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A CLASS OF HIGH ORDER TVD SCHEMES FOR SYSTEMS OF CONSERVATION LAWS

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Abstract: The purpose of this paper is to develop improved finite difference schemes. We firstly propose to use the fourth order TVD flux presented in [21] as the building block in ENO schemes for scalar problems. A way to extend this scheme and the scheme [20] to general systems of nonlinear hyperbolic conservation laws, in one and two dimensions, is discussed in details. A semi-discrete version of the two schemes are presented. The performance of the schemes is assessed by solving test problems for the Euler equations of gas dynamics in one and two dimensions. We use exact solutions and other results to validate the results.

1 – Introduction

In recent years there has been a substantial and productive effort to develop computational techniques for partial differential equations, particularly the case for conservation laws.

In [21], we established a centered TVD fourth order finite difference scheme for solving linear and nonlinear scalar hyperbolic conservation laws. It is a combination of MUSCL-Hancock (upwind) approach and centered second order scheme [19].

In [20] a second order TVD scheme is used as a building block for designing high order essentially non-oscillatory (ENO) method for scalar problems. It is called TVD-ENO scheme. The resulting scheme improve upon the original ENO and TVD schemes in terms of better convergence, higher overall accuracy and better resolution of discontinuities.

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In this paper we propose to use the fourth order TVD flux presented in [21] as the building block in ENO schemes for scalar problems. A way to extend this scheme and the scheme [20] to general systems of nonlinear hyperbolic conservation laws, in one and two dimensions, is discussed in details. For constant coefficients linear hyperbolic systems the extension is straightforward. Recall that these finite difference schemes are high resolution schemes and can accommodate arbitrary wave directions, so they can automatically deal with linear systems with eigenvalues of mixed sign. For nonlinear hyperbolic systems, however, the matrix of eigenvalues is not constant, which means that the numerical flux cannot be defined directly. Hence the approach for linear systems no longer holds when solving nonlinear systems.

In this paper we discuss the extension of the two schemes to nonlinear systems of conservation laws in one and two dimensions.

It is well known that when using a second (or higher) order accurate schemes, accuracy is gained in smooth parts of the solution, but the waves are accompanied by spurious oscillations and near discontinuities overshoots or undershoots are accompanied by spurious oscillations are produced.

This problem had frustrated people for many years until the concept and theory of total variation diminishing (TVD) schemes was introduced by Harten [6]. The main property of TVD schemes is that it can be second (or higher) order and oscillations free across discontinuities.

In order to make the schemes oscillations free (TVD) we present a technique that is different to this of original schemes. We add some limiter functions to the flux. [15], [14].

It is well known that the semi-discrete schemes are especially effective when they combine high resolution, non-oscillatory spatial discretisation with high order, large step size ordinary differential equations solvers for their time evolution.

In this paper we propose the semi-discrete version of our schemes presented here with Runge-Kutta method for time discretisation.

These schemes are validated by applications to the Euler equations of gas dynamics in one and two dimensions.

The rest of the paper is organized in the following manner: section 2 briefly reviews the ENO reconstruction for the scalar case. Section 3 reviews the numerical fluxes used in our methods. In section 4 Extension of the schemes to one dimensional conservation laws is presented in section 5. Section 6 contained the application of our methods to the Euler equations. The semi-discrete version of our schemes is presented in section 7. Numerical results for one dimensional problems are presented in section 8. In section 9 the extension to two dimensional problems is presented.

2 – Numerical methods

We are concerned with the approximations of scalar hyperbolic conservation law

$$u_t + f(u)_x = 0 \ , \qquad -\infty < x < \infty, \quad t \geq 0 \ , \tag{2.1a}$$

subjected to the initial condition

$$u(x,0) = u_0(x) \ . \tag{2.1b}$$

To approximate solution of (2.1) we discretise both space and time assuming uniform mesh spacing Δx and Δt respectively. We denote the spatial grid points by $x_j = j\,\Delta x$ and the time steps by $t^n = n\,\Delta t$. Assuming that the solutions u_j^n at time $t^n = n\,\Delta t$ are known, our goal is to compute the solution at the next time step t^{n+1}.

From the values $\{\bar{u}_j^n\}_{j=0}^N$ we reconstruct the point-values of the function $u(x,t^n)$ via a suitable nonlinear piecewise polynomial interpolation $P_j(x)$ of degree at most $(k-1)$, for each cell I_j where $I_j = \big[x_{j-1/2}, x_{j+1/2}\big]$. We use here the ENO reconstruction [7]. We will describe our implementation of the reconstruction step in section (2.1) for completeness. As a result, at each cell interface $x_{j+1/2}$ the reconstruction produces two different values of the function $u(x)$, namely the left extrapolated values and right extrapolated value: $u^{\mathrm{L}}_{j+1/2} = P_j(x_{j+1/2})$, $u^{\mathrm{R}}_{j+1/2} = P_{j+1}(x_{j+1/2})$.

A sequence of Riemann Problems (RP) with initial data $u^{\mathrm{L}}_{j+1/2}$, $u^{\mathrm{R}}_{j+1/2}$ is then constructed. The evolution of the discontinuous data $u(x,t^n)$ can be computed by solving (exactly or approximately) these Riemann problems.

This is the framework of upwind methods.

On the other hand, in central schemes, the solution is updated by the fully discrete scheme

$$u_j^{n+1} = u_j^n - \lambda\big[f_{j+1/2} - f_{j-1/2}\big] \ , \qquad \lambda = \Delta t/\Delta x \ , \tag{2.2}$$

where $f_{j+1/2}$ is the numerical flux at $x_{j+1/2}$.

The numerical flux function at the cell boundaries $x_{j+1/2}$ is defined as a monotone function of left and right extrapolated values $u^{\mathrm{L}}_{j+1/2}$, $u^{\mathrm{R}}_{j+1/2}$:

$$f_{j+1/2} \;=\; f(u_{j+1/2}, t) \;=\; f_{j+1/2}(u^{\mathrm{L}}_{j+1/2}, u^{\mathrm{R}}_{j+1/2}) \ . \tag{2.3}$$

In the next subsection, we will present the ENO reconstruction which supplies the required piecewise polynomial $P_j(x)$.

2.1. ENO reconstruction

In this section, we present the ENO reconstruction, which will be then utilized in the next sections to construct our methods.

Given the location I_j and the order of accuracy k, we first choose "stencil", based on r cells to the left, s cells to the right and I_j itself, if $r, s \geq 0$, with $r + s = k - 1$:

$$S(j) = \left\{ I_{j-r}, ..., I_{j+s} \right\} . \tag{2.4}$$

The third order ENO reconstruction ($k = 3$) is given by [7]

$$P_j(x_{j+1/2}) \;=\; u^{\mathrm{L}}_{j+1/2} \;=\; -\frac{1}{6}\, u_{j-1} + \frac{5}{6}\, u_j + \frac{1}{3}\, u_{j+1} \tag{2.5}$$

and the fourth ENO reconstruction is given by

$$P_j(x_{j+1/2}) \;=\; u^{\mathrm{L}}_{j+1/2} \;=\; -\frac{1}{12}\, u_{j-1} + \frac{7}{12}\, u_j + \frac{7}{12}\, u_{j+1} - \frac{1}{12}\, u_{j+2} \;. \tag{2.6}$$

The right values $u^{\mathrm{R}}_{j+1/2}$ is obtained by symmetry.

3 – Numerical fluxes

In this section we review fluxes associated with fully discrete second and fourth order TVD schemes which will be then used as the building block for the high order ENO schemes.

3.1. Fourth order TVD difference scheme

The initial value problem (IVP) for the one-dimensional scalar hyperbolic conservation law is considered, namely

$$u_t + f(u)_x = 0 \;, \qquad -\infty < x < \infty, \quad t \geq 0 \;, \tag{3.1a}$$

$$u(x, 0) = u_0(x) \;, \tag{3.1b}$$

where f is the flux and $f'(u) = a$ is constant wave (characteristic) speed.

If the initial data is consisting of two constant states,

$$u_0 \;=\; \begin{cases} u_{\mathrm{L}} & x < 0 \;, \\ u_{\mathrm{R}} & x > 0 \;, \end{cases} \tag{3.1c}$$

equation (3.1) is called the usual Riemann problem (RP).

In [21] the MUSCL-Hancock approach was used to construct centered scheme, whereby the Godunov first order upwind method is replaced by the second order centered scheme [19], thus eliminating the need of Riemann problem altogether. The resulting scheme takes the conservative form

$$u_j^{n+1} = u_j^n - \lambda[f_{j+1/2} - f_{j-1/2}] \tag{3.2}$$

with the numerical flux

$$\begin{aligned} f_{j+1/2} &= \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2}|a|\,\Delta_{j+1/2}u \\ &\quad + |a|\Big\{A_0\,\Delta_{j+1/2}u + A_1\,\Delta_{j+L+1/2}u + A_2\,\Delta_{j+M+1/2}u\Big\} \\ &\quad + |a|\Big\{A_3\,\Delta_{j+S+1/2}u + A_4\,\Delta_{j+Q+1/2}u\Big\}\,, \end{aligned} \tag{3.3}$$

where

$$\begin{cases} A_0 = \dfrac{1}{192}\Big\{59 + 96\,\omega + 48c + 8c^2 - 48c^2\omega\Big\}, \\ A_1 = \dfrac{1}{192}\Big\{176 + 24c - 36\,\omega + 12c^2\omega\Big\}, \\ A_2 = \dfrac{1}{192}\Big\{24 + 24c - 8c^2 - 24\,c\,\omega\Big\}, \\ A_3 = \dfrac{1}{192}\Big\{-6 + 6c^2 + 6\,\omega - 6c^2\omega\Big\}, \\ A_4 = \dfrac{1}{192}\Big\{6 - 6c^2 - 6\,\omega + 6c^2\omega\Big\}\,, \end{cases} \tag{3.4}$$

$L=-1$, $M=1$, $S=-2$, $Q=2$ for $c>0$ and $L=1$, $M=-1$, $S=2$, $Q=-2$ for $c<0$. Here ω is a free parameter in the real interval $[-1,1]$. Where $c=\lambda\,a$ is the Courant number and $\Delta_{j+1/2}u = u_{j+1} - u_j$. The scheme (3.2)–(3.4) is stable for $|c| \le 1$.

Theorem 1. *Scheme (3.2)–(3.4) is third order accurate in space and time for any value $\omega$and it is fourth order accurate in space and time when*

$$\omega = \frac{64c^3 - 32c^2 - 256c + 157}{384 - 192c^2}\,.$$

Proof: See [21]. ∎

This scheme, being third and fourth order accurate scheme, is not TVD. It can be made TVD by replacing (2.3) with the more general form

$$
\begin{aligned}
F_{j+1/2} = {} & \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2}|a|\,\Delta_{j+1/2}u \\
& + |a|\Big\{A_0\,\Delta_{j+1/2}u + A_1\,\Delta_{j+L+1/2}u\Big\}\,\phi_j \\
& + |a|\Big\{A_2\,\Delta_{j+M+1/2}u + A_3\,\Delta_{j+S+1/2} + A_4\,\Delta_{j+Q+1/2}u\Big\}\,\phi_{j+M}\;,
\end{aligned}
\tag{3.5}
$$

where ϕ_j and ϕ_{j+M} are flux limiter functions.

3.2. Second order TVD

The second order TVD scheme is introduced in [19] has the form (3.2) with the numerical flux

$$
\begin{aligned}
f_{j+1/2} = {} & \frac{1}{2}(a u_j + a u_{j+1}) - \frac{1}{2}|a|\,\Delta_{j+1/2}u \\
& + |a|\Big\{A_0\,\Delta_{j+1/2}u + A_1\,\Delta_{j+L+1/2}u\Big\}\,\varphi_j \\
& + |a|\,A_2\,\Delta_{j+M+1/2}u\;\phi_{j+M}\;,
\end{aligned}
\tag{3.6a}
$$

$$
A_0 = \frac{1}{2} - \frac{|c|}{4}\;, \qquad A_1 = -\frac{1}{8} - \frac{|c|}{8}\;, \qquad A_2 = \frac{1}{8} - \frac{|c|}{8}\;,
\tag{3.6b}
$$

$L=-1$, $M=1$ for $c>0$ and $L=1$, $M=-1$ for $c<0$. Here ϕ_j and ϕ_{j+M} are flux limiter functions.

4 – TVD-ENO scheme

Very high order methods such as ENO methods, use high order polynomial reconstruction of the solution and a lower (first) order monotone flux as the building block. Toro [16] proposed to use the second order centred TVD flux, instead of first order fluxes, as a building block for designing high order schemes. In [20], we proposed to use the second order TVD flux (3.6) as a building block in the high order ENO schemes.

In this paper we use the fourth order flux (3.3) as a building block in the high order ENO schemes as follows:

i) Compute the ENO procedure to obtain the reconstructed value $u^{\mathrm{L}}_{j+1/2}, u^{\mathrm{R}}_{j+1/2}$

ii) Compute the flux (3.3) with $\Delta_{j+1/2}u = u^{\mathrm{R}}_{j+1/2} - u^{\mathrm{L}}_{j+1/2}$.

iii) Use the finite difference scheme (2.2) to compute the numerical solution of (2.1).

5 – Extension to systems of hyperbolic conservation laws

Extension of the scalar schemes mentioned in the last section to systems of conservation laws can be accomplished by defining at each point a "local" system of characteristic field, and applying the schemes to each of the m scalar characteristic equations. Here m is the dimension of the hyperbolic system.

Consider a system of hyperbolic conservation laws

$$U_t + \big[F(U)\big]_x = 0 \tag{5.1}$$

where U and $F(U)$ are column vectors of m components and $B = \partial F/\partial U$ is the Jacobian matrix. The assumption that (5.1) is hyperbolic implies that $B(U)$ has real eigenvalues $\{a^\ell(U)\}$ and a complete set of right eigenvectors $R^\ell(U)$, $\ell = 1, 2, ..., m$. Hence the matrix

$$R(U) = (R^1, R^2, ..., R^m) \tag{5.2a}$$

is invertible. Thus

$$R^{-1}AR = \operatorname{diag}(a^\ell)\ . \tag{5.2b}$$

Here $\operatorname{diag}(a^\ell)$ denotes a diagonal matrix with diagonal elements a^ℓ.

We define characteristic variables W with respect to the state U by

$$W = R^{-1}U\,. \tag{5.3}$$

In the constant coefficient, i.e., B is a constant matrix, (5.1) decouples into m scalar equations for the characteristic variables

$$\frac{\partial w^\ell}{\partial t} + a^\ell\,\frac{\partial w^\ell}{\partial x} = 0\ , \qquad a^\ell = \text{constant}\ . \tag{5.4}$$

This offers a natural way of extending a scalar scheme to a constant coefficient system by applying it "scalarly" to each of the m scalar characteristic equations (5.4).

We now come to the situation where B is not constant. The trouble is that now all the matrices $R(U)$, $R^{-1}(U)$ are dependent upon U. We must "freeze" them locally in order to carry out a similar procedure as in the constant coefficient case.

Let $U_{j+1/2}$ denote some symmetric average of U_j and U_{j+1} (see Roe [10]). Let $a^\ell_{j+1/2}$, $R_{j+1/2}$, $R^{-1}_{j+1/2}$ denote the quantities a^ℓ, R, R^{-1} evaluated at $U_{j+1/2}$.

Let w^ℓ be the vector elements of W, and let $\alpha^\ell_{j+1/2} = w^\ell_{j+1} - w^\ell_{j+1}$ be the component of in the ℓ-th characteristic direction, i.e., define

$$\alpha_{j+1/2} = R^{-1}_{j+1/2}\,\Delta_{j+1/2}U\,, \qquad \Delta_{j+1/2}U = R_{j+1/2}\,\alpha_{j+1/2}\,, \tag{5.5}$$

where $\alpha_{j+1/2}$ is called the wave strength vector with component $\alpha^\ell_{j+1/2}$ across the wave travelling at speed $a^\ell_{j+1/2}$.

To extend the TVD-ENO scheme to nonlinear systems of hyperbolic problems we do the following steps:

1) at each fixed $x_{j+1/2}$, do the following:

- **a**) compute an average states $U_{j+1/2}$, using Roe average,
- **b**) compute the eigenvalues of the Jacobian $B(U_{j+1/2})$, right eigenvectors and left eigenvectors and denote them by $a^\ell_{j+1/2}$, $R_{j+1/2}$, $R^{-1}_{j+1/2}$,
- **c**) transform all those variables to the local characteristic fields by using (5.3), i.e., $W_j = R_j^{-1}U_j$,
- **d**) perform the scalar ENO reconstructions, for each component of the characteristic variables W, to obtain the corresponding component of the reconstruction $W^{L,R}_{j+1/2}$,
- **e**) transform back into physical space by using $U = RW$.

2) Construct the numerical flux

$$\begin{aligned} F_{j+1/2} \;=\; & \frac{1}{2}\,(F_j + F_{j+1}) - \frac{1}{2}\sum_{\ell=1}^{m} |a^\ell_{j+1/2}|\;\alpha^\ell_{j+1/2}\,R^\ell_{j+1/2} \\ & + \sum_{\ell=1}^{m} A^\ell_{0,j+1/2}\,|a^\ell_{j+1/2}|\;\alpha^\ell_{j+1/2}\,R^\ell_{j+1/2} \\ & + \sum_{\ell=1}^{m} A^\ell_{1,j+L+1/2}\,|a^\ell_{j+L+1/2}|\;\alpha^\ell_{j+L+1/2}\,R^\ell_{j+L+1/2} \\ & + \sum_{\ell=1}^{m} A^\ell_{2,j+M+1/2}\,|a^\ell_{j+M+1/2}|\;\alpha^\ell_{j+M+1/2}\,R^\ell_{j+M+1/2} \\ & + \sum_{\ell=1}^{m} A^\ell_{3,j+S+1/2}\,|a^\ell_{j+S+1/2}|\;\alpha^\ell_{j+S+1/2}\,R^\ell_{j+S+1/2} \\ & + \sum_{\ell=1}^{m} A^\ell_{4,j+Q+1/2}\,|a^\ell_{j+Q+1/2}|\;\alpha^\ell_{j+Q+1/2}\,R^\ell_{j+Q+1/2} \end{aligned} \tag{5.6a}$$

where $a^\ell_{j+1/2}$, $R^\ell_{j+1/2}$ and $\alpha^\ell_{j+1/2}$ $(\ell = 1, 2, ..., m)$ are Roe-averaged eigenvalues, eigenvectors and wave strengths respectively, $A^\ell = f(c^\ell)$ are functions of the cell Courant number (see (3.4)).

d) Form the scheme

$$U_j^{n+1} = U_j^n - \lambda\big[F_{j+1/2} - F_{j-1/2}\big] \tag{5.6b}$$

to compute U_j^{n+1}.

In order to make the scheme (5.6) oscillations free (TVD), we propose the flux (5.6a) in general form [19] as

$$\begin{aligned} F_{j+1/2} = {} & \frac{1}{2}(F_j + F_{j+1}) - \frac{1}{2}\sum_{\ell=1}^{m} |a^\ell_{j+1/2}|\, \alpha^\ell_{j+1/2}\, R^\ell_{j+1/2} \\ & + \sum_{\ell=1}^{m} A^\ell_{0,j+1/2}\, |a^\ell_{j+1/2}|\, \alpha^\ell_{j+1/2}\, R^\ell_{j+1/2} \\ & + \sum_{\ell=1}^{m} A^\ell_{1,j+L+1/2}\, |a^\ell_{j+L+1/2}|\, \alpha^\ell_{j+L+1/2}\, R^\ell_{j+L+1/2} \\ & + \sum_{\ell=1}^{m} A^\ell_{2,j+M+1/2}\, |a^\ell_{j+M+1/2}|\, \alpha^\ell_{j+M+1/2}\, R^\ell_{j+M+1/2} \\ & + \sum_{\ell=1}^{m} A^\ell_{3,j+S+1/2}\, |a^\ell_{j+S+1/2}|\, \alpha^\ell_{j+S+1/2}\, R^\ell_{j+S+1/2} \\ & + \sum_{\ell=1}^{m} A^\ell_{4,j+Q+1/2}\, |a^\ell_{j+Q+1/2}|\, \alpha^\ell_{j+Q+1/2}\, R^\ell_{j+Q+1/2} - \frac{1}{2} R_{j+1/2}\, \Psi_{j+1/2} \end{aligned} \tag{5.7}$$

where $\Psi_{j+1/2}$ is a limiter vector whose elements denoted by $\psi^\ell_{j+1/2}$, $\ell = 1, 2, ..., m$ and

$$\psi^\ell_{j+1/2} = |a^\ell_{j+1/2}|\,(1 - \phi^\ell_{j+1/2})\, \alpha^\ell_{j+1/2}\ , \tag{5.8a}$$

where

$$\phi^\ell_{j+1/2} = \phi^\ell_{j+1/2}\,(r^\ell_{j+1/2})\ , \tag{5.8b}$$

$$r^\ell_{j+1/2} = \frac{\alpha^\ell_{j-1/2}}{\alpha^\ell_{j+1/2}}\ . \tag{5.8c}$$

The limiter function $\phi\,(r)$ is given by [15]

$$\phi(r) = \max\Big\{0,\ \min(Qr, 1),\ \min(r, Q)\Big\}\ , \qquad 1 \le Q \le 2\ . \tag{5.8d}$$

If $Q = 1$, we get

$$\phi(r) = \max\Big\{0,\ \min(r, 1)\Big\} \tag{5.9a}$$

and it is called minimod limiter. If $Q = 2$ we get

$$\phi(r) = \max\Big\{0,\ \min(2r, 1),\ \min(r, 2)\Big\} \tag{5.9b}$$

and it is called superbee limiter.

6 – Application to one-dimensional Euler equations of gas dynamics

In this section we describe how to apply the TVD scheme, discussed above, to the compressible inviscid equations of gas dynamics (Euler equations) which can be written in the conservation form as

$$U_t + \big[F(U)\big]_x = 0 \tag{6.1a}$$

where

$$U = (\rho, \rho\, u, E)^{\mathrm{T}}\ , \qquad F(U) = \big(\rho\, u,\ P + \rho\, u^2,\ u(E+P)\big)^{\mathrm{T}}\ . \tag{6.1b}$$

Here, u is the velocity, ρ the density, P the pressure and E the total energy,

$$E = \rho\, e + 0.5\, \rho\, u^2 \tag{6.1c}$$

sum of internal energy and kinetic energy; and e the specific internal energy. Equations (6.1) are closed by an equation of state for a fluid which can be written as

$$P = \rho\, e\, (\gamma - 1) \tag{6.1d}$$

where γ is the ratio of specific heat capacities of the fluid.

Let A denote the Jacobian matrix whose eigenvalues are $(u - S, u, u + S)$, where S is the local speed of sound

$$S^2 = (\gamma - 1)\left(e + \frac{P}{\rho}\right). \tag{6.2}$$

The eigenvectors of A form the matrix $R = (R^1, R^2, R^3)$ given by

$$R = \begin{bmatrix} 1 & 1 & 1 \\ u - S & u & u + S \\ H - u\, S & \frac{1}{2}\, u^2 & H + u\, S \end{bmatrix} \tag{6.3}$$

where the enthalpy, H, is defined by

$$H = \frac{E + P}{\rho}\ . \tag{6.4}$$

It is now a question of choosing a suitable average state to evaluate $a^{\ell}_{j+1/2}$, $R_{j+1/2}$, $R^{-1}_{j+1/2}$. Roe [10] suggested the following average values for the density, velocity, enthalpy and sound speed

$$\rho_{j+1/2} = \sqrt{\rho_j\,\rho_{j+1}}\;, \qquad u_{j+1/2} = \frac{u_j\sqrt{\rho_j} + u_{j+1}\sqrt{\rho_{j+1}}}{\sqrt{\rho_j}+\sqrt{\rho_{j+1}}}\;,$$

$$H_{j+1/2} = \frac{H_j\sqrt{\rho_j} + H_{j+1}\sqrt{\rho_{j+1}}}{\sqrt{\rho_j}+\sqrt{\rho_{j+1}}}\;, \quad S_{j+1/2} = \sqrt{(\gamma-1)\,(H_{j+1/2} - 0.5\,u^2_{j+1/2})}\;. \tag{6.5}$$

In terms of the above average values, the average of the eigenvalues become

$$\Big(a^1_{j+1/2},\, a^2_{j+1/2},\, a^3_{j+1/2}\Big) \;=\; \Big(u_{j+1/2} - S_{j+1/2},\; u_{j+1/2},\; u_{j+1/2} + S_{j+1/2}\Big)\,. \tag{6.6}$$

The wave strengths are

$$\Big(\alpha^1_{j+1/2},\, \alpha^2_{j+1/2},\, \alpha^3_{j+1/2}\Big) \;= \tag{6.7}$$

$$= \left(\frac{1}{2\,S^2_{j+1/2}}\Big(\Delta P - \rho_{j+1/2}\,S_{j+1/2}\,\Delta u\Big),\; \Delta\rho - \frac{\Delta P}{S^2_{j+1/2}},\; \frac{1}{2\,S^2_{j+1/2}}\Big(\Delta P + \rho_{j+1/2}\,S_{j+1/2}\,\Delta u\Big)\right)$$

where $\Delta(\cdot) = (\cdot)_{j+1} - (\cdot)_j$.

7 – Semi-discrete formulation

Another way to discretise (2.1) is to keep the time variable t continuous and consider semi-discrete schemes. Integrating (2.1) with respect to x only we obtain the following system of ordinary differential equations

$$\frac{d}{dt}\big(u_j(t)\big) \;=\; -\frac{1}{\Delta x}\,\big\{F_{j+1/2} - F_{j-1/2}\big\} \;=\; L_j(u)\;. \tag{7.1}$$

The numerical solution of (7.1) is advanced in time by means of a third order TVD Runge–Kutta method [9] as follows (here we dropped the index j)

$$\begin{aligned} u^{(1)} &= u^n + \Delta t\,L(u^n)\;,\\ u^{(2)} &= \frac{3}{4}\,u^n + \frac{1}{4}\,u^{(1)} + \frac{1}{4}\,\Delta t\,L(u^{(1)})\;,\\ u^{n+1} &= \frac{1}{3}\,u^n + \frac{2}{3}\,u^{(2)} + \frac{2}{3}\,\Delta t\,L(u^{(2)})\;. \end{aligned} \tag{7.2}$$

In [9], it has been shown that, even with a very nice second order TVD spatial discretization, if the time discretization is by a non-TVD but linearly stable Runge–Kutta method, the result may be oscillatory. Thus it would always be safer to use TVD Runge–Kutta methods for hyperbolic problems.

8 – Numerical experiments

In this section we compare numerical results of our schemes proposed here with Toro scheme. We consider spatially fourth ENO reconstruction (2.6) and for time discretisation we use the third order TVD Runge–Kutta method.

We compare the following schemes:

1 – TVEN2: is the second order flux (3.6) with the fourth ENO reconstruction.

2 – TVEN4: is the fourth order flux (3.5) with the fourth ENO reconstruction.

3 – TVENTORO: is the Toro second order flux [15] with the fourth ENO reconstruction.

In all tests we use the superbee limiter.

Here we present some numerical experiments to show the performance of our TVD schemes of the Euler equations of gas dynamics (6.1). The boundary conditions used are transmissive. The exact solution is represented by full line and the numerical solutions by symbols.

Example 1. Here we choose a sonic test problem with initial data [17]

$$(\rho, u, P) = \begin{cases} (1, 0.75, 1), & 0 \le x \le 0.3 \ , \\ (0.125, 0, 0.1), & 0.3 < x \le 1.0 \ . \end{cases} \tag{8.1}$$

The ratio of specific heats is chosen to be $\gamma = 1.4$.

The exact and numerical solutions are found in spatial domain [0,1]. The numerical solutions will be displayed after $t = 0.2$, we used $\Delta x = 0.01$ and Δt is chosen to satisfy the Courant condition $\Delta t = C_S \, \Delta x / a_{\max}$. Where $a_{\max}$ is the maximum eigenvalues wave speed at time t^n and the Courant number $C_S = 0.9$ [13].

This problem is a modified version of the popular Sod's test, the solution consists of a right shock wave, a right travelling contact wave and a left sonic rarefaction wave, a feature that is very useful for assessing the entropy satisfaction property of numerical methods. Therefore it is a good problem to test the entropy satisfying property of a numerical scheme.

Figures 1, 2 and 3 show the comparison between the computed results (symbols) and the exact solution (full lines) using the TVENTORO, TVEN2 and TVEN4 schemes respectively. Figure 1 shows the results obtained by the TVENTORO scheme. The results look satisfactory for both smooth parts and shocks.

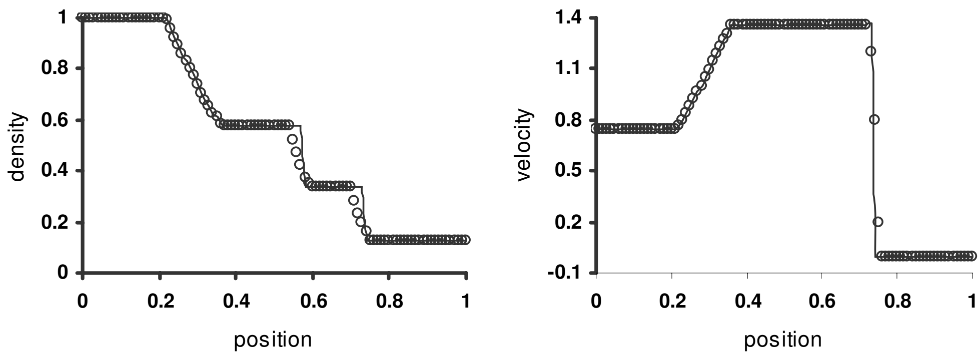

Figure 1 – Solution of equation (8.1) using TVENTORO scheme.

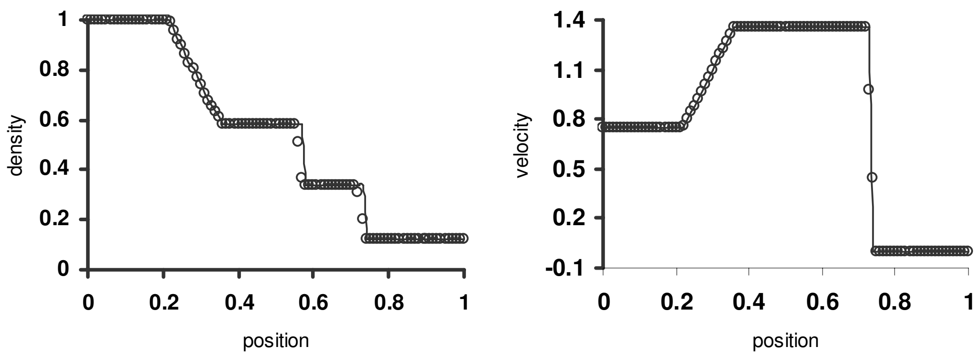

Figure 2 – Solution of equation (8.1) using TVEN2 scheme.

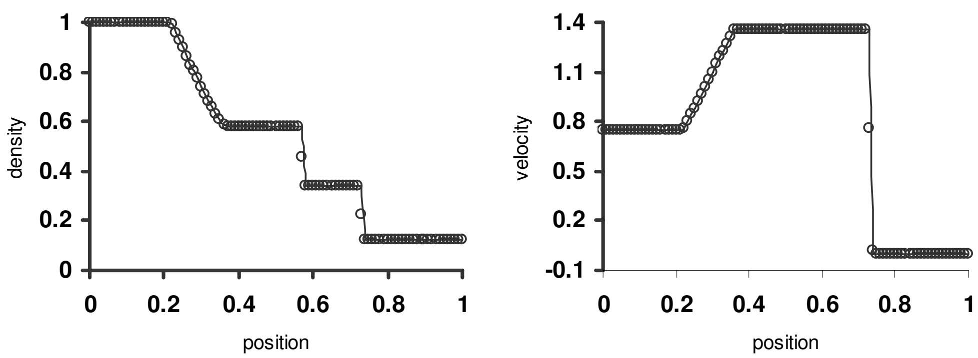

Figure 3 – Solution of equation (8.1) using TVEN4 scheme.

However, the contact has four to five points. Figure 2 show the results by TVEN2 scheme. Comparing with figure 1, the TVEN2 scheme shows an obvious improvement in capturing the contacts with two points. Figure 3 shows the solution of TVEN4 scheme. The results are superior to the others. Both shocks and contacts are presented with one point only. □

Example 2. We solve the Euler equations with initial data [17]

$$(\rho, u, P) = \begin{cases} (1, -19.59745, 1000), & 0 \le x \le 0.8 \ , \\ (1, -19.59745, 0.01), & 0.8 < x \le 1.0 \ . \end{cases} \tag{8.2}$$

This problem is a very severe test problem and in our experience causes several well known schemes to fail. The exact solution consists of a left travelling rarefaction, a right slow-moving strong shock and a stationary contact discontinuity. This test is most useful in assessing not only the robustness of schemes but also the ability of these to resolve slow-moving contact discontinuities.

Figures 4, 5 and 6 show the results obtained by TVENTORO, TVEN2 and TVEN4 schemes respectively with 100 points at $t = 0.012$. For this problem there is a clear difference between the three schemes, particularly in resolving the contact discontinuity, with TVEN4 (figure 6) being the most accurate. The TVEN2 scheme is better than the TVENTORO scheme at resolving the shock wave. □

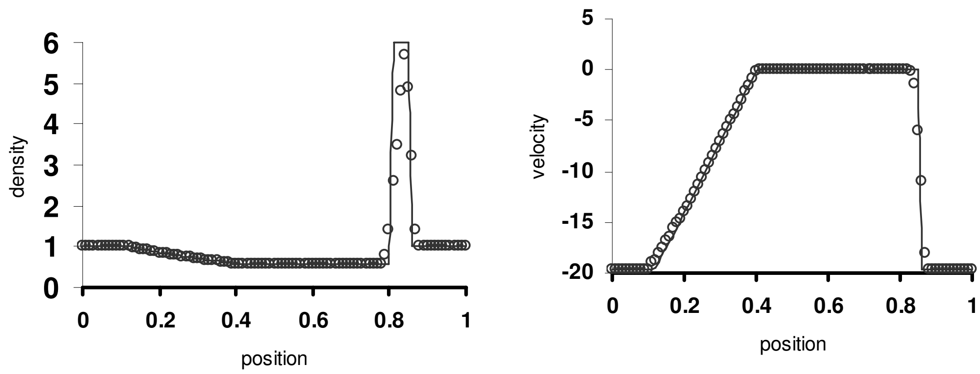

Figure 4 – Solution of equation (8.2) using TVENTORO scheme.

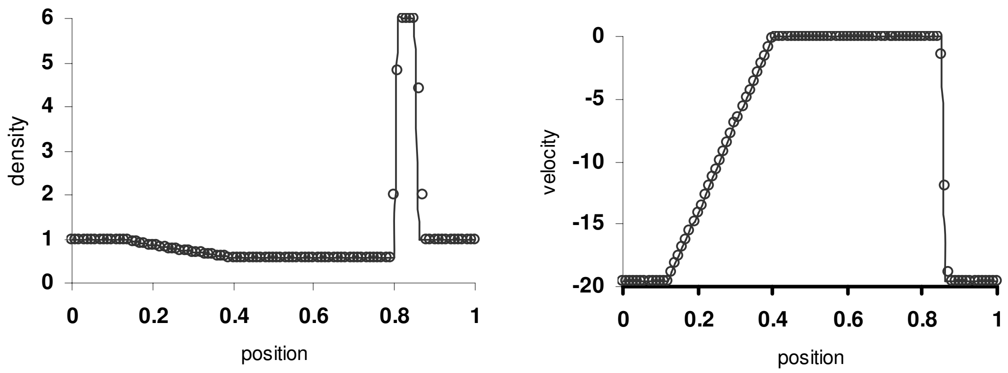

Figure 5 – Solution of equation (8.2) using TVEN2 scheme.

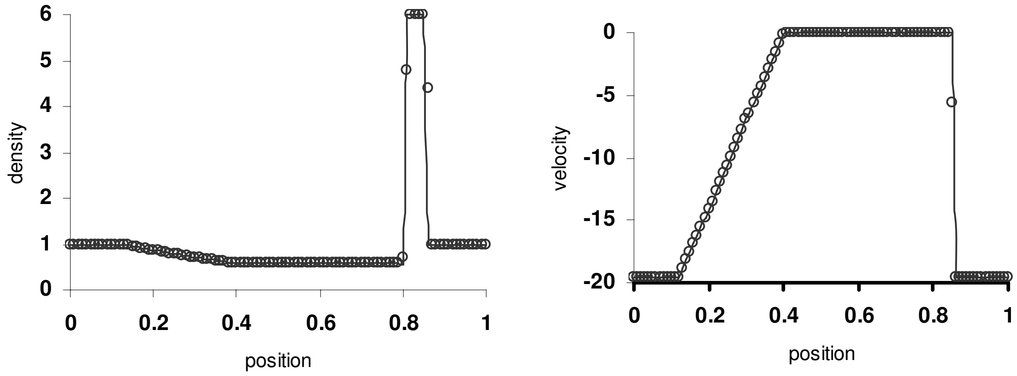

Figure 6 – Solution of equation (8.2) using TVEN4 scheme.

Example 3. We compare the performance of a different schemes on a problem with a rich smooth structure and a shock wave.

We solve the Euler equations (6.1) with a moving Mach$=$3 shock interacting with sine waves in density; i.e., initially [9]

$$\tag{8.3} \begin{cases} (\rho_{\rm L}, u_{\rm L}, P_{\rm L}) \;=\; (3.857143, 2.629369, 10.3333), & \text{for } x < -4\ , \\ (\rho_{\rm R}, u_{\rm R}, P_{\rm R}) \;=\; (1+0.2\sin 5x,\ 0,\ 1), & \text{for } x > -4\ . \end{cases}$$

The flow contains physical oscillations which have to be resolved by the numerical method. We compute the solution at $t=1.8$ against the reference solution, which is converged solution computed by the fifth order finite difference WENO scheme with 2000 grid points [9].

Figures 7–9 show the density by TVENTORO, TVEN2 and TVEN4 schemes respectively with 200 points. We observe the clear improvements in accuracy as we move from TVENTORO scheme to TVEN2 and to TVEN4 scheme. The TVEN4 scheme produces the most accurate, which is very close to the reference solution. □

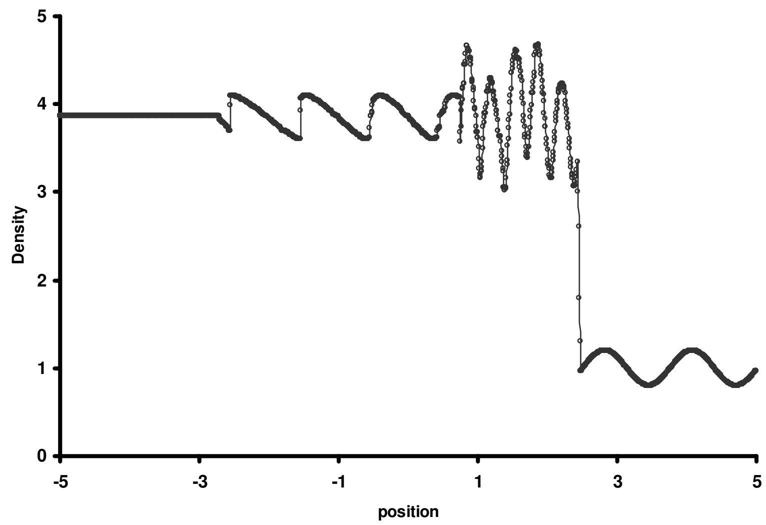

Figure 7 – Solution of equation (8.3) using TVENTORO scheme.

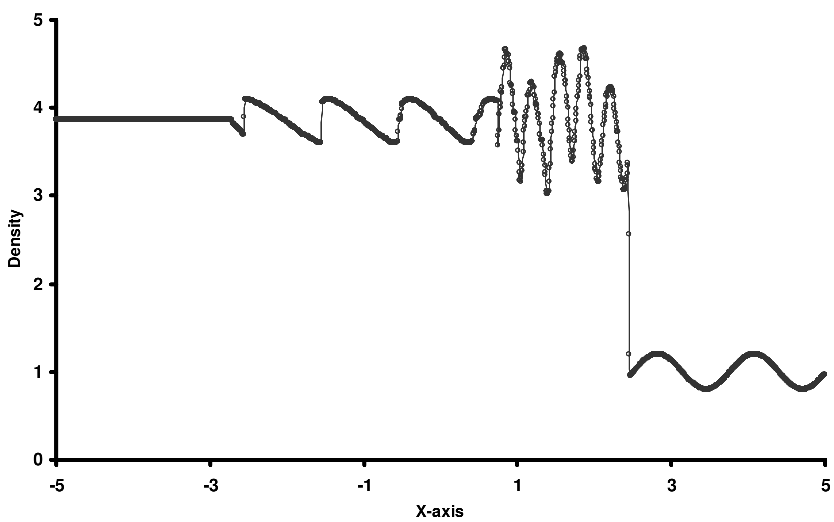

Figure 8 – Solution of equation (8.3) using TVEN2 scheme.

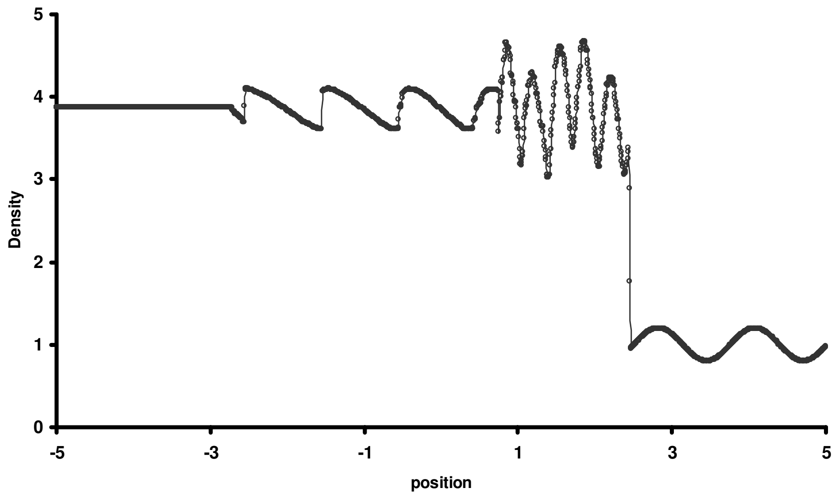

Figure 9 – Solution of equation (8.3) using TVEN4 scheme.

9 – Extension to multidimensional problems

The present schemes can be applied to multidimensional problems by means of space operator splitting. The original idea is attributed to Strang [14]. As an example we consider the two dimensional, Euler equations

$$U_t + \big[F(U)\big]_x + \big[G(U)\big]_y = 0 \tag{9.1}$$

where

$$\begin{aligned} U &= \big(\rho, \rho u, \rho v, E\big)^{\mathrm{T}} , \\ F(U) &= \big(\rho u,\, P+\rho u^2,\, \rho u v,\, u(P+E)\big)^{\mathrm{T}} , \\ G(U) &= \big(\rho v,\, \rho u v,\, P+\rho v^2,\, v(P+E)\big)^{\mathrm{T}} . \end{aligned}$$

There are several versions of space splitting. Here we take the simplest one, whereby the two dimensional problem (9.1) is replaced by the sequence of two one-dimensional problems

$$U_t + \big[F(U)\big]_x = 0 , \tag{9.2a}$$

$$U_t + \big[G(U)\big]_y = 0 . \tag{9.2b}$$

If the data U^n at time level n for problem (9.1) are given, the solution U^{n+1} at time level $n+1$ is obtained in the following two steps:

a) solve equation (9.2a) with data U^n to obtain an intermediate solution $\bar{U}^{n+1}$ (x-sweep);

b) solve equation (9.2b) with data $\bar{U}^{n+1}$ to obtain the complete solution U^{n+1} (y-sweep);

For three dimensional problems there is an extra z-sweep.

Example 4 (Example Double Mach reflection problem)**.** The governing equation for this problem is the two dimensional Euler equations (9.1). The computational domain is $[0,4]\times[0,1]$. The reflecting wall lies at the bottom of the computational domain starting from $x=\frac{1}{6}$. Initially a right moving Mach 10 shock is positioned at $(x,y)=(\frac{1}{6},0)$ and makes 60° angle with the x-axis. For the bottom boundary, the exact postshock condition is imposed from $x=0$ to $x=\frac{1}{6}$ and a reflective boundary condition is used for the rest of the x-axis. At the top boundary of the computational domain, the data is set to describe the exact motion of the Mach 10 shock; consult [22] for a detailed discussion of this problem.

Figures 10–11 show the numerical results of TVEN2 and TVEN4 schemes on the 480×120 cells. We observe that the schemes produce the flow pattern generally accepted in the present literature [18] as correct. All discontinuities are well resolved and correctly positioned. □

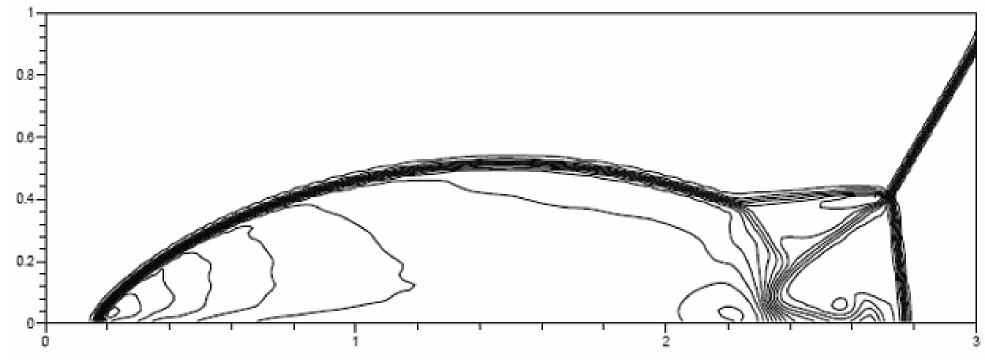

Figure 10 – Solution of the double Mach reflection problem by using TVEN2 scheme.

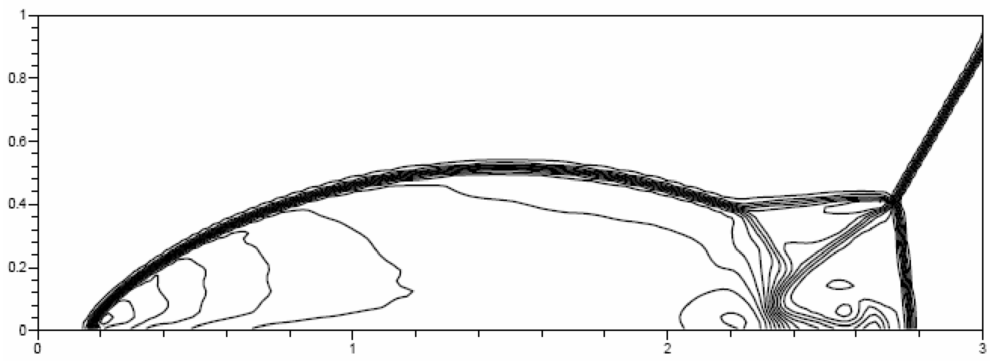

Figure 11 – Solution of the double Mach reflection problem by using TVEN4 scheme.

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