power,
Zhengzhou 450045, Henan Province – P. R. CHINA