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THE INFLUENCE OF DOMAIN GEOMETRY IN THE BOUNDARY BEHAVIOR OF LARGE SOLUTIONS OF DEGENERATE ELLIPTIC PROBLEMS

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Abstract: In this paper we study the asymptotic boundary behavior of large solutions of the equation $\Delta u = d^\alpha u^p$ in a regular bounded domain Ω in $\mathbb{R}^N$, $N \geq 2$, where $d(x)$ denotes the distance from x to $\partial\Omega$, $p>1$ and $\alpha>0$. We precise the expansion which depends on the mean curvature of the boundary.

1 – Introduction: notations and main results

Let Ω be a regular bounded domain in $\mathbb{R}^N$, $N \geq 2$, $p>1$ and $\alpha>0$. We denote by $d(x)$ the distance from x to $\partial\Omega$, the boundary of Ω. In this paper we consider the semilinear degenerate equation

$$\Delta u = d^\alpha u^p \quad \text{in} \quad \Omega \tag{1}$$

and we are interesting in the large solutions of (1), that is solutions of (1) which blow up at the boundary:

$$u(x) \to +\infty \quad \text{as} \quad d(x) \to 0 \; . \tag{2}$$

Note already that the maximum principle implies that the solutions $u \in C^2(\Omega)$ of (1)–(2) are positive in Ω.

Equation (1) registers in problems of the form

$$\Delta u = p(x)\, f(u) \quad \text{in} \quad \Omega \; . \tag{3}$$

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Those problems were first studied by Bieberbach [4] for the case $p(x)=1$, $f(u)=e^u$ and $N=2$, in the context of Riemannian surfaces of constant negative curvature, and the theory of automorphic functions. The case $p(x)>0$ for all $x\in\overline{\Omega}$ has been, largely dealt with in the literature (see [7], [12], [8], [5] for example).

Existence of solutions of (1)–(2) was established by Lair and Wood [9]. The question of the uniqueness of solutions of (1)–(2) is more delicate. When $\alpha=0$ and $p>1$, it is well know that problem (1)–(2) has a unique solution which satisfies

$$\lim_{d(x)\to 0} u(x)\, d(x)^{\frac{2}{p-1}} = \left(\frac{2\,(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}} . \tag{4}$$

This was first established by Loewner and Nirenberg [10] for the case $p=(N+2)/(N-2)$. Later we can find many extensions, see for example [1], [2] and [14] and the references cited there. The case $\alpha<0$ and $p>0$ is studied in [6]. In the general case $\alpha\geq 0$, Marcus and Véron proved the uniqueness of the solutions of (1)–(2) under the condition $1<p<(N+1+\alpha)/(N-1)$. Our first theorem completes this result and gives the rate of the blow-up.

Theorem 1.1. *Let $u\in C^2(\Omega)$ be a solution of (1)–(2). Then it satisfies*

$$\lim_{d(x)\to 0} d(x)^{\frac{\alpha+2}{p-1}}\, u(x) = l \tag{5}$$

where l is given by

$$l = \left[\frac{(\alpha+2)\,(\alpha+p+1)}{(p-1)^2}\right]^{\frac{1}{p-1}} . \tag{6}$$

This theorem allows us to establish the uniqueness of solutions of (1)–(2) with no conditions on p and α.

Theorem 1.2. *Problem (1) possesses a unique large solution.*

In the second time we are interested in the influence of the geometry of Ω in the boundary behavior. When $\alpha=0$, this problem was first studied by Bandle and Marcus [3] for the radially symmetric solutions of (1)–(2) in a ball. Later their result was extended by del Pino and Letelier [13] for general solutions. They proved that a lower-order term, still explosive, appears in the expansion of u which depends linearly of the mean curvature of the boundary of Ω. More precisely, if $1<p<3$ and $\alpha=0$, then on a sufficiently small neighborhood of $\partial\Omega$ we have the expansion

$$u(x) = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}} d(x)^{-\frac{2}{p-1}} \left\{1+\frac{N+1}{p+3}\,H(\overline{x})\,d(x)+o\big(d(x)\big)\right\}. \tag{7}$$

Here, for all x in a neighborhood of $\partial\Omega$, $\overline{x}$ denotes the unique point of the boundary such that $d(x)=|x-\overline{x}|$ and $H(\overline{x})$ the mean curvature of the boundary at that point. Estimate (7) generalizes to our case $\alpha\geq 0$ in the following way.

Theorem 1.3. *Let $u\in C^2(\Omega)$ a large solution of (1). Then, on a sufficiently small neighborhood of $\partial\Omega$:*

$$u(x) = l\,d(x)^{-\frac{2+\alpha}{p-1}} \left\{1+\frac{N-1}{\alpha+p+3}\,H(\overline{x})\,d(x)+o\big(d(x)\big)\right\}. \tag{8}$$

This theorem implies that on a sufficiently small neighborhood of $\partial\Omega$:

$$u(x)-l\,d(x)^{-\frac{2+\alpha}{p-1}} = \frac{N-1}{\alpha+p+3}\,H(\overline{x})\,d(x)^{-\frac{\alpha+3-p}{p-1}} + o\Big(d(x)^{-\frac{\alpha+3-p}{p-1}}\Big). \tag{9}$$

Therefore, we obtain that

– if $p>\alpha+3$, then the first member of (9) tends to 0 at the boundary,
– if $p=\alpha+3$, then $u(x)-l\,d(x)^{-\frac{2+\alpha}{p-1}}=\frac{N-1}{\alpha+p+3}\,H(\overline{x})+o(1)$,
– if $p<\alpha+3$, then the first member of (9) is not bounded and the blow-up depends on the mean curvature. Roughly, the "more curved" or "sharper" towards the exterior of Ω is around a given point of $\partial\Omega$, the higher the explosion rate at that point is.

That is a generalization of the results of Bandle and Marcus [3] for the radially symmetric solutions of (1)–(2) in a ball $\Omega=B(0,R)$:

– if $p>3$, then $u(r)-l(R-r)^{-\frac{2}{p-1}}\to 0$ when $r\to R$,
– if $p=3$, then $u(r)-l(R-r)^{-\frac{2}{p-1}}\to\frac{C(N)}{R}$ when $r\to R$, which represents the mean curvature of the ball,
– if $p<3$, then $u(r)-l(R-r)^{-\frac{2}{p-1}}\to\infty$ when $r\to R$.

Our paper is organized as follows:

1. Introduction
2. Asymptotic behavior and uniqueness
3. Boundary influence in the explosion rate.

2 – Asymptotic behavior and uniqueness

We begin this section by proving a classical estimate for all solution u of (1) (see [12]).

Proposition 2.1 (Osserman estimate)**.** *There exist two positive constants $a=a(\partial\Omega)$ and $C=C(\Omega,\alpha,p)$ such that for all solution $u\in C^2(\Omega)$ of equation (1), we have:*

$$u(x) \le C\, d(x)^{-\frac{2+\alpha}{p-1}} \tag{10}$$

for all $x\in\Omega$ such that $d(x)<a$.

Proof: Since Ω is regular there exist $\tilde a=\tilde a(\Omega)>0$ and $M=M(\Omega)>0$ such that

$$|\Delta d(x)| \le M\ , \qquad |\nabla d(x)| = 1 \tag{11}$$

for all $x\in\Omega$ such that $d(x)<\tilde a$. Set $a=\min(1,\frac{\tilde a}{2})$. Let $x_0\in\Omega$ such that $d(x_0)<a$ and $r_0=d(x_0)/2$. We denote by B_0 the ball centered at x_0 of radius r_0 and we define the function w in B_0 as follows:

$$w(x) \;=\; \lambda\, d(x)^{-\frac{\alpha}{p-1}} \Big(r_0^2-|x-x_0|^2\Big)^{-\frac{2}{p-1}} \tag{12}$$

with $\lambda>0$ to determine such that

$$-\Delta w + d^\alpha w^p \ge 0 \qquad \text{in} \quad B_0\ . \tag{13}$$

A straightforward computation gives:

$$\begin{aligned}
-\Delta w + d^\alpha w^p \;=\; &\lambda\, d^{-\frac{\alpha}{p-1}} \Big(r_0^2-|x-x_0|^2\Big)^{-\frac{2p}{p-1}} \;\times \\
&\Bigg[-\frac{\alpha\,(\alpha+p-1)}{(p-1)^2}\Big(r_0^2-|x-x_0|^2\Big)^2 d^{-2} + \frac{\alpha}{p-1}\Big(r_0^2-|x-x_0|^2\Big)^2 d^{-1}\Delta d \\
&+\frac{8\alpha}{(p-1)^2}\Big(r_0^2-|x-x_0|^2\Big)\, d^{-1}\nabla d.(x-x_0) \;-\; \frac{8(p+1)}{(p-1)^2}\,|x-x_0|^2 \\
&-\frac{4N}{p-1}\Big(r_0^2-|x-x_0|^2\Big) + \lambda^{p-1}\Bigg]\ .
\end{aligned}$$

Since $|x-x_0| < d(x_0) \le 1$, $d(x) \ge d(x_0)/2$ and $r_0^3 < r_0^2$, there exists a constant $L = L(\alpha, p, M) > 0$ such that

$$-\Delta w + d^\alpha w^p \;\ge\; \lambda d^{-\frac{\alpha}{p-1}} \Big(r_0^2 - |x-x_0|^2\Big)^{-\frac{2p}{p-1}} \big(-L r_0^2 + \lambda^{p-1}\big)$$

in B_0. Therefore, we choose $\lambda = L^{\frac{1}{p-1}} r_0^{\frac{2}{p-1}}$ and we obtain (13). Note that $w(x) = +\infty$ if $x \in \partial B_0$ because $-2/(p-1) < 0$. The comparison principle implies $u \le w$ in B_0 and in particular

$$u(x_0) \;\le\; w(x_0) \;=\; L^{\frac{1}{p-1}} \left(\frac{d(x_0)}{2}\right)^{\frac{2}{p-1}} \big(d(x_0)\big)^{-\frac{\alpha}{p-1}} \left(\frac{d(x_0)}{2}\right)^{-\frac{4}{p-1}}$$

which gives inequality (10). ∎

We now establish an estimate from below for the solutions of (1)–(2). The results of [1] and [2] can't be used because the distance function d is not positive in $\overline{\Omega}$. Nevertheless we can adapt them as follows.

Proposition 2.2. *Let $u \in C^2(\Omega)$ be a solution of (1)–(2). Then*

$$\liminf_{d(x)\to 0} d(x)^{\frac{\alpha+2}{p-1}} u(x) \;\ge\; l \tag{14}$$

where l is defined in (6).

Proof: Let $\varepsilon > 0$, $\tilde{a}$ be as the proof of Proposition 2.1 and $\beta \in (0,1)$. We define

$$\underline{u}(x) \;=\; \beta l \Big(\big(d(x)+\varepsilon\big)^{-\frac{\alpha+2}{p-1}} - (\overline{a}+\varepsilon)^{-\frac{\alpha+2}{p-1}} \Big)$$

where $\overline{a}$ will be determined such that $\overline{a} < \tilde{a}$. We have $\underline{u} > 0$ on $\partial\Omega$ and $\underline{u}(x) = 0$ for all x such that $d(x) = \overline{a}$. Moreover a straightforward computation yields

$$\begin{aligned} -\Delta \underline{u} + d^\alpha \underline{u}^p \;=\; \beta \Bigg[&\Delta d \left(\frac{\alpha+2}{p-1}\right) l (d+\varepsilon)^{-\frac{\alpha+p+1}{p-1}} \;-\; l^p (d+\varepsilon)^{-\frac{\alpha+2p}{p-1}} \\ &+ d^\alpha \beta^{p-1} l^p \Big((d+\varepsilon)^{-\frac{\alpha+2}{p-1}} - (\overline{a}+\varepsilon)^{-\frac{\alpha+2}{p-1}} \Big)^p \Bigg] \end{aligned}$$

in $0 < d(x) < \overline{a}$. Using inequality (11), we obtain

$$-\Delta \underline{u} + d^\alpha \underline{u}^p \;\le\; \beta l^p (d+\varepsilon)^{-\frac{\alpha+2p}{p-1}} \left[M \left(\frac{\alpha+2}{p-1}\right) l^{1-p} (d+\varepsilon) - 1 + \beta^{p-1} \left(\frac{d}{d+\varepsilon}\right)^\alpha \right]$$

which implies

$$-\Delta \underline{u} + d^\alpha \underline{u}^p \;\le\; \beta l^p (d+\varepsilon)^{-\frac{\alpha+2p}{p-1}} \left[\overline{M}(d+\varepsilon) - (1-\beta^{p-1})\right]$$

with $\overline{M} = M\big(\frac{\alpha+2}{p-1}\big)\, l^{1-p}$. We now choose $\overline{a} = \frac{1}{2}\min\big(\tilde{a}, \frac{1-\beta^{p-1}}{\overline{M}}\big)$ and impose $\varepsilon < \frac{1}{2}\big(\frac{1-\beta^{p-1}}{\overline{M}}\big)$.

Then $\overline{u}$ is a subsolution of (1) in $0 < d(x) < \overline{a}$. By the maximum principle we derive $\underline{u} \le u$ in $0 < d(x) < \overline{a}$. Letting ε tend to 0, this implies for all $\beta \in (0,1)$ and x such that $d(x) < \overline{a}$:

$$\beta l \left[1 - \left(\frac{d(x)}{\overline{a}}\right)^{\frac{\alpha+2}{p-1}}\right] \;\le\; d(x)^{\frac{\alpha+2}{p-1}}\, u(x)\ .$$

Therefore for all $\beta \in (0,1)$:

$$\beta l \;\le\; \liminf_{d(x)\to 0} d(x)^{\frac{\alpha+2}{p-1}}\, u(x)$$

which ends the proof. ∎

Because of Proposition 2.2, we can describe the asymptotic behavior of radially symmetric solutions of (1)–(2).

Proposition 2.3. *Let $R > 0$ and $v \in C^2(0,R)$ a solution of*

$$-\,v'' - \frac{N-1}{r}\, v' + (R-r)^\alpha v^p \;=\; 0 \qquad (15)$$

in $(0,R)$ such that

$$\lim_{r\to R} v(r) \;=\; +\infty\ .$$

Then

$$\lim_{r\to R} (R-r)^{\frac{\alpha+2}{p-1}}\, v(r) \;=\; l \qquad (16)$$

where l is defined in (6). ∎

We omit the proof of this proposition because it follows the idea of [14]: the function $w(t) = (R-r)^{\frac{\alpha+2}{p-1}}\, v(r)$ with $R - r = e^{-t}$ is bounded and satisfies a second order differential equation in a neighborhood of infinity and the ω-limit set of a trajectory of that equation is $\{0\}$ or $\{l\}$. Therefore Proposition 2.2 implies Proposition 2.3. Those results allows us to prove Theorem 1.1.

Proof of Theorem 1.1: In view of (14) we must only prove that

$$\limsup_{d(x)\to 0} d(x)^{\frac{\alpha+2}{p-1}} u(x) \;\leq\; l \;. \tag{17}$$

Still the results of [1], [2] or [14] don't apply directly but we can adapt them. Let $y \in \partial\Omega$. Since $\partial\Omega$ is smooth, there exists a ball B_y centered at a point Y of radius R_y such that $B_y \subset \Omega$ and $\overline{B_y} \cap \partial\Omega = \{y\}$. We introduce the function V defined by $V(x) = v(|x|)$ for all $x \in B_{R_y}$ where v is a function as in Proposition 2.3 with $R = R_y$. The function v exists because it is the radial solution of (1)–(2) for $\Omega = B$ (see [9]). Let $k > 1$. Finally we introduce the function V_k defined by $V_k(x) = k^{\frac{2}{p-1}}\, V\big(k(x-Y)\big)$ for all $x \in B\big(Y, \frac{R_y}{k}\big)$. Note that $B\big(Y, \frac{R_y}{k}\big) \subset B_y$ and V_k is solution of

$$-\Delta V_k + \Big(R_y - k|x-Y|\Big)^{\alpha}\, V_k^p \;=\; 0$$

in $B\big(y, \frac{R_y}{k}\big)$ and satisfies

$$\lim_{|x-Y|\to \frac{R_y}{k}} V_k(x) \;=\; +\infty \;.$$

Since $x \in B\big(Y, \frac{R_y}{k}\big)$ implies $d(x) \geq R_y - |x-Y| \geq R_y - k|x-Y|$, the comparison principle involves $u \leq V_k$ in $B\big(Y, \frac{R_y}{k}\big)$. Letting k tend to 1, we obtain

$$u(x) \;\leq\; v\big(|x-Y|\big) \qquad \text{in} \quad B_y \;. \tag{18}$$

Because of Proposition 2.3, for all $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\Big| s^{\frac{\alpha+2}{p-1}}\, v(R_y - s) - l \Big| < \varepsilon \qquad \forall s \in (0, \eta) \;. \tag{19}$$

Let $\tilde\eta > 0$ be sufficiently small so that for all $x \in \Omega$ with $d(x) < \tilde\eta$ there exists a unique $y \in \partial\Omega$ such that $|x-y| = d(x)$. Then for all $x \in \Omega$ such that $d(x) < \min(\eta, \tilde\eta)$, both inequalities (18) and (19) imply

$$d(x)^{\frac{\alpha+2}{p-1}}\, u(x) \;\leq\; d(x)^{\frac{\alpha+2}{p-1}}\, v\big(R_y - d(x)\big) \;<\; l + \varepsilon$$

and inequality (17) holds. ∎

Proof of Theorem 1.2: Large solutions of (1) satisfy (5). Then two large solutions u_1 and u_2 of (1) are such that

$$\lim_{d(x)\to 0} \frac{u_1(x)}{u_2(x)} = 1 \;.$$

and the result follows as in [1] or [11]. ∎

3 – Boundary influence in the explosion rate

In this section we prove Theorem 1.3. As in [13] we construct suitable sub- and supersolutions of (1) in a neighborhood of $\partial\Omega$ which are inspired of the radial study that we omit here.

Since Ω is regular there exists $\overline{b} > 0$ such that d is a function of class C^2 in $\{x \in \Omega \ / \ d(x) < \overline{b}\}$, $|\nabla d(x)| = 1$ and

$$\Delta d(x) = -(N-1)\,H(\overline{x}) + o(1) \qquad \text{as} \quad d(x) \to 0\,. \tag{20}$$

Let $b_0 \in (0, \overline{b})$, $b \in (0, b_0)$ and $\varepsilon > 0$. We introduce the function Ψ defined in $E_{b,b_0} = \{x \in \Omega \ / \ b < d(x) < b_0\}$ by

$$\Psi(x) \;=\; l\Big(d(x)-b\Big)^{-\frac{\alpha+2}{p-1}} + \frac{l}{\alpha+p+3}\Big((N-1)\,H(\overline{x})+\varepsilon\Big)\Big(d(x)-b\Big)^{-\frac{\alpha+3-p}{p-1}}\,.$$

We claim that if b_0 is chosen sufficiently small, independently of ε and b, then Ψ is a supersolution in E_{b,b_0}. Indeed, a straightforward computation using (20) gives:

$$\begin{aligned}\Delta\Psi \;=\; & l^p\Big(d(x)-b\Big)^{-\frac{\alpha+2p}{p-1}} + l\Big(d(x)-b\Big)^{-\frac{\alpha+p+1}{p-1}}\Bigg[\frac{\alpha+2}{p-1}\,(N-1)\,H(\overline{x})\\ & + \frac{(\alpha+3-p)\,(\alpha+2)}{(\alpha+p+3)\,(p-1)^2}\Big((N-1)\,H(\overline{x})+\varepsilon\Big) + o(1)\\ & + \frac{\alpha+3-p}{(p-1)\,(\alpha+p+3)}\Big((N-1)\,H(\overline{x})+\varepsilon\Big)\Big((N-1)\,H(\overline{x})+o(1)\Big)\Big(d(x)-b\Big)\Bigg]\,.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}d(x)^\alpha\,\Psi^p \;\geq\; & \Big(d(x)-b\Big)^{\alpha}\,\Psi^p\\ \geq\; & l^p\Big(d(x)-b\Big)^{-\frac{\alpha+2p}{p-1}} \;\times\\ & \Bigg[1+\frac{p}{\alpha+p+3}\Big((N-1)\,H(\overline{x})+\varepsilon\Big)\Big(d(x)-b\Big)+o\Big(d(x)-b\Big)\Bigg]\,.\end{aligned}$$

Then

$$
\begin{aligned}
-\Delta\Psi + d^\alpha \Psi^p \;\geq&\\
\geq\; & l\Big(d(x)-b\Big)^{-\frac{\alpha+p+1}{p-1}} \times \\
& \Bigg[-\frac{\alpha+2}{p-1}\,(N-1)\,H(\overline{x}) - \frac{(\alpha+3-p)\,(\alpha+2)}{(\alpha+p+3)\,(p-1)^2}\Big((N-1)\,H(\overline{x})+\varepsilon\Big) \\
& - \frac{\alpha+3-p}{(p-1)\,(\alpha+p+3)}\Big((N-1)\,H(\overline{x})+\varepsilon\Big)\Big((N-1)\,H(\overline{x})+o(1)\Big)\Big(d(x)-b\Big) \\
& + \frac{l^{p-1}p}{\alpha+p+3}\Big((N-1)\,H(\overline{x})+\varepsilon\Big) + o(1)\Bigg] .
\end{aligned}
$$

Since

$$
-\frac{\alpha+2}{p-1} - \frac{(\alpha+3-p)\,(\alpha+2)}{(\alpha+p+3)\,(p-1)^2} + \frac{l^{p-1}p}{\alpha+p+3} \;=\; 0 \;,
$$

and since the coefficient of ε is $(\alpha+2)/(p-1)$, it implies that there exists $b_0 = b_0(\varepsilon) \in (0,\overline{b})$ such that for all $0<b<b_0$:

$$
-\Delta\Psi + d^\alpha\,\Psi^p \geq 0 \qquad \text{in} \quad E_{b,b_0} \;.
$$

Consider the solution u of (1)–(2). We claim that there exists a positive number K independent of $b \in (0,b_0)$ such that:

$$
\Psi(x) + K \;\geq\; u(x) \tag{21}
$$

for all $x\in\Omega$ with $d(x)=b_0$. In fact, if we define

$$
M_0 \;=\; \max_{d(x)=b_0} u(x) \;,
$$

we can compute for all x such that $d(x)=b_0$:

$$
\Psi(x) \;=\; l\big(b_0-b\big)^{-\frac{\alpha+2}{p-1}} + \frac{l}{\alpha+p+3}\Big((N-1)\,H(\overline{x})+\varepsilon\Big)\big(b_0-b\big)^{-\frac{\alpha+3-p}{p-1}} \;.
$$

Since $\partial\Omega$ is regular, there exists a real $b_1\in(0,b_0)$ such that

$$
\left|\frac{1}{\alpha+p+3}\Big((N-1)\,H(\overline{x})+\varepsilon\Big)\,(b_0-b)\right| \;\leq\; \frac{1}{2}
$$

for all $b\in(b_1,b_0)$, where $\overline{x}$ is such that $d(x)=|x-\overline{x}|$. Therefore

$$
1 + \frac{1}{\alpha+p+3}\Big((N-1)\,H(\overline{x})+\varepsilon\Big)\,(b_0-b) \;\geq\; \frac{1}{2}
$$

and then

$$\Psi(x) \;\geq\; \frac{l}{2}\big(b_0-b\big)^{-\frac{\alpha+2}{p-1}} \;\geq\; \frac{l}{2}\big(b_0-b_1\big)^{-\frac{\alpha+2}{p-1}}$$

for all $b \in (b_1, b_0)$, where $\overline{x}$ is such that $d(x) = |x-\overline{x}|$. On the other hand, for all $b \in (0, b_1]$ and $d(x) = b_0$:

$$\begin{aligned}\Psi(x) \;&=\; l\big(b_0-b\big)^{-\frac{\alpha+2}{p-1}} + \frac{l}{\alpha+p+3}\Big((N-1)\,H(\overline{x})+\varepsilon\Big)\big(b_0-b\big)^{-\frac{\alpha+3-p}{p-1}}\\ &\geq\; l\,b_0^{-\frac{\alpha+2}{p-1}} - C\big(b_0-b_1\big)^{-\frac{\alpha+3-p}{p-1}}\end{aligned}$$

with $C>0$ and because the assumption if we assume $\alpha+3-p>0$ (we omit the proof in the case $\alpha+3-p \leq 0$ which is simpler). Finally we obtain for all $b \in (0, b_0)$:

$$\Psi(x) \;\geq\; L \;=\; \min\left(\frac{l}{2}\big(b_0-b_1\big)^{-\frac{\alpha+2}{p-1}},\; l\,b_0^{-\frac{\alpha+2}{p-1}} - C\big(b_0-b_1\big)^{-\frac{\alpha+3-p}{p-1}}\right),$$

then, for all x such that $d(x) = b_0$,

$$u \;\leq\; M_0 \;\leq\; \max(1, M_0-L) + L \;\leq\; \max(1, M_0-L) + \psi$$

which implies (21).

On the other hand the function $\Psi + K$ is itself a supersolution of equation (1) in E_{b,b_0}. Therefore the comparison principle implies (21) in E_{b,b_0}. Letting b tend to 0, we obtain

$$u(x) \;\leq\; l\,d(x)^{-\frac{\alpha+2}{p-1}} + \frac{l}{\alpha+p+3}\Big((N-1)\,H(\overline{x})+\varepsilon\Big)\,d(x)^{-\frac{\alpha+3-p}{p-1}} + K$$

for all $x \in \Omega$ such that $0 < d(x) < b_0$. In the same way, by considering subsolutions in the form

$$\phi(x) \;=\; l\Big(d(x)+b\Big)^{-\frac{2+\alpha}{p-1}} + \frac{l}{\alpha+p+3}\Big((N-1)\,H(\overline{x})-\varepsilon\Big)\,\Big(d(x)+b\Big)^{-\frac{\alpha+3-p}{p-1}} - \overline{K}$$

we obtain expansion (8). ∎

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