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EULER CONSTANTS FOR THE RING OF S-INTEGERS OF A FUNCTION FIELD

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Abstract: The Euler constant γ may be defined as the limit for n tending to $+\infty$, of the difference $\sum_{j=1}^{n} \frac{1}{j} - \log n$. Alternatively, it may be defined as the limit at 1 of the difference $\sum_{n=1}^{\infty} \frac{1}{j^s} - \frac{1}{s-1}$, s being a complex number in the half-plane $\Re(s) > 1$. Mertens theorem states that for x real number tending to $+\infty$, $\prod_{p \le x} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x}$, the product being over prime numbers $\le x$. We prove analog results for the ring of S-integers of a function field. However, in the function field case, the three approaches lead to different constants.

1 – Introduction and main results

Let q be a power of a prime number p and let K be a function field with genus g and field of constants k a finite field with q elements. Let V denote the set of places of K and let h be the number of divisor classes of K. For S a finite and non-empty set of s places of K, let $O = O_S$ denote the ring of S-integers of K, that is to say the set of elements $a \in K$ such that $v(a) \ge 0$ for any place v out of S. Let $\mathcal{I}$, resp. $\mathcal{P}$, denote the set of non-zero ideals of O, resp. the set of prime ideals of the ring O. The set V is the union of the set S and the set of the P-adic places v_P for $P \in \mathcal{P}$. If $v \in V$, let f_v denote the residual degree of v. In order to reduce notations, we set $f_{v_P} = f_P$ for each $P \in \mathcal{P}$. If $H \in \mathcal{I}$, let $|H|$ denote the

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norm of the ideal H. We note that $|H|$ is a power of q. We set $|H| = q^{f_H}$ and we note that this notation agrees with the notation f_P used for prime ideals P. The number f_H will be called the degree of the ideal H.

The zeta-function ζ_K of the field K may be defined on the complex half-plane $\Re(s) > 1$ by:

$$\zeta_K(s) = \prod_{v \in V} \left(1 - q^{-f_v s}\right)^{-1} . \tag{1.1}$$

The following facts are well-known, [6]:

$$\zeta_K(s) = \frac{P_K(q^{-s})}{(1 - q^{-s})\,(1 - q^{1-s})} , \tag{1.2}$$

where $P_K(u) \in \mathbb{Z}[u]$ is a polynomial of degree $2g$ such that

$$P_K(1) = h . \tag{1.3}$$

If $g > 0$, there exist algebraic numbers $\rho_1, ..., \rho_g$ such that

$$P_K(u) = \prod_{i=1}^{g} (1 - \rho_i u)\,(1 - \bar{\rho}_i u) , \tag{1.4}$$

with

$$|\rho_i| = q^{1/2} . \tag{1.5}$$

Moreover, P_K verifies the functional equation

$$P_K(u) = q^g u^{2g} P_K\left(\frac{1}{qu}\right) . \tag{1.6}$$

The aim of this paper is to prove the following theorems. The first one deals with the finite sum

$$\sum_{\substack{H \in \mathcal{I} \\ f_H < N}} \frac{1}{|H|}$$

and leads to the definition of our first Euler constant $\gamma_{K,S}$. The second one deals with the Laurent expansion at $s = 1$ of the sum

$$\sum_{H \in \mathcal{I}} \frac{1}{|H|^s}$$

which is an analog of the classical Dedekind zeta-function for number fields and leads to the definition of a second Euler constant $\delta_{K,S}$. Although they are different, these two first Euler constants are of the same kind. They are related to the Euler–Kronecker constant of a function field defined by Ihara, [4]. The third theorem deals with the product

$$\prod_{\substack{P\in\mathcal{P}\\ f_P\le N}}\left(1-\frac{1}{|P|}\right).$$

It provides an analog to the classical Mertens theorem and leads to the definition of a third Euler constant $\Gamma_{K,S}$ different from the two first ones. This theorem generalizes a theorem of Conrad proved in [2] for the ring $k[T]$.

Theorem 1.1. *Let*

$$A_{K,S} = \frac{h\,q^{1-g}}{q-1}\prod_{v\in S}\left(1-q^{-f_v}\right) \tag{1.7}$$

and

$$\gamma_{K,S} = \sum_{i=1}^{g}\left(\frac{1}{\rho_i-1}+\frac{1}{\bar\rho_i-1}\right)+\sum_{v\in S}\frac{f_v}{q^{f_v}-1}-\frac{1}{q-1}\ . \tag{1.8}$$

Then,

$$\sum_{\substack{H\in\mathcal{I}\\ f_H<N}}\frac{1}{|H|} = A_{K,S}(N+\gamma_{K,S})+O(Nq^{-N})\ , \tag{1.9}$$

the constants involved in the O depending only on K and S.

Theorem 1.2. *Let*

$$\delta_{K,S} = \log q\left(\sum_{i=1}^{g}\left(\frac{1}{\rho_i-1}+\frac{1}{\bar\rho_i-1}\right)+\sum_{v\in S}\frac{f_v}{q^{f_v}-1}+\frac{1}{2}-\frac{1}{q-1}\right). \tag{1.10}$$

For z complex number such that $\Re(z)>1$, let

$$\zeta_{K,S}(z) = \sum_{H\in\mathcal{I}}\frac{1}{|H|^z}\ .$$

Then,

$$\lim_{z\to1}\left(\zeta_{K,S}(z)-\frac{A_{K,S}/\log q}{z-1}\right) = A_{K,S}\,\delta_{K,S}\ . \tag{1.11}$$

Theorem 1.3. *For N tending to $+\infty$,*

$$\prod_{\substack{P\in\mathcal{P}\\ f_P\le N}}\left(1-\frac{1}{|P|}\right)=\frac{e^{-\gamma}A_{K,S}}{N}\left(1+O\left(\frac{1}{N}\right)\right), \tag{1.12}$$

γ being the classical Euler constant, the constants involved in the O depending only on K and S. In other words, if

$$\Gamma_{K,S}\;=\;\gamma-\log\left(1-\frac{1}{q}\right)+\sum_{v\in S}\log\left(1-\frac{1}{q^{f_v}}\right)+\log(h\,q^{-g})\;, \tag{1.13}$$

$$\prod_{\substack{P\in\mathcal{P}\\ f_P\le N}}\left(1-\frac{1}{|P|}\right)\;=\;\frac{e^{-\Gamma_{K,S}}}{N}\left(1+O\left(\frac{1}{N}\right)\right), \tag{1.14}$$

the constants involved in the O depending only on K and S.

If γ_K denotes the Euler–Kronecker constant of the function field K defined in [4], then

$$\frac{\gamma_K}{\log q}\;=\;(q-1)\sum_{i=1}^{g}\frac{1}{(1-\rho_i)\,(1-\bar{\rho}_i)}-(g-1)-\frac{q+1}{2(q-1)}\;.$$

We note that constants $\gamma_{K,S}$, $\delta_{K,S}$ and γ_K are connected by the relations

$$\gamma_{K,S}\;=\;\frac{\gamma_K}{\log q}-\frac{1}{2}+\sum_{v\in S}\frac{f_v}{q^{f_v}-1}\;, \tag{1.15}$$

$$\gamma_{K,S}\;=\;\frac{\delta_{K,S}}{\log q}-\frac{1}{2}\;, \tag{1.16}$$

$$\frac{\delta_{K,S}}{\log q}\;=\;\frac{\gamma_K}{\log q}+\sum_{v\in S}\frac{f_v}{q^{f_v}-1}\;. \tag{1.17}$$

We also note that Euler constants $\gamma_{K,S}$, $\delta_{K,S}$, $\Gamma_{K,S}$ are pairwise distinct. In [2] the authors mentioned without proof a Mertens formula for prime divisors of a function field due to Conrad. Our Euler–Mertens constant $\Gamma_{K,S}$ occurring in the Mertens formula for prime ideals of the ring O_S is related to Conrad constant as the Euler constant $\gamma_{K,S}$ is related to Ihara constant γ_K. By (1.13) and the Hermite–Lindeman theorem, [5], when $\Gamma_{K,S}\neq\gamma$, the difference $\Gamma_{K,S}-\gamma$ is transcendental. Hence it is an open question to decide if or if not the Euler–Mertens constant $\Gamma_{K,S}$ is an algebraic number. On the other hand, since $P_K(u)\in\mathbb{Z}[u]$, the Euler constant $\gamma_{K,S}$ is a rational number.

2 – Proofs of theorems 1.1 and 1.2

We begin this section by an obvious remark concerning the polynomial case. Every non-zero ideal of $k[T]$ being generated by a monic polynomial, a sum over non-zero ideals may be replaced by a sum over monic polynomials. Let $\mathcal{M}$ denote the set of monic polynomials.

Remark 2.1.

(i) *Let N be a positive integer. Then,*

$$\sum_{\substack{H\in\mathcal{M}\\ \deg H<N}} \frac{1}{|H|} = N\ , \tag{2.1}$$

where $|H|=q^{\deg H}$.

(ii) *For s a complex number such that $\Re(s)>1$, let*

$$\zeta_{k[T]}(s) = \sum_{H\in\mathcal{M}} \frac{1}{|H|^s}\ .$$

Then,

$$\lim_{s\to 1}\left(\zeta_{k[T]}(z) - \frac{1}{(z-1)\,\log q}\right) = \frac{1}{2}\ .$$

Proof: There are exactly q^n monic polynomials of degree n in $k[T]$, whence (i). For the same reasons, for $\Re(s)>1$,

$$\zeta_{k[T]}(s) = \frac{1}{1-q^{1-s}}\ .$$

The Laurent expansion of $\zeta_{k[T]}$ at $s=1$ gives (ii). ∎

This remark proves that theorems 1.1 and 1.2 are true in the case where $K=k(T)$ and S reduced to $\{\infty\}$, where ∞ is the $\frac{1}{T}$-adic place, with $\gamma_{k(T),\{\infty\}}=0$ and $\delta_{k(T),\{\infty\}}=\log q/2$. Even in the rational field case, these constants are different. (In the rational field case Ihara's constant is $\gamma_{k(T)}=\frac{q-3}{2(q-1)}$.)

Nevertheless, it is possible to obtain the same Euler constant. It suffices to modify the definition of the zeta function of the ring $k[T]$. Indeed, the above definition of $\zeta_{k[T]}(s)$ follows the analogy with the Riemann zeta function.

The change of variable $u = q^{-s}$ allows us to define the zeta function of the ring $k[T]$ as the powers series

$$Z_{k[T]}(u) \;=\; \sum_{H\in\mathcal{M}} u^{\deg H}\,,$$

absolutely convergent in the open disk $D_{1/q} = \left\{z\in\mathbb{C};\ |z| < \frac{1}{q}\right\}$. In this setting

$$Z_{k[T]}(u) \;=\; \frac{1}{1-qu}$$

and

$$\lim_{u\to 1/q}\left(Z_{k[T]}(u) - \frac{\frac{1}{q}}{\frac{1}{q}-u}\right) \;=\; 0 \;=\; \gamma_{k(T),\infty}\ .$$

We return to the general case. For N a non-negative integer, let

$$S_N \;=\; \sum_{\substack{H\in\mathcal{I}\\ f_H<N}} \frac{1}{|H|} \tag{2.2}$$

and let i_N denote the number of ideals of degree N in the ring O_S.

Proposition 2.2. *Let*

$$A_{K,S} \;=\; \frac{h\,q^{1-g}}{q-1}\prod_{v\in S}\bigl(1-q^{-f_v}\bigr) \tag{2.3}$$

and

$$B_{K,S} \;=\; A_{K,S}\left(\sum_{i=1}^{g}\left(\frac{1}{\rho_i-1}+\frac{1}{\bar\rho_i-1}\right)+\sum_{v\in S}\frac{f_v}{q^{f_v}-1}-\frac{1}{q-1}\right). \tag{2.4}$$

Then, for any integer $N\ge 0$,

$$\bigl|S_N - A_{K,S}\,N - B_{K,S}\bigr| \;\le\; M_{K,S}\,q^{-N}\,, \tag{2.5}$$

with $M_{K,S}$ a constant depending only on K and S.

Proof: We want to estimate the sum

$$S_N \;=\; \sum_{n=0}^{N-1} i_n\,q^{-n}\ . \tag{1}$$

For r a positive real number, let D_r denote the open disk $\{z \in \mathbb{C};\ |z| < r\}$. The series

$$G_{K,S}(u) \;=\; \sum_{n=0}^{+\infty} i_n \left(\frac{u}{q}\right)^n = \sum_{Y \in \mathcal{I}} \left(\frac{u}{q}\right)^{f_Y} \tag{2}$$

is absolutely convergent in the disk D_1 and for $u \in D_1$,

$$G_{K,S}(u) \;=\; \prod_{\substack{v \in V \\ v \notin S}} \left(1 - \left(\frac{u}{q}\right)^{f_v}\right)^{-1} . \tag{3}$$

In view of (1.1) and (1.2), as for the proof of Proposition 3.5 in [1], we get that

$$G_{K,S}(u) \;=\; \frac{H(u)}{1 - u} \;, \tag{4}$$

with

$$H(u) \;=\; \frac{P_K\left(\frac{u}{q}\right)}{1 - \frac{u}{q}} \prod_{v \in S} \left(1 - \left(\frac{u}{q}\right)^{f_v}\right) . \tag{5}$$

We note that H is holomorphic in the disk D_q. Let

$$H(u) \;=\; \sum_{n=0}^{+\infty} h_n\, u^n \tag{6}$$

be its power series expansion at the origin and let coefficients b_k be defined by

$$P_K(u) \prod_{v \in S} \left(1 - u^{f_v}\right) \;=\; \sum_{i=0}^{d} b_i\, u^i \;. \tag{7}$$

Let

$$m_{K,S} \;=\; \max\Big(|b_0|, |b_0 + b_1|, \ldots, |b_0 + b_1 + \cdots + b_d|\Big) . \tag{8}$$

Then, in view of (5), (6), (7) and (8), for any $n \geq 0$,

$$|h_n| \leq m_{K,S}\, q^{-n} \;. \tag{9}$$

By (1), (2) and (4),

$$S_N \;=\; \sum_{n=0}^{N-1} \sum_{i=0}^{n} h_i \;=\; \sum_{i=0}^{N-1} (N - i)\, h_i \;.$$

Since H is holomorphic in the disk D_q,

$$S_N = NH(1) - H'(1) + \sum_{i=N}^{+\infty}(i-N)\,h_i \ .$$

By (9),

$$\left|S_N - \big(NH(1) - H'(1)\big)\right| \le m_{K,S}\,\frac{q}{(q-1)^2}\,q^{-N} \ . \tag{10}$$

It remains to compute $H(1)$ and $H'(1)$ to get the announced result. By (5), then (1.3) and (1.6),

$$H(1) = \frac{h\,q^{1-g}}{q-1}\prod_{v\in S}\big(1-q^{-f_v}\big) \ .$$

By (5), then (1.4),

$$\frac{H'(1)}{H(1)} = -\sum_{i=1}^{g}\left(\frac{\rho_i}{q-\rho_i}+\frac{\bar\rho_i}{q-\bar\rho_i}\right) - \sum_{v\in S}\frac{f_v}{q^{f_v}-1} + \frac{1}{q-1}\psi$$

We conclude noting that in view of (1.5) $\rho_i\,\bar\rho_i = q$. ∎

This proposition gives theorem 1.1. We note that if $K = k(T)$ and $S = \{\infty\}$, then $h=1$, $g=0$ and $f_\infty = 1$. Thus, $A_{k(T),\{\infty\}} = 1$ and $B_{k(T),\{\infty\}} = 0$, as it has been remarked at the beginning of the section.

Proposition 2.3. *We have*

$$\lim_{\substack{z\to 1\\ \Re(z)>1}}\left(\zeta_{K,S}(z) - \frac{A_{K,S}}{\log q(z-1)}\right) = \frac{A_{K,S}}{2} - B_{K,S} \ . \tag{2.6}$$

Proof: In the half-plane $\Re(z)>1$, the zeta-function of the ring O_S may be expanded as an eulerian product absolutely convergent

$$\zeta_{K,S}(z) = \sum_{H\in\mathcal{I}}\frac{1}{|H|^s} = \prod_{\substack{v\in V\\ v\notin S}}\big(1-q^{-f_v z}\big)^{-1} \ .$$

Using notations of the above proposition we get that

$$\zeta_{K,S}(z) = G_{K,S}\big(q^{1-z}\big) \ . \tag{1}$$

The Laurent series expansion at $z=1$ of $\zeta_{K,S}(z)$ is obtained from the expansion of $G_{K,S}$ in series of powers. ∎

As in the polynomial case, using the change of variable $u = q^{-s}$, we may define the zeta-function of the ring O_S as the powers series.

$$Z_{K,S}(u) \;=\; \sum_{Y\in\mathcal{I}} u^{f_Y} \;.$$

In this setting,

$$Z_{K,S}(u) \;=\; G_{K,S}(qu) \;.$$

Hence,

$$\lim_{u\to 1}\left(Z_{K,S}(u) - \frac{H(1)}{1-qu}\right) \;=\; H'(1)$$

with H defined as in the proof of proposition 1.1. With this definition of the zeta function, the Euler constant occurring in the Laurent expansion at $1/q$ of the zeta function of the ring O_S is the same that the Euler constant occurring in the finite sum over ideals.

3 – Proof of the theorem 1.3

The proof follows that of theorem 429 in [3]. Firstly, we need some results about the distribution of prime ideals. For n a positive integer, let p_n denote the number of prime ideals of degree n in the ring O_S. Let

(3.1)
$$a_n \;=\; \sum_{j\mid n} j\,p_j \;.$$

In [1] it has been proved that

(3.2)
$$a_n \;=\; q^n + 1 - \sum_{\substack{v\in S\\ f_v\mid n}} f_v \;-\; \sum_{i=1}^{g} ({\rho_i}^n + {\bar\rho_i}^{\,n}) \;.$$

Proposition 3.1. *If n is a positive integer, then*

(3.3)
$$n\,p_n \;\le\; (q + 2\,g\,q^{1/2} + 1)\,q^{n-1} \;.$$

Proof: For n a positive integer, let $\epsilon(n)$ be defined by

$$n\,p_n \;=\; q^n + 1 - \sum_{\substack{v\in S\\ f_v\mid n}} f_v \,+\, \epsilon(n) \;. \qquad (1)$$

In [1] it has been proved that

$$\epsilon(n) \le 2\, g\, q^{n/2} \ . \tag{2}$$

This gives (3.3). ∎

The following proposition will be crucial for proving theorem 3.

Proposition 3.2. *Let*

$$\Gamma_{K,S} \;=\; \gamma - \log(1-q^{-1}) + \sum_{v\in S} \log(1-q^{-f_v}) + \log(h\, q^{-g}) \ , \tag{3.4}$$

where γ is the classical Euler constant and for N a positive integer, let

$$X_N \;=\; \sum_{n=1}^{N} \frac{a_n}{n\, q^n} \ . \tag{3.5}$$

Then,

$$X_N \;=\; \log(N) + \Gamma_{K,S} + Y_N \ , \tag{3.6}$$

with

$$|Y_N| \;\le\; \left(1 + \frac{g}{\sqrt{q}\,\big(\sqrt{q}-1\big)} + \sum_{v\in S} \frac{f_v\, q^{f_v-1}}{q^{f_v}-1}\right) \frac{1}{N} \ . \tag{3.7}$$

Proof: By (3.5) and (3.2),

$$X_N \;=\; \sum_{n=1}^{N} \frac{1}{n} \left(1 + q^{-n}\left(1 - \sum_{\substack{v\in S\\ f_v|n}} f_v - \sum_{i=1}^{g} ({\rho_i}^n + {\bar\rho_i}^n)\right)\right) \ . \tag{1}$$

The Euler constant verifies

$$\sum_{n=1}^{N} \frac{1}{n} \;=\; \log(N) + \gamma + \lambda(N) \ , \tag{2}$$

with

$$\frac{1}{N(N+1)} \;\le\; \lambda(N) \;\le\; \frac{1}{N} \ . \tag{3}$$

A proof of this may be found in [3], theorem 422. For $\theta \in D_q$, let

$$\Phi_N(\theta) = \sum_{n=1}^{N} \frac{1}{n}\left(\frac{\theta}{q}\right)^n \tag{4}$$

and

$$\Phi(\theta) = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\theta}{q}\right)^n , \tag{5}$$

this last series being absolutely convergent. Then,

$$\Phi_N(\theta) = \Phi(\theta) - \varphi_N(\theta) , \tag{6}$$

with

$$|\varphi_N(\theta)| \le \varphi_N(|\theta|) \le \sum_{n=N+1}^{\infty} \frac{1}{n}\left(\frac{|\theta|}{q}\right)^n \le \frac{1}{N+1} \sum_{n=N+1}^{\infty} \frac{1}{n}\left(\frac{|\theta|}{q}\right)^n .$$

Hence,

$$|\varphi_N(\theta)| \le \varphi_N(|\theta|) \le \frac{1}{(N+1)} \times \frac{|\theta|}{q-|\theta|}\left(\frac{|\theta|}{q}\right)^N . \tag{7}$$

On the other hand,

$$\Phi(\theta) = -\log\left(1-\frac{\theta}{q}\right) .$$

Thus, by (1),

$$X_N = \log(N) + \gamma - \log\left(1-\frac{1}{q}\right) + \sum_{i=1}^{g}\left(\log\left(1-\frac{\rho_i}{q}\right) + \log\left(1-\frac{\bar{\rho}_i}{q}\right)\right) - \Omega_N + x_N ,$$

with

$$\Omega_N = \sum_{n=1}^{N} \frac{1}{n\,q^n} \sum_{\substack{v\in S\\ f_v|n}} f_v \tag{8}$$

and

$$x_N = \lambda(N) - \varphi_N(1) + \sum_{i=1}^{g}\left(\varphi_N\left(\frac{\rho_i}{q}\right) + \varphi_N\left(\frac{\bar{\rho}_i}{q}\right)\right) . \tag{9}$$

Now,

$$\sum_{i=1}^{g}\left(\log\left(1-\frac{\rho_i}{q}\right) + \log\left(1-\frac{\bar{\rho}_i}{q}\right)\right) = \log\left(\prod_{i=1}^{g}\left(1-\frac{\rho_i}{q}\right)\left(1-\frac{\bar{\rho}_i}{q}\right)\right)$$

and by (1.3) and (1.6),

$$\sum_{i=1}^{g}\left(\log\left(1-\frac{\rho_i}{q}\right)+\log\left(1-\frac{\bar{\rho}_i}{q}\right)\right) = \log\bigl(h\,q^{-g}\bigr) .$$

Then,

$$X_N = \log(N)+\gamma-\log\left(1-\frac{1}{q}\right)+\log(h\,q^{-g})-\Omega_N+x_N . \tag{10}$$

By (8),

$$\Omega_N = \sum_{v\in S} f_v \sum_{j=1}^{[N/f_v]}\frac{1}{j\,f_v\,q^{jf_v}} = \sum_{v\in S}\sum_{j=1}^{[N/f_v]}\frac{1}{j\,q^{jf_v}} .$$

Hence, as above,

$$\Omega_N = -\sum_{v\in S}\log\bigl(1-q^{-f_v}\bigr)-\omega_N , \tag{11}$$

with

$$0 \le \omega(N) \le \frac{1}{N\,q^N}\sum_{v\in S}\frac{f_v\,q^{f_v}}{q^{f_v}-1} , \tag{12}$$

and

$$X_N = \log(N)+\Gamma_{K,S}+Y_N ,$$

with

$$Y_N = \lambda(N)-\varphi_N(1)+\sum_{i=1}^{g}\left(\varphi_N\left(\frac{\rho_i}{q}\right)+\varphi_N\left(\frac{\bar{\rho}_i}{q}\right)\right)+\omega_N .$$

This gives the announced result, the bound (3.7) being given by (3), (7) and (12). ∎

Proposition 3.3. *Let*

$$\theta(K,S) = 1+\frac{g}{\sqrt{q}\,\bigl(\sqrt{q}-1\bigr)}+\sum_{v\in S}\frac{f_v q^{f_v}}{q^{f_v}-1}+\frac{2\bigl(q+2g\sqrt{q}+1\bigr)}{q^{1/2}(q-1)^2} . \tag{3.8}$$

Then, for any positive integer N,

$$\exp\left(-\frac{\theta(K,S)}{N}\right) \le \exp(\Gamma_{K,S})\,N\prod_{\substack{P\in\mathcal{P}\\ f_P\le N}}\left(1-\frac{1}{|P|}\right) \le \exp\left(\frac{\theta(K,S)}{N}\right) . \tag{3.9}$$

Proof: For N a positive integer, let T_N, U_N and V_N be defined by:

$$T_N = \sum_{\substack{P\in\mathcal{P}\\ f_P\le N}} \log\left(1-\frac{1}{|P|}\right) , \tag{1}$$

$$U_N = \sum_{\substack{P\in\mathcal{P}\\ f_P\le N}} \frac{1}{|P|} , \tag{2}$$

$$U_N + T_N = -V_N . \tag{3}$$

Then,

$$U_N = \sum_{n=1}^{N} p_n\, q^{-n} .$$

By (3.1) and (3.5),

$$U_N = X_N - R_N , \tag{4}$$

with

$$R_N = \sum_{n=1}^{N} \frac{1}{n\,q^n} \sum_{\substack{j|n\\ j\ne n}} j\,p_j . \tag{5}$$

Interchanging the order of summation in (5), we get that

$$R_N = \sum_{j=1}^{[N/2]} j\,p_j \sum_{\substack{1\le n\le N\\ j|n\\ j\ne n}} \frac{1}{n\,q^n} = \sum_{j=1}^{[N/2]} p_j \sum_{m=2}^{[N/j]} \frac{1}{m\,q^{jm}} = \sum_{\substack{P\in\mathcal{P}\\ f_P\le N/2}} \sum_{m=2}^{[N/f_P]} \frac{1}{m\,|P|^m} .$$

Whence, in view of (3),

$$R_N = V_N - W'_N - W''_N , \tag{6}$$

with

$$W'_N = \sum_{\substack{P\in\mathcal{P}\\ f_P\le N/2}} \sum_{m=1+[N/f_P]}^{\infty} \frac{1}{m\,|P|^m} \tag{7}$$

and

$$W''_N = \sum_{\substack{P\in\mathcal{P}\\ N/2<f_P\le N}} \sum_{m=2}^{\infty} \frac{1}{m\,|P|^m} . \tag{8}$$

We bound W'_N and W''_N. By (7),

$$W'_N \;\leq\; \sum_{\substack{P\in\mathcal{P}\\ f_P\leq N/2}} \frac{1}{1+[N/f_P]} \sum_{m=1+[N/f_P]}^{\infty} \frac{1}{|P|^m} \;\leq\; q^{-N} \sum_{\substack{P\in\mathcal{P}\\ f_P\leq N/2}} \frac{|P|}{\big(1+[N/f_P]\big)\big(|P|-1\big)} \ .$$

Hence,

$$N\,q^N\,W'_N \;\leq\; \sum_{\substack{P\in\mathcal{P}\\ f_P\leq N/2}} \frac{f_P\,|P|}{|P|-1} \ .$$

By (3.3),

$$N\,q^N\,W'_N \;\leq\; \big(q+2g\,q^{1/2}+1\big) \sum_{j=1}^{[N/2]} \frac{q^{2j-1}}{q^j-1} \;\leq\; \frac{q+2g\,q^{1/2}+1}{q-1} \sum_{j=1}^{[N/2]} q^j \ ,$$

$$N\,q^N\,W''_N \;\leq\; \frac{q+2g\,q^{1/2}+1}{(q-1)^2}\; q^{[N/2]+1} \ . \tag{9}$$

By (8),

$$W''_N \;\leq\; \sum_{\substack{P\in\mathcal{P}\\ f_P>N/2}} \sum_{m=2}^{\infty} \frac{1}{|P|^m} \;\leq\; \frac{1}{2} \sum_{\substack{P\in\mathcal{P}\\ f_P>N/2}} \sum_{m=2}^{\infty} |P|^{-m} \;=\; \frac{1}{2} \sum_{\substack{P\in\mathcal{P}\\ f_P>N/2}} \frac{1}{|P|\big(|P|-1\big)} \ .$$

Hence,

$$W''_N \;\leq\; \frac{q}{2\,(q-1)} \sum_{\substack{P\in\mathcal{P}\\ f_P>N/2}} |P|^{-2}$$

and by (3.3)

$$W''_N \;\leq\; \frac{q+2g\,q^{1/2}+1}{2\,(q-1)} \sum_{j=1+[N+2]}^{\infty} \frac{1}{j\,q^j} \;\leq\; \frac{q+2g\,q^{1/2}+1}{N(q-1)} \sum_{j=1+[N+2]}^{\infty} q^{-j} \ .$$

Hence,

$$N\,W''_N \;\leq\; \frac{q\big(q+2g\,q^{1/2}+1\big)}{(q-1)^2}\; q^{-N/2} \ . \tag{10}$$

By (3), (4) and (6),

$$T_N \;=\; -X_N - W'_N - W''_N$$

and, in view of (3.6)

$$T_N \;=\; -\log(N) - \Gamma_{K,S} - Y_N - W'_N - W''_N \ . \tag{11}$$

By (9), (10) and (3.7),

$$|W_N + Y_N| \leq \frac{\theta(K,S)}{N}\ , \tag{12}$$

with $\theta(K,S)$ defined by (3.8). By (11),

$$\exp\left(-\frac{\theta(K,S)}{N}\right) \leq \exp\big(T_N + \log(N) + \Gamma_{K,S}\big) \leq \exp\left(\frac{\theta(K,S)}{N}\right).$$

This is (3.9). ∎

Corollary 3.4. *We have*

$$N\, e^{\Gamma_{K,S}} \prod_{\substack{P\in\mathcal{P}\\ f_P\leq N}} \left(1 - \frac{1}{|P|}\right) = 1 + O\left(\frac{1}{N}\right), \tag{3.10}$$

the constants involved in the O depending only on K and S. ∎

This last corollary gives theorem 1.3.

If we restrict theorem 1.3 to the polynomial ring $k[T]$, we obtain the following theorem already proved in [2].

Theorem 3.5. *Let $\mathcal{I}$ denote the set of monic irreducible polynomials of the ring $k[T]$. Then, for N tending to $+\infty$,*

$$\prod_{\substack{P\in\mathcal{I}\\ \deg(P)\leq N}} \left(1 - \frac{1}{|P|}\right) = \frac{e^{-\gamma}}{N}\left(1 + O\left(\frac{1}{N}\right)\right), \tag{3.11}$$

with $|P| = q^{\deg P}$, the constants involved in the O depending only on q. ∎

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