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A MONOTONE METHOD FOR FOURTH ORDER BOUNDARY VALUE PROBLEMS INVOLVING A FACTORIZABLE LINEAR OPERATOR

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Abstract: We consider the nonlinear fourth order beam equation

$$u^{\mathrm{iv}} = f(t, u, u'') ,$$

with boundary conditions corresponding to the periodic or the hinged beam problem. In presence of upper and lower solutions, we consider a monotone method to obtain solutions. The main idea is to write the equation in the form

$$u^{\mathrm{iv}} - c\,u'' + d\,u = g(t, u, u'') ,$$

where c, d are adequate constants, and use maximum principles and a suitable decomposition of the operator appearing in the left-hand side.

1 – Introduction

The existence and approximation of solutions to boundary value problems of the fourth order has been the object of a lot of recent works. Beam theory provides a strong motivation to study the nonlinear equation

$$u^{\mathrm{iv}} = f\big(t, u, u', u'', u'''\big) ,$$

with various types of linear or nonlinear boundary conditions. Some results of rather general character have been given by Senkyřik [27].

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In this paper we are interested in the case where f depends only on u and u'', i.e.

$$u^{iv} = f(t, u, u'') \ , \tag{1.1}$$

and consider the periodic boundary conditions

$$u(0) = u(2\pi), \;\; u'(0) = u'(2\pi), \;\; u''(0) = u''(2\pi), \;\; u'''(0) = u'''(2\pi) \tag{1.2}$$

as well as the simply supported beam conditions

$$u(0) = u''(0) = u(\pi) = u''(\pi) = 0 \ . \tag{1.3}$$

We should note that various other boundary conditions have been considered: we refer the reader among others to Graef and Yang [14] or Yuji and Weigao [30].

The periodic problem has been studied by Cabada [6], via maximum principles and the monotone method, for equations where the right hand side depends on t and u only. A monotone method for the full nonlinear problem (1.1)–(1.2) was proposed by Jiang, Gao and Wan [15]. Using fixed point theory in cones, Li [18] and Liu and Li [20] have obtained existence results for (1.1)–(1.2), allowing a linear dependence of f on u''. With similar techniques, nonlinear dependence has been considered by Liu [19] and other authors. Conti, Terracini and Verzini [10] have considered linear dependence in u'', superlinearity in u and a min-max method to obtain solutions to (1.1)–(1.2) with an arbitrarily large even number of zeros.

Concerning important models as the extended Fisher–Kolmogorov's equation, or the Swift–Hohenberg equation, which have a linear term in u'', we can refer the reader to the variational approach by J.B. van den Berg [5], Mizel, Peletier and Troy [21] or Vandervorst and van den Berg [28]. Similar non-autonomous equations have been considered by Chaparova, Peletier and Tersian [8]. Unfortunately such approaches based on variational arguments do not apply to nonlinear dependence on u''.

For (1.1)–(1.3), Bai and Wang [3] have obtained existence and multiplicity results without dependence on u''. With linear dependence on u'', we can find results of existence in Li [17] and existence and multiplicity in Yao [29]. It is not clear if the monotone method proposed in [1, 2] allows nonlinear dependence. Recently, the superlinear case has also been dealt with by B.R. Rynne [24] using a bifurcation technique.

The above quoted papers together with the references therein provide a view of the literature dedicated to fourth order boundary value problems for (1.1).

Our purpose is to give a monotone method that works for the nonlinear problems (1.1)–(1.2) and (1.1)–(1.3). In the approach of Jiang, Gao and Wan [15], at each iteration step one needs to solve a fixed point equation using a contraction mapping. Monotonicity was obtained via a maximum principle. We propose a modification of the method based on a factorization of linear operators. This approach proceeds more directly and is therefore simpler. It does not cover all cases where the monotone method applies, but it does provide monotonicity behaviour of the successive approximations. In addition, we apply the method to a variety of situations not covered in [15] by using maximum and anti-maximum principles. In particular, we are able to deal with lower and upper solutions in reverse order for the periodic problem.

We should remark that the factorization technique has been used in a variety of instances. It appears in arguments of Omari and Trombetta [22] and also in [5], combined with maximum principles. On the other hand, it has been used recently by Rynne [25, 26] in the study of $2m$ boundary value problems via bifurcation. In this connection, factorization is closely related to the disconjugacy theory of U. Elias (see e.g. [13] and also the monograph by Coppel [11]).

We provide some examples that prove both the applicability of our results and their interest when seeking multiplicity or positive solutions in the presence of the trivial one. It is worth noting that the examples we give have a variational structure, but they do not seem to be easily reducible to the cases that have been studied by variational methods; on the other hand the lower and upper solutions provide easy localization of solutions.

It turns out that the ideas we used run as well (in a simpler way) for second order periodic problems with a derivative dependent nonlinearity. We have found it useful to include the treatment of this case in a final section. In this connection, our procedure is quite close to a method devised by Bellen [4]. The results we find (which, essentially, are not new) may also be viewed as variants of Theorems 4.6 and 4.11 in [12] (see also [9]).

The authors are indebted to C. De Coster for useful discussions and remarks.

2 – Auxiliary results

Our results are strongly based on the monotone method, a classical result that goes back to Kantorovich [16] (see also Zeidler [31]). For convenience we state here a version of this principle which can be found in De Coster and Habets [12].

Let Z be a Banach space. An *order cone* $K \subset Z$ is a closed set such that

for all x and $y \in K$, $x+y \in K$,
for all $t \in \mathbb{R}^+$ and $x \in K$, $tx \in K$,
if $x \in K$ and $-x \in K$ then $x = 0$.

Such an order cone K induces an *order* on Z:

$$x \leq y \quad \text{if and only if} \quad y-x \in K .$$

We write equivalently $x \leq y$ or $y \geq x$. The order cone is said to be *normal* if there exists $c > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq c\|y\|$.

Theorem 2.1. *Let $X \subset Z$ be continuously included Banach spaces so that Z has a normal order cone. Let $\hat\alpha$ and $\hat\beta \in X$, $\hat\alpha \leq \hat\beta$, $\mathcal{E} = \{x \in X \mid \hat\alpha \leq x \leq \hat\beta\}$ and let $T\colon \mathcal{E} \to X$ be continuous and monotone increasing, i.e. $x \leq y$ implies $Tx \leq Ty$. Assume $T(\mathcal{E})$ is relatively compact in X,*

$$\hat\alpha \leq T\hat\alpha \quad \text{and} \quad T\hat\beta \leq \hat\beta .$$

Then, the sequences $(\hat\alpha_n)_n$ and $(\hat\beta_n)_n$ defined by

$$\hat\alpha_0 = \hat\alpha\,, \quad \hat\alpha_n = T\hat\alpha_{n-1}\,,$$

and

$$\hat\beta_0 = \hat\beta\,, \quad \hat\beta_n = T\hat\beta_{n-1}\,,$$

converge monotonically in X to fixed points $x_{\min}$ and $x_{\max}$ of T such that

$$\hat\alpha \leq x_{\min} \leq x_{\max} \leq \hat\beta .$$

Further, any fixed point $x \in \mathcal{E}$ of T verifies

$$x_{\min} \leq x \leq x_{\max} .$$

Proof: See Kantorovich [16], Zeidler [31] or De Coster and Habets [12]. ∎

The monotonicity of the operators we consider relies on maximum and anti-maximum principles. We present here such results and provide proofs for the sake of completeness. First, we work second order operators associated with the periodic problem and give conditions to ensure they are inverse monotone. Our first result is Lemma 2 in [15].

Proposition 2.2 (Maximum principle)**.** *Let $p > 0$ and $q \in \mathbb{R}$. Assume $u \in W^{2,1}(0, 2\pi)$ is such that*

$$u'' - p\,u_+ + q u_- = f(t) \geq 0 \;,$$
$$u(0) = u(2\pi)\,, \qquad u'(0) \geq u'(2\pi) \;.$$

Then $u \leq 0$ on $[0, 2\pi]$.

Proof:

Claim 1. *u must take non-positive values.*

Assume that $u = u_+ \neq 0$. We obtain then the contradiction

$$0 \;\geq\; u'(2\pi) - u'(0) \;=\; \int_0^{2\pi} \Big(f(s) + p\,u_+(s)\Big)\, ds \;>\; 0 \;.$$

Claim 2. $u \leq 0$.

If the claim is wrong, there exist t_1 and $t_2 \neq t_1$ such that $u(t) > 0$ if t is between t_1 and t_2, $u'(t_1)\,(t_2 - t_1) \geq 0$ and $u'(t_2) = 0$. Assume $t_1 < t_2$. This implies $u'(t_1) \geq 0$. We compute then

$$0 \;\geq\; u'(t_2) - u'(t_1) \;=\; \int_{t_1}^{t_2} \Big[f(s) + p\,u(s)\Big]\, ds \;>\; 0 \;,$$

which is a contradiction. The same argument applies if $t_1 > t_2$. ∎

Another result of the same type is an Anti-maximum principle.

Proposition 2.3 (Anti-maximum principle)**.** *Let $p < \frac{1}{4}$ and $q > 0$. Assume $u \in W^{2,1}(0, 2\pi)$ is such that*

$$u'' + p\,u_+ - q u_- = f(t) \geq 0 \;,$$
$$u(0) = u(2\pi)\,, \qquad u'(0) \geq u'(2\pi) \;.$$

Then $u \geq 0$ on $[0, 2\pi]$.

Proof:

Claim 1. *If u is non-trivial, it must take non-negative values.*

Assume that $u = -u_- \neq 0$. We obtain then the contradiction

$$0 \;\geq\; u'(2\pi) - u'(0) \;=\; \int_0^{2\pi} \Big(f(s) + q u_-(s)\Big)\, ds \;>\; 0 \;.$$

Claim 2. $u \ge 0$.

If the claim is wrong, extending u by periodicity if necessary, there exist t_1 and $t_2 > t_1$ such that $t_2 - t_1 \le 2\pi$, $u(t) > 0$ if $t \in]t_1, t_2[$, $u(t_1) = 0$ and $u(t_2) = 0$. Let $t_0 = \frac{t_2+t_1}{2}$, $v(t) = \cos\frac{t-t_0}{2}$ and compute

$$0 \ \ge\ (u'v - v'u)\Big|_{t_1}^{t_2} \ \ge \int_{t_1}^{t_2} \Big[f\,v + \big(\tfrac{1}{4} - p\big)\,u\,v\Big]\,ds \ >\ 0\ ,$$

which is a contradiction. Notice that the inequality in the integration by parts is due to the fact that 2π might be in the interval $]t_1, t_2[$. ∎

The previous propositions can be adapted to deal with Dirichlet problems.

Proposition 2.4 (Maximum principle). *Assume* $p < 1$, $q \in \mathbb{R}$ *and* $u \in W^{2,1}(0,\pi)$ *is a function such that*

$$u'' + p\,u_+ - q\,u_- = f(t) \ge 0\ ,$$

$$u(0) \le 0\,, \qquad u(\pi) \le 0\ ,$$

Then $u \le 0$ *on* $[0,\pi]$.

Proof: If the claim is wrong, there exist t_1 and $t_2 > t_1$ such that $t_2 - t_1 \le \pi$, $u(t) > 0$ if $t \in]t_1, t_2[$, $u(t_1) = 0$ and $u(t_2) = 0$. Let $t_0 = \frac{t_2+t_1}{2}$, $v(t) = \cos(t - t_0)$ and compute

$$0 \ \ge\ (u'v - v'u)\Big|_{t_1}^{t_2} \ = \int_{t_1}^{t_2} \Big[f\,v + (1-p)\,u\,v\Big]\,ds \ >\ 0\ ,$$

which is a contradiction. ∎

Remark. The previous propositions can be improved as follows. If the function u is non trivial, we have in Proposition 2.2, $u < 0$, in Proposition 2.3, $u > 0$, and in Proposition 2.4, $u < 0$ on $]0,\pi[$. □

Similar results hold for first order operators.

Proposition 2.5 (Maximum principle). *Let* $p > 0$ *and* $q \in \mathbb{R}$. *Assume* $u \in W^{1,1}(0,2\pi)$ *is such that*

$$u' - p\,u_+ + q\,u_- \ge 0\,, \qquad u(0) \ge u(2\pi)\ .$$

Then $u \le 0$ *on* $[0,2\pi]$.

Proof: If the claim is wrong, there exists $t_1 < t_2$ such that $u(t) > 0$ on $[t_1, t_2]$ and $u(t_1) \geq u(t_2)$. Integrating on $[t_1, t_2]$, we obtain the contradiction

$$0 \;>\; -p \int_{t_1}^{t_2} u_+ \, ds \;\geq\; u(t_1) - u(t_2) \;\geq\; 0 \;. \blacksquare$$

In a similar way, we obtain the following dual result.

Proposition 2.6 (Maximum principle). *Let* $p \in \mathbb{R}$ *and* $q > 0$. *Assume* $u \in W^{1,1}(0, 2\pi)$ *is such that*

$$u' + p\, u_+ - q\, u_- \geq 0\,, \qquad u(0) \geq u(2\pi) \;.$$

Then $u \geq 0$ *on* $[0, 2\pi]$. $\blacksquare$

3 – Fourth order periodic problem

Consider the problem

$$\begin{aligned} &u^{\text{iv}} = f(t, u, u'') \;, \\ &u(0) = u(2\pi)\,,\;\; u'(0) = u'(2\pi)\,,\;\; u''(0) = u''(2\pi)\,,\;\; u'''(0) = u'''(2\pi) \;. \end{aligned} \tag{3.1}$$

Our aim is to define iteration schemes that converge toward solutions of (3.1). First approximations will be given by lower and upper solutions. A *lower solution* of (3.1) is a function $\alpha \in W^{4,1}(0, 2\pi)$ such that

$$\alpha^{\text{iv}} \leq f(t, \alpha, \alpha'') \;,$$
$$\alpha(0) = \alpha(2\pi)\,,\;\; \alpha'(0) = \alpha'(2\pi)\,,\;\; \alpha''(0) = \alpha''(2\pi)\,,\;\; \alpha'''(0) \leq \alpha'''(2\pi) \;.$$

Similarly, we define an *upper solution* of (3.1) as a function $\beta \in W^{4,1}(0, 2\pi)$ such that

$$\beta^{\text{iv}} \geq f(t, \beta, \beta'') \;,$$
$$\beta(0) = \beta(2\pi)\,,\;\; \beta'(0) = \beta'(2\pi)\,,\;\; \beta''(0) = \beta''(2\pi)\,,\;\; \beta'''(0) \geq \beta'''(2\pi) \;.$$

To build the iteration scheme, we write (3.1) as

$$Lu \;=\; u^{\text{iv}} + a\, u'' + b\, u \;=\; h(t, u, u'') \;,$$

where L is defined on the space $\{u \in W^{4,1}(0, 2\pi) \mid u(0) = u(2\pi),\; u'(0) = u'(2\pi),\; u''(0) = u''(2\pi),\; u'''(0) = u'''(2\pi)\}$. The idea is to factorize L as a product of two second order operators $L = L_1 L_2$. Four cases are then possible according as the Maximum or the Anti-maximum principle applies to L_1 or L_2.

3.1. Well-ordered lower and upper solutions

Let us first factorize L into two operators that verify the Maximum principle, i.e.

$$Lu = (D^2 - \lambda)\,(D^2 - \kappa)\,u \ ,$$

where $\kappa > 0$ and $\lambda > 0$ are given. It turns out that in such a case the first approximations α and β have to be "well-ordered": $\alpha \le \beta$. The factorization implies we can write (3.1) as

$$u'' - \kappa\, u = v\,, \qquad u(0) = u(2\pi)\,, \quad u'(0) = u'(2\pi)\ , \tag{3.2}$$

$$v'' - \lambda v = g(t,u,v)\,, \qquad v(0) = v(2\pi)\,, \quad v'(0) = v'(2\pi)\ , \tag{3.3}$$

where

$$g(t,u,v) = f(t,u,v+\kappa\, u) - (\lambda+\kappa)v - \kappa^2\, u\ . \tag{3.4}$$

Let us assume $f(t,u,v)$ is an L^1-Carathéodory function, i.e. measurable in t for all (u,v), continuous in (u,v) for almost all t, and for any compact set $K \subset \mathbb{R}^2$ there exists $h \in L^1(0,2\pi)$ such that for all $(u,v) \in K$, $|g(t,u,v)| \le h(t)$. With such assumptions, it is well-known that solutions of

$$u'' - \kappa\, u = \hat v\,, \qquad u(0) = u(2\pi)\,, \quad u'(0) = u'(2\pi)\ , \tag{3.5}$$

$$v'' - \lambda v = g(t,\hat u,\hat v)\,, \qquad v(0) = v(2\pi)\,, \quad v'(0) = v'(2\pi)\ , \tag{3.6}$$

read

$$u = R(\hat v) \quad \text{and} \quad v = S(\hat u, \hat v)\ ,$$

where

$$R\colon \mathcal{C}([0,2\pi]) \to \mathcal{C}([0,2\pi]) \quad \text{and} \quad S\colon \mathcal{C}([0,2\pi])\times\mathcal{C}([0,2\pi]) \to \mathcal{C}([0,2\pi])\ .$$

It follows that solutions of (3.2) and (3.3) are such that

$$u = R(v) \quad \text{and} \quad v = S(u,v)\ .$$

The following theorem describes iteration schemes that converge to extremal solutions. As a by-product, we have both existence of at least one solution of (3.1) and its localization between the lower and upper solutions.

Theorem 3.1. *Let $f\colon [0,2\pi]\times\mathbb{R}^2 \to \mathbb{R}$ be an L^1-Carathéodory function and $\alpha,\beta \in W^{4,1}(0,2\pi)$ be respectively lower and upper solutions such that $\alpha \le \beta$. Assume there exist $0<A\le B$ such that*

$$A(v_2-v_1) \;\le\; f(t,u,v_2)-f(t,u,v_1) \;\le\; B(v_2-v_1)$$

for a.e. $t\in[0,2\pi]$, all $u\in[\alpha(t),\beta(t)]$ and all $v_1,v_2\in\mathbb{R}$ with $v_2\ge v_1$, and

$$f(t,u_2,v)-f(t,u_1,v) \;\ge\; -\tfrac{A^2}{4}\,(u_2-u_1)$$

for a.e. $t\in[0,2\pi]$, all $u_1,u_2\in[\alpha(t),\beta(t)]$ with $u_2\ge u_1$, and all $v\in\mathbb{R}$. Let $\kappa=\frac{A}{2}>0$ and $\lambda=B-\frac{A}{2}$. Then the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ defined from

$$\begin{gathered}\alpha_0=\alpha\,,\qquad \mu_0=\alpha''-\kappa\alpha\,,\\ \alpha_n=R(\mu_{n-1})\,,\quad \mu_n=S(\alpha_{n-1},\mu_{n-1})\,,\qquad n\in\mathbb{N}^*\,,\end{gathered}$$

$$\begin{gathered}\beta_0=\beta\,,\qquad \nu_0=\beta''-\kappa\beta\,,\\ \beta_n=R(\nu_{n-1})\,,\quad \nu_n=S(\beta_{n-1},\nu_{n-1})\,,\qquad n\in\mathbb{N}^*\,,\end{gathered}$$

converge uniformly and monotonically to solutions $u_{\min}$ and $u_{\max}$ of (3.1) such that

$$\alpha\le u_{\min}\le u_{\max}\le\beta\,.$$

Further, any solution u of (3.1) such that $\alpha\le u\le\beta$, verifies

$$u_{\min}\le u\le u_{\max}\,.$$

Remark. The iterations defined in this theorem are such that the converging sequences $(\mu_n)_n$ and $(\nu_n)_n$ are respectively monotone decreasing and monotone increasing. ◻

Example. Consider the problem

$$\begin{gathered}u^{\mathrm{iv}}=f(t,u,u'') \;=\; Au''+k^2\sin u+h(t)\,,\\ u(0)=u(2\pi)\,,\quad u'(0)=u'(2\pi)\,,\quad u''(0)=u''(2\pi)\,,\quad u'''(0)=u'''(2\pi)\,,\end{gathered}$$

where $A\ge 2k>0$.

If $h\in L^\infty(0,2\pi)$ and $\|h\|_\infty\le k^2$, it is easy to see that $\alpha(t)=\frac{\pi}{2}$ and $\beta(t)=\frac{3\pi}{2}$ are respectively lower and upper solutions of the problem. Further, $\alpha(t)\le u_1\le u_2\le\beta(t)$, implies

$$\begin{aligned}f(t,u_2,v)-f(t,u_1,v) \;&=\; k^2(\sin u_2-\sin u_1)\\ &\ge\; -k^2(u_2-u_1)\;\ge\; -\tfrac{A^2}{4}(u_2-u_1)\,.\end{aligned}$$

Hence, Theorem 3.1 applies and we have existence of a solution $u \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. Notice that the theorem applies if we choose $\alpha(t) = \frac{\pi}{2}$ and $\beta(t) = \frac{7\pi}{2}$. In this case, u and $u+2\pi$ are solutions, which implies that $u_{\min}$ and $u_{\max}$ do not always coincide.

A more elaborate analysis is needed if $h \in L^1(0,2\pi)$ is unbounded. In such a case, we shall look for a lower solution of the form $\alpha(t) = \frac{\pi}{2} + w(t)$. Given a function $u \in L^1(0,2\pi)$, we define

$$\bar{u} = \frac{1}{2\pi}\int_0^{2\pi} u(t)\,dt \quad \text{and} \quad \tilde{u}(t) = u(t) - \bar{u} \; .$$

We then choose $w(t)$ to be such that

$$\begin{gathered} w^{iv} - Aw'' = \tilde{h}(t) \; , \\ w(0) = w(2\pi), \quad w'(0) = w'(2\pi), \quad w''(0) = w''(2\pi), \quad w'''(0) = w'''(2\pi) \; , \\ \bar{w} = 0 \; . \end{gathered}$$

Such a solution exists and is such that

$$\|w\|_\infty \le K\,\|\tilde{h}\|_{L^1} \; ,$$

where we can chose $K = \frac{\pi}{6}\min\bigl(\frac{1}{A}, \frac{\pi^2}{5!}\bigr)$. It follows then that $\alpha(t)$ is a lower solution if $K\|\tilde{h}\|_{L^1} \le \pi$ and $k^2\cos\bigl(K\|\tilde{h}\|_{L^1}\bigr) + \bar{h} \ge 0$. Similarly, we show that $\beta(t) = \frac{3\pi}{2} + w(t)$ is an upper solution if $K\|\tilde{h}\|_{L^1} \le \pi$ and $k^2\cos\bigl(K\|\tilde{h}\|_{L^1}\bigr) - \bar{h} \ge 0$. Hence, Theorem 3.1 applies if

$$K\,\|\tilde{h}\|_{L^1} \le \tfrac{\pi}{2} \quad \text{and} \quad k^2\cos\bigl(K\,\|\tilde{h}\|_{L^1}\bigr) \ge |\bar{h}| \; . \;\square$$

Proof of Theorem 3.1: Let

$$\begin{aligned} W^{2,1}_{\rm per}(0,2\pi) &= \Bigl\{u \in W^{2,1}(0,2\pi) \mid u(0) = u(2\pi),\; u'(0) = u'(2\pi)\Bigr\}, \\ X &= W^{2,1}_{\rm per}(0,2\pi) \times W^{2,1}_{\rm per}(0,2\pi) \; , \\ Z &= \mathcal{C}([0,2\pi]) \times \mathcal{C}([0,2\pi]) \end{aligned}$$

and define the order cone $K = \bigl\{(u,v) \in Z \mid u \ge 0,\; v \le 0\bigr\}$. Next, we consider $\hat{\alpha} = (\alpha_0, \mu_0)$ and $\hat{\beta} = (\beta_0, \nu_0)$.

Claim 1. $\hat{\alpha} \le \hat{\beta}$.

First, we notice that the function $g(t,u,v)$ defined by (3.4) is non-decreasing in u, i.e. $u_2 \geq u_1$ implies

$$\begin{aligned} g(t,u_2,v)-g(t,u_1,v) &= f(t,u_2,v+\kappa\,u_2)-f(t,u_1,v+\kappa\,u_2) \\ &\quad + f(t,u_1,v+\kappa\,u_2)-f(t,u_1,v+\kappa\,u_1)-\kappa^2(u_2-u_1) \\ &\geq \left[-\tfrac{A^2}{4}+A\kappa-\kappa^2\right](u_2-u_1) \;=\; 0\;. \end{aligned}$$

Next, we verify that $g(t,u,v)$ is non-increasing in v and Lipschitz in v with Lipschitz constant $L=B-A$. Let $v_2 \geq v_1$ and check that

$$\begin{aligned} -L(v_2-v_1) &= (A-\lambda-\kappa)\,(v_2-v_1) \\ &\leq g(t,u,v_2)-g(t,u,v_1) \\ &= f(t,u,v_2+\kappa\,u)-f(t,u,v_1+\kappa\,u)-(\lambda+\kappa)\,(v_2-v_1) \\ &\leq (B-\lambda-\kappa)\,(v_2-v_1) \;=\; 0\;. \end{aligned}$$

Notice now that

$$\mu_0''-\lambda\,\mu_0 \;\leq\; f(t,\alpha,\alpha'')-(\kappa+\lambda)\mu_0-\kappa^2\alpha \;=\; g(t,\alpha,\mu_0)$$

and

$$\nu_0''-\lambda\,\nu_0 \;\geq\; f(t,\beta,\beta'')-(\kappa+\lambda)\nu_0-\kappa^2\beta \;=\; g(t,\beta,\nu_0)\;.$$

It follows that

$$\begin{aligned} (\nu_0-\mu_0)''-\lambda(\nu_0-\mu_0) &\geq g(t,\beta,\nu_0)-g(t,\alpha,\mu_0) \\ &\geq g(t,\alpha,\nu_0)-g(t,\alpha,\mu_0) \;\geq\; -L\,|\nu_0-\mu_0|\;, \end{aligned}$$

$$\nu_0(0)-\mu_0(0)=\nu_0(2\pi)-\mu_0(2\pi)\,, \qquad \nu_0'(0)-\mu_0'(0)\geq \nu_0'(2\pi)-\mu_0'(2\pi)\;.$$

We deduce then from the Maximum principle (Proposition 2.2) that $\nu_0-\mu_0 \leq 0$ and the claim follows.

Notations. Let us write $\mathcal{E}=\big\{(u,v)\in X \mid \alpha_0\leq u\leq\beta_0,\ \nu_0\leq v\leq\mu_0\big\}$ and

$$T\colon\ \mathcal{E}\to X,\ (u,v)\mapsto T(u,v)=\big(R(v),S(u,v)\big)\,.$$

Claim 2. *T is monotone increasing, i.e. $u_1\leq u_2$ and $v_1\geq v_2$ implies $R(v_1)\leq R(v_2)$ and $S(u_1,v_1)\geq S(u_2,v_2)$.*

Assume $u_1\leq u_2$ and $v_1\geq v_2$. The function $w=R(v_1)-R(v_2)$ is such that

$$w''-\kappa\,w \;=\; v_1-v_2 \;\geq\; 0\;,$$
$$w(0)=w(2\pi)\,,\qquad w'(0)=w'(2\pi)\;,$$

and it follows from the Maximum principle (Proposition 2.2) that $w = R(v_1) - R(v_2) \le 0$.

As g is non-decreasing in u and non-increasing in v, the function $z = S(u_1, v_1) - S(u_2, v_2)$ verifies

$$z'' - \lambda z \,=\, g(t, u_1, v_1) - g(t, u_2, v_2) \,\le\, 0 \;,$$
$$z(0) = z(2\pi)\,, \qquad z'(0) = z'(2\pi) \;,$$

and it follows as above that $z = S(u_1, v_1) - S(u_2, v_2) \ge 0$.

Claim 3. *$\hat\alpha \le T\hat\alpha$ and $T\hat\beta \le \hat\beta$, i.e. $\alpha_0 \le R(\mu_0)$, $\mu_0 \ge S(\alpha_0, \mu_0)$, $\beta_0 \ge R(\nu_0)$ and $\nu_0 \le S(\beta_0, \nu_0)$.*

Recall that $u = R(\mu_0)$ is such that

$$u'' - \kappa\, u \,=\, \alpha_0'' - \kappa\, \alpha_0 \;,$$
$$u(0) - \alpha_0(0) = u(2\pi) - \alpha_0(2\pi)\,, \qquad u'(0) - \alpha_0'(0) = u'(2\pi) - \alpha_0'(2\pi) \;.$$

We deduce then that $\alpha_0 = u = R(\mu_0)$. Also $v = S(\alpha_0, \mu_0)$ is such that

$$\begin{aligned} v'' - \lambda v \,&=\, f(t, \alpha_0, \alpha_0'') - (\lambda + \kappa)\,(\alpha_0'' - \kappa\, \alpha_0) - \kappa^2 \alpha_0 \\ &\ge\, \alpha_0^{iv} - \kappa\, \alpha_0'' - \lambda(\alpha_0'' - \kappa\, \alpha_0) \,=\, \mu_0'' - \lambda \mu_0 \;, \end{aligned}$$
$$v(0) - \mu_0(0) = v(2\pi) - \mu_0(2\pi)\,, \qquad v'(0) - \mu_0'(0) \ge v'(2\pi) - \mu_0'(2\pi) \;,$$

so that we deduce from the Maximum principle (Proposition 2.2) that $\mu_0 \ge v = S(\alpha_0, \mu_0)$. In a similar way, we prove $\beta_0 = R(\nu_0)$ and $\nu_0 \le S(\beta_0, \nu_0)$.

Conclusion. As $T(\mathcal{E})$ is relatively compact in X, the proof follows from Theorem 2.1. The sequences $(\alpha_n)_n$ and $(\beta_n)_n$ converge towards solutions $u_{\min}$ and $u_{\max}$ that are extremal in $\mathcal{E}$. To prove these are also extremal between α and β, let u be a solution of (3.1) such that $\alpha \le u \le \beta$. We have $u = R(v)$, $v = S(u, v)$, where $v = u'' - \kappa u$. We prove then as in Claim 1

$$(v - \mu_0)'' - \lambda(v - \mu_0) \,\ge\, -L|v - \mu_0| \;,$$
$$(v - \mu_0)(0) = (v - \mu_0)\,(2\pi)\,, \qquad (v - \mu_0)'\,(0) \ge (v - \mu_0)'\,(2\pi)$$

and it follows from the Maximum principle (Proposition 2.2) that $v \le \mu_0$. In a similar way, we prove $v \ge \nu_0$ so that $(u, v) \in \mathcal{E}$ which implies $u_{\min} \le u \le u_{\max}$. ∎

Remark. Theorem 3.1 holds, with the same proof, if we assume that $\alpha'(0) \ge \alpha'(2\pi)$ and $\beta'(0) \le \beta'(2\pi)$, instead of equality. □

An alternative result can be obtained if we decompose the differential operator into a product $L_1 L_2$ such that the Anti-maximum principle applies to both operators L_1 and L_2. To this end we write problem (3.1) as

$$u'' + \kappa\, u = v\,, \qquad u(0) = u(2\pi)\,, \quad u'(0) = u'(2\pi)\;, \tag{3.7}$$

$$v'' + \lambda v = g(t,u,v)\,, \qquad v(0) = v(2\pi)\,, \quad v'(0) = v'(2\pi)\;, \tag{3.8}$$

where κ, $\lambda > 0$ and

$$g(t,u,v) \;=\; f(t,u,v-\kappa\, u) + (\lambda+\kappa) v - \kappa^2 u\;.$$

In case $f(t,u,v)$ is an L^1-Carathéodory function and λ and κ are not square of integers, we can proceed as for (3.5), (3.6) and define

$$R\colon \mathcal{C}([0,2\pi]) \to \mathcal{C}([0,2\pi]) \quad \text{and} \quad S\colon \mathcal{C}([0,2\pi]) \times \mathcal{C}([0,2\pi]) \to \mathcal{C}([0,2\pi])\;,$$

so that solutions of (3.7) and (3.8) read

$$u = R(v) \quad \text{and} \quad v = S(u,v)\;.$$

Theorem 3.2. *Let $f\colon [0,2\pi]\times\mathbb{R}^2 \to \mathbb{R}$ be an L^1-Carathéodory function and α, $\beta \in W^{4,1}(0,2\pi)$ be respectively lower and upper solutions such that $\alpha \le \beta$. Assume there exist $A \le B < 0$ such that $\frac{3B}{2} - 2A < \frac{1}{4}$,*

$$A(v_2-v_1) \;\le\; f(t,u,v_2) - f(t,u,v_1) \;\le\; B(v_2-v_1)$$

for a.e. $t \in [0,2\pi]$, all $u \in [\alpha(t),\beta(t)]$ and all $v_1, v_2 \in \mathbb{R}$ with $v_2 \ge v_1$, and

$$f(t,u_2,v) - f(t,u_1,v) \;\ge\; -\tfrac{B^2}{4}\,(u_2-u_1)$$

for a.e. $t \in [0,2\pi]$, all $u_1, u_2 \in [\alpha(t),\beta(t)]$ with $u_2 \ge u_1$, and all $v \in \mathbb{R}$. Let $\kappa = -\frac{B}{2} > 0$ and $\lambda = -A + \frac{B}{2} > 0$. Then the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ defined from

$$\begin{gathered} \alpha_0 = \alpha\,, \qquad \mu_0 = \alpha'' + \kappa\alpha\;, \\ \alpha_n = R(\mu_{n-1})\,, \quad \mu_n = S(\alpha_{n-1},\mu_{n-1})\,, \qquad n \in \mathbb{N}^*\,, \end{gathered}$$

$$\begin{gathered} \beta_0 = \beta\,, \qquad \nu_0 = \beta'' + \kappa\,\beta\;, \\ \beta_n = R(\nu_{n-1})\,, \quad \nu_n = S(\beta_{n-1},\nu_{n-1})\,, \qquad n \in \mathbb{N}^*\,, \end{gathered}$$

converge uniformly and monotonically to solutions $u_{\min}$ and $u_{\max}$ of (3.1) such that

$$\alpha \le u_{\min} \le u_{\max} \le \beta\;.$$

Further, any solution u of (3.1) such that $\alpha \le u \le \beta$, verifies

$$u_{\min} \le u \le u_{\max}\;.$$

Idea of the proof. The proof is similar to the proof of Theorem 3.1. Here, we introduce the order cone $K = \{(u,v) \in Z \mid u \geq 0,\ v \geq 0\}$ and use the Anti-maximum principle (Proposition 2.3) rather than the Maximum principle. The function $g(t,u,v)$ turns out to be non-decreasing in u, non-decreasing in v and lipschitzian in v with Lipschitz constant $L = B - A$. ∎

3.2. Lower and upper solutions in the reversed order

In this section, we use a factorization $L = L_1 L_2$ such that the Maximum principle applies to one of the factors L_i and the Anti-maximum principle to the other. In such instances, the iteration scheme is such that the first approximations α and β have to be in the reversed order: $\beta \leq \alpha$.

Let us first write problem (3.1) as

$$u'' + \kappa u = v, \qquad u(0) = u(2\pi), \quad u'(0) = u'(2\pi)\ , \tag{3.9}$$

$$v'' - \lambda v = g(t,u,v), \qquad v(0) = v(2\pi), \quad v'(0) = v'(2\pi)\ , \tag{3.10}$$

where κ, $\lambda > 0$ and

$$g(t,u,v) \;=\; f(t,u,v-\kappa u) - (\lambda-\kappa) v - \kappa^2 u\ .$$

It is known that if $f(t,u,v)$ is an L^1-Carathéodory function and κ is not the square of an integer, we can define as above

$$R\colon \mathcal{C}([0,2\pi]) \to \mathcal{C}([0,2\pi]) \quad \text{and} \quad S\colon \mathcal{C}([0,2\pi]) \times \mathcal{C}([0,2\pi]) \to \mathcal{C}([0,2\pi])\ ,$$

so that solutions of (3.9) and (3.10) read

$$u = R(v) \quad \text{and} \quad v = S(u,v)\ .$$

The following result describes the iteration scheme based on (3.9), (3.10). Notice that, as we use a monotone method, this theorem provides a localization of solutions $\beta \leq u \leq \alpha$ which is stronger than corresponding results on lower and upper solutions in the reversed order and are based on other methods (see Theorem 3.2 in [7] for a fourth order problem or Chapter 3 in [12] for second order ones).

Theorem 3.3. *Let $f\colon [0,2\pi] \times \mathbb{R}^2 \to \mathbb{R}$ be an L^1-Carathéodory function and $\alpha, \beta \in W^{4,1}(0,2\pi)$ be respectively lower and upper solutions such that $\beta \leq \alpha$. Assume there exist A, B and C such that $A < 0 < B$, $0 \leq C \leq \frac{1}{16} + \frac{A}{4}$,*

$$A(v_2 - v_1) \;\leq\; f(t,u,v_2) - f(t,u,v_1) \;\leq\; B(v_2 - v_1)$$

for a.e. $t \in [0, 2\pi]$, *all* $u \in [\beta(t), \alpha(t)]$ *and all* $v_1, v_2 \in \mathbb{R}$ *with* $v_2 \geq v_1$, *and*

$$f(t, u_2, v) - f(t, u_1, v) \;\leq\; C(u_2 - u_1)$$

for a.e. $t \in [0, 2\pi]$, *all* $u_1, u_2 \in [\beta(t), \alpha(t)]$ *with* $u_2 \geq u_1$, *and all* $v \in \mathbb{R}$. *Let* $\kappa \in \left]\frac{-A+\sqrt{A^2+4C}}{2}, \frac{1}{4}\right[$ *and* $\lambda = B+\kappa$. *Then the sequences* $(\alpha_n)_n$ *and* $(\beta_n)_n$ *defined from*

$$\alpha_0 = \alpha\,, \qquad \mu_0 = \alpha'' + \kappa\, \alpha\;,$$
$$\alpha_n = R(\mu_{n-1})\,, \quad \mu_n = S(\alpha_{n-1}, \mu_{n-1})\,, \qquad n \in \mathbb{N}^*\,,$$

$$\beta_0 = \beta\,, \qquad \nu_0 = \beta'' + \kappa\, \beta\;,$$
$$\beta_n = R(\nu_{n-1})\,, \quad \nu_n = S(\beta_{n-1}, \nu_{n-1})\,, \qquad n \in \mathbb{N}^*\,,$$

converge uniformly and monotonically to solutions $u_{\max}$ *and* $u_{\min}$ *of (3.1) such that*

$$\beta \leq u_{\min} \leq u_{\max} \leq \alpha\;.$$

Further, any solution u *of (3.1) such that* $\beta \leq u \leq \alpha$, *verifies*

$$u_{\min} \leq u \leq u_{\max}\;.$$

Idea of the proof. The argument is similar to the proof of Theorem 3.1. It uses the cone $K = \bigl\{(u, v) \in Z \mid u \leq 0,\ v \leq 0\bigr\}$, $\hat{\alpha} = (\alpha_0, \mu_0)$, $\hat{\beta} = (\beta_0, \nu_0)$, and both the Maximum and Anti-maximum principle (Propositions 2.2 and 2.3). Here, the function $g(t, u, v)$ is non-increasing in u, non-increasing in v and lipschitzian in v with Lipschitz constant $L = B - A$. ∎

Example. The theorems we described can be used to prove some multiplicity results. Consider for instance the problem

$$u^{\text{iv}} = f(t, u, u'') \;=\; u'' + k(u^4 - u^2) + h(t)\;,$$
$$u(0) = u(2\pi)\,, \quad u'(0) = u'(2\pi)\,, \quad u''(0) = u''(2\pi)\,, \quad u'''(0) = u'''(2\pi)\;,$$

where $k > 0$ and $h \in L^\infty(0, 2\pi)$ is such that $0 \lneqq h(t) \lneqq k/4$. It is straightforward that the constants -1, 0 and 1 are lower solutions and $-1/\sqrt{2}$ and $1/\sqrt{2}$ are upper solutions. If k is small enough, we deduce then from Theorem 3.3 the existence of two solutions u_2 and u_4 and from Theorem 3.1 the existence of two other solutions u_1 and u_3 such that

$$-1 \leq u_1 \leq -1/\sqrt{2} \leq u_2 \leq 0 \leq u_3 \leq -1/\sqrt{2} \leq u_4 \leq 1\;.$$

Notice that no two such solutions can coincide since the lower and upper solutions are not solutions.

An alternative result interchanges the use of the Maximum and Anti-maximum principle. To this end we write problem (3.1) as

$$u'' - \kappa u = v, \qquad u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \;, \tag{3.11}$$

$$v'' + \lambda v = g(t, u, v), \qquad v(0) = v(2\pi), \quad v'(0) = v'(2\pi) \;, \tag{3.12}$$

where $\kappa > 0$, $\lambda > 0$ and

$$g(t, u, v) \;=\; f(t, u, v + \kappa u) + (\lambda - \kappa) v - \kappa^2 u \;.$$

If $f(t, u, v)$ is an L^1-Carathéodory function and λ is not the square of an integer, we define as previously

$$R\colon \mathcal{C}([0, 2\pi]) \to \mathcal{C}([0, 2\pi]) \quad \text{and} \quad S\colon \mathcal{C}([0, 2\pi]) \times \mathcal{C}([0, 2\pi]) \to \mathcal{C}([0, 2\pi]) \;,$$

so that solutions of (3.11) and (3.12) read

$$u = R(v) \quad \text{and} \quad v = S(u, v) \;. \;\square$$

Theorem 3.4. *Let $f\colon [0, 2\pi] \times \mathbb{R}^2 \to \mathbb{R}$ be an L^1-Carathéodory function and $\alpha, \beta \in W^{4,1}(0, 2\pi)$ be respectively lower and upper solutions such that $\beta \le \alpha$. Assume there exist $A < 0 < B$, and $C > 0$ such that $\frac{3B + \sqrt{B^2 + 4C}}{2} - 2A < \frac{1}{4}$,*

$$A(v_2 - v_1) \;\le\; f(t, u, v_2) - f(t, u, v_1) \;\le\; B(v_2 - v_1)$$

for a.e. $t \in [0, 2\pi]$, all $u \in [\beta(t), \alpha(t)]$ and all $v_1, v_2 \in \mathbb{R}$ with $v_2 \ge v_1$, and

$$f(t, u_2, v) - f(t, u_1, v) \;\le\; C(u_2 - u_1)$$

for a.e. $t \in [0, 2\pi]$, all $u_1, u_2 \in [\beta(t), \alpha(t)]$ with $u_2 \ge u_1$, and all $v \in \mathbb{R}$. Let $\kappa = \frac{B + \sqrt{B^2 + 4C}}{2}$ and $\lambda = \kappa - A$. Then the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ defined from

$$\alpha_0 = \alpha\,, \quad \mu_0 = \alpha'' - \kappa\, \alpha \;,$$
$$\alpha_n = R(\mu_{n-1})\,, \quad \mu_n = S(\alpha_{n-1}, \mu_{n-1})\,, \qquad n \in \mathbb{N}^* \,,$$

$$\beta_0 = \beta\,, \quad \nu_0 = \beta'' - \kappa\, \beta \;,$$
$$\beta_n = R(\nu_{n-1})\,, \quad \nu_n = S(\beta_{n-1}, \nu_{n-1})\,, \qquad n \in \mathbb{N}^* \,,$$

converge uniformly and monotonically to solutions $u_{\max}$ and $u_{\min}$ of (3.1) such that

$$\beta \le u_{\min} \le u_{\max} \le \alpha \;.$$

Further, any solution u of (3.1) such that $\beta \le u \le \alpha$, verifies

$$u_{\min} \le u \le u_{\max} \;.$$

Idea of the proof. The proof uses the previous arguments with the cone $K = \{(u,v) \in Z \mid u \le 0,\ v \ge 0\}$. Here, the function $g(t,u,v)$ is non-increasing in u, non-decreasing in v and lipschitzian in v with Lipschitz constant $L = B - A$. ∎

4 – The simply supported beam

The approximation method we worked for the periodic problem can also be developed for other boundary value problems such as the simply supported beam problem

$$\tag{4.1} \begin{gathered} u^{\mathrm{iv}} = f(t,u,u'') , \\ u(0) = u(\pi) = 0 , \qquad u''(0) = u''(\pi) = 0 . \end{gathered}$$

A lower solution $\alpha \in W^{4,1}(0,\pi)$ of (4.1) is defined as follows

$$\begin{gathered} \alpha^{\mathrm{iv}} \le f(t,\alpha,\alpha'') , \\ \alpha(0) \le 0, \quad \alpha(\pi) \le 0, \quad \alpha''(0) \ge 0, \quad \alpha''(\pi) \ge 0 . \end{gathered}$$

Similarly, an upper solution $\beta \in W^{4,1}(0,2\pi)$ of (4.1) is such that

$$\begin{gathered} \beta^{\mathrm{iv}} \ge f(t,\beta,\beta'') , \\ \beta(0) \ge 0, \quad \beta(\pi) \ge 0, \quad \beta''(0) \le 0, \quad \beta''(\pi) \le 0 . \end{gathered}$$

Given real numbers κ and λ, we can write problem (4.1) as

$$\tag{4.2} u'' - \kappa\, u = v , \qquad u(0) = 0, \quad u(\pi) = 0 ,$$

$$\tag{4.3} v'' - \lambda v = g(t,u,v), \qquad v(0) = 0, \quad v(\pi) = 0 ,$$

where

$$\tag{4.4} g(t,u,v) = f(t,u,v + \kappa\, u) - (\lambda + \kappa) v - \kappa^2 u .$$

If $f(t,u,v)$ is an L^1-Carathéodory function, $-\kappa < 1$ and $-\lambda < 1$, we define as in the previous section

$$R\colon\ \mathcal{C}([0,\pi]) \to \mathcal{C}([0,\pi]) \quad \text{and} \quad S\colon\ \mathcal{C}([0,\pi]) \times \mathcal{C}([0,\pi]) \to \mathcal{C}([0,\pi]) ,$$

so that solutions of (4.2) and (4.3) read

$$u = R(v) \quad \text{and} \quad v = S(u,v) .$$

Our first result assumes the function $f(t,u,v)$ to be increasing in v.

Theorem 4.1. *Let $f\colon [0,\pi]\times\mathbb{R}^2\to\mathbb{R}$ be an L^1-Carathéodory function and $\alpha,\beta\in W^{4,1}(0,\pi)$ be respectively lower and upper solutions such that $\alpha\le\beta$. Assume there exist $A\le B$ and C such that*

$$A(v_2-v_1)\ \le\ f(t,u,v_2)-f(t,u,v_1)\ \le\ B(v_2-v_1)$$

for a.e. $t\in[0,\pi]$, all $u\in[\alpha(t),\beta(t)]$ and all $v_1,v_2\in\mathbb{R}$ with $v_2\ge v_1$, and

$$f(t,u_2,v)-f(t,u_1,v)\ \ge\ C(u_2-u_1)$$

for a.e. $t\in[0,\pi]$, all $u_1,u_2\in[\alpha(t),\beta(t)]$ with $u_2\ge u_1$, and all $v\in\mathbb{R}$. Assume further

(A-1) $A\ge 0,\ C=-\frac{A^2}{4}$

and define

(A-2) $\kappa=\frac{A}{2}$ *and* $\lambda=B-\kappa$.

Then the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ defined from

$$\begin{gathered}\alpha_0=\alpha\,,\qquad \mu_0=\alpha''-\kappa\,\alpha\ ,\\ \alpha_n=R(\mu_{n-1})\,,\quad \mu_n=S(\alpha_{n-1},\mu_{n-1})\,,\qquad n\in\mathbb{N}^*\,,\end{gathered}$$

$$\begin{gathered}\beta_0=\beta\,,\qquad \nu_0=\beta''-\kappa\,\beta\ ,\\ \beta_n=R(\nu_{n-1})\,,\quad \nu_n=S(\beta_{n-1},\nu_{n-1})\,,\qquad n\in\mathbb{N}^*\,,\end{gathered}$$

converge uniformly and monotonically to solutions $u_{\min}$ and $u_{\max}$ of (4.1) such that

$$\alpha\le u_{\min}\le u_{\max}\le\beta\ .$$

Further, any solution u of (4.1) such that $\alpha\le u\le\beta$, verifies

$$u_{\min}\le u\le u_{\max}\ .$$

Proof: Let

$$\begin{aligned}&W_0^{2,1}(0,\pi)=\Big\{u\in W^{2,1}(0,\pi)\ \big|\ u(0)=0,\ u(\pi)=0\Big\}\,,\\ &X=W_0^{2,1}(0,\pi)\times W_0^{2,1}(0,\pi)\ ,\\ &Z=\mathcal{C}([0,\pi])\times\mathcal{C}([0,\pi])\end{aligned}$$

and define the order cone $K=\big\{(u,v)\in Z\mid u\ge 0,\ v\le 0\big\}$. Next, we consider $\hat\alpha=(\alpha_0,\mu_0)$ and $\hat\beta=(\beta_0,\nu_0)$.

Claim 1. $\hat\alpha\le\hat\beta$.

First, as in the proof of Theorem 3.1, we can show that the function $g(t,u,v)$ defined by (4.4) is non-decreasing in u and is non-increasing in v and Lipschitz in v with Lipschitz constant $L = B - A$. Notice now that

$$\mu_0'' - \lambda\mu_0 \;\le\; f(t,\alpha,\alpha'') - (\kappa+\lambda)\,\mu_0 - \kappa^2\,\alpha \;=\; g(t,\alpha,\mu_0)$$

and

$$\nu_0'' - \lambda\nu_0 \;\ge\; f(t,\beta,\beta'') - (\kappa+\lambda)\,\nu_0 - \kappa^2\,\beta \;=\; g(t,\beta,\nu_0)\ .$$

It follows, using the properties of g and the boundary conditions of α and β, that

$$\begin{aligned}(\nu_0-\mu_0)'' - \lambda(\nu_0-\mu_0) \;&\ge\; g(t,\beta,\nu_0) - g(t,\alpha,\mu_0)\\ &\ge\; g(t,\alpha,\nu_0) - g(t,\alpha,\mu_0) \;\ge\; -L\,|\nu_0-\mu_0|\ ,\\ \nu_0(0)-\mu_0(0)\le 0\,,\quad &\nu_0(\pi)-\mu_0(\pi)\le 0\ .\end{aligned}$$

Since $L < \lambda+1$ we deduce then from the Maximum principle (Proposition 2.4) that $\nu_0-\mu_0 \le 0$ and the claim follows.

Notations. Let us write $\mathcal{E} = \big\{(u,v)\in X \mid \alpha_0\le u\le \beta_0,\ \nu_0\le v\le \mu_0\big\}$ and

$$T\colon\ \mathcal{E}\to X,\ (u,v)\mapsto T(u,v) = \big(R(v), S(u,v)\big)\ .$$

Claim 2. *T is monotone increasing, i.e. $u_1\le u_2$ and $v_1\ge v_2$ implies $R(v_1)\le R(v_2)$ and $S(u_1,v_1)\ge S(u_2,v_2)$.*

Assume $u_1\le u_2$ and $v_1\ge v_2$. The function $w=R(v_1)-R(v_2)$ is such that

$$\begin{aligned}w''-\kappa\,w \;=\; v_1-v_2 \;\ge\; 0\ ,\\ w(0)=0\,,\quad w(\pi)=0\ ,\end{aligned}$$

and it follows from the Maximum principle (Proposition 2.4) that $w = R(v_1) - R(v_2)\le 0$.

As g is non-decreasing in u and non-increasing in v, the function $z=S(u_1,v_1)-S(u_2,v_2)$ verifies

$$\begin{aligned}z''-\lambda z \;=\; g(t,u_1,v_1)-g(t,u_2,v_2) \;\le\; 0\ ,\\ z(0)=0\,,\quad z(\pi)=0\ ,\end{aligned}$$

and it follows as above that $z = S(u_1,v_1)-S(u_2,v_2)\ge 0$.

Claim 3. *$\hat\alpha\le T\hat\alpha$ and $T\hat\beta\le\hat\beta$, i.e. $\alpha_0\le R(\mu_0)$, $\mu_0\ge S(\alpha_0,\mu_0)$, $\beta_0\ge R(\nu_0)$ and $\nu_0\le S(\beta_0,\nu_0)$.*

Recall that $u = R(\mu_0)$ is such that

$$\begin{aligned}u''-\kappa\,u \;=\; \alpha_0''-\kappa\,\alpha_0\ ,\\ u(0)-\alpha_0(0) = -\alpha(0)\ge 0\,,\quad u(\pi)-\alpha_0(\pi) = -\alpha(\pi)\ge 0\ .\end{aligned}$$

We deduce then that $\alpha_0 \le u = R(\mu_0)$. Also $v = S(\alpha_0, \mu_0)$ is such that

$$\begin{aligned} v'' - \lambda v \;&=\; f(t, \alpha_0, \alpha_0'') - (\lambda + \kappa)\,(\alpha_0'' - \kappa\,\alpha_0) - \kappa^2\,\alpha_0 \\ &\ge\; \alpha_0^{\mathrm{iv}} - \kappa\,\alpha_0'' - \lambda(\alpha_0'' - \kappa\,\alpha_0) \;=\; \mu_0'' - \lambda\mu_0 \;, \end{aligned}$$

$$v(0) - \mu_0(0) \;=\; -\alpha''(0) + \kappa\,\alpha(0) \le 0\,, \qquad v(\pi) - \mu_0(\pi) \;=\; -\alpha''(\pi) + \kappa\alpha(\pi) \le 0\;,$$

so that we deduce from the Maximum principle (Proposition 2.4) that $\mu_0 \ge v = S(\alpha_0, \mu_0)$. In a similar way, we prove $\beta_0 \ge R(\nu_0)$ and $\nu_0 \le S(\beta_0, \nu_0)$.

Conclusion. As $T(\mathcal{E})$ is relatively compact in X, the proof follows from Theorem 2.1. The final assertion may be checked in the same way as we did in the proof of Theorem 3.1. ∎

Remark. The above theorem still holds if the lower and upper solutions are such that

$$\kappa\,\alpha(0) = 0\,, \quad \kappa\,\alpha(\pi) = 0\,, \quad \kappa\,\beta(0) = 0\,, \quad \kappa\,\beta(\pi) = 0\;,$$

and we replace (A-1) and (A-2) by

(B-1) $-2 < A < 0$, $\hat{B} = \min(B, 0) < 2(A+1)$ and $C = -\frac{\hat{B}^2}{4}$;

(B-2) $\kappa = \frac{\hat{B}}{2}$ and $\lambda = B - \kappa$.

An alternative replaces (A-1) and (A-2) by

(C-1) $-2 < A < 0$, $\min(B, 0) \ge 2(A+1)$ and $C > (A+1)\,(A+1-B)$;

(C-2) $\kappa \in \,]-1, A+1[$ such that $C \ge \kappa(\kappa - B)$ and $\lambda = B - \kappa$. □

The proof of these remarks is identical to the proof of Theorem 4.1.

Example. Consider the boundary value problem

$$\begin{gathered} u^{\mathrm{iv}} = \phi(u'') + g(u)\;, \\ u(0) = u(\pi) = 0\,, \qquad u''(0) = u''(\pi) = 0\;. \end{gathered} \tag{4.5}$$

and assume ϕ and g are real functions with the following properties:

(i) $\phi(0) = g(0) = 0$ and $\phi'(0) + 1 < g'(0)$;

(ii) there exist $B > A \ge 0$ such that $A \le \phi'(v) \le B$ for all $v \in \mathbb{R}$ and $g'(u) \ge -\frac{A^2}{4}$ for all $u \ge 0$;

(iii) $\max\limits_{u \in [0, \pi^2/4]} g(u) + \phi(-2) \le 0$.

Then it is easy to see that $\alpha(t) = \epsilon \sin t$, for $\epsilon > 0$ sufficiently small, and $\beta(t) = \pi t - t^2$ are lower and upper solutions of (4.5). From Theorem 4.1 it follows that this problem has a strictly positive solution that can be found by iteration starting from β which is explicitely known. □

5 – Second order periodic problems

In this final section we remark that the operator decomposition procedure allows also to deal with a class of second order periodic problems. In fact, following steps analogue to those of Section 3, we may construct an iterative process quite similar to the method devised by Bellen [4].

Let $f\colon [0,2\pi]\times\mathbb{R}^2 \to \mathbb{R}$ be a L^1-Carathéodory function. Consider the problem

$$\tag{5.1} \begin{gathered} u'' = f(t,u,u')\ , \\ u(0) = u(2\pi)\,, \qquad u'(0) = u'(2\pi)\ . \end{gathered}$$

A *lower solution* $\alpha \in W^{2,1}(0,2\pi)$ of (5.1) is defined as follows

$$\alpha'' \ge f(t,\alpha,\alpha')\,, \qquad \alpha(0) = \alpha(2\pi)\,, \quad \alpha'(0) \ge \alpha'(2\pi)\ .$$

Similarly, an *upper solution* $\beta \in W^{2,1}(0,2\pi)$ of (5.1) is such that

$$\beta'' \le f(t,\beta,\beta')\,, \qquad \beta(0) = \beta(2\pi)\,, \quad \beta'(0) \le \beta'(2\pi)\ .$$

Given κ and $\lambda > 0$, we can write problem (5.1) as

$$\tag{5.2} u' - \kappa\, u \,=\, v\,, \qquad\qquad u(0) = u(2\pi)\ ,$$

$$\tag{5.3} v' + \lambda v \,=\, g(t,u,v)\,, \qquad v(0) = v(2\pi)\ ,$$

where

$$\tag{5.4} g(t,u,v) \,=\, f(t,u,v+\kappa\, u) + (\lambda-\kappa)v - \kappa^2 u\ .$$

In case $f(t,u,v)$ is L^1-Carathéodory, we define as previously

$$R\colon \mathcal{C}([0,2\pi]) \to \mathcal{C}([0,2\pi]) \quad \text{and} \quad S\colon \mathcal{C}([0,2\pi])\times\mathcal{C}([0,2\pi]) \to \mathcal{C}([0,2\pi])\ ,$$

so that solutions of (5.2) and (5.3) read

$$u = R(v) \quad \text{and} \quad v = S(u,v)\ .$$

Theorem 5.1. *Let $f\colon [0,2\pi]\times\mathbb{R}^2 \to \mathbb{R}$ be a L^1-Carathéodory function and $\alpha,\beta \in W^{2,1}(0,2\pi)$ be respectively lower and upper solutions such that $\alpha \le \beta$. Assume there exist $A \le B$ and C such that*

$$A(v_2 - v_1) \,\le\, f(t,u,v_2) - f(t,u,v_1) \,\le\, B(v_2 - v_1)$$

for a.e. $t \in [0, 2\pi]$*, all* $u \in [\alpha(t), \beta(t)]$ *and all* $v_1, v_2 \in \mathbb{R}$ *with* $v_2 \geq v_1$*, and*

$$f(t, u_2, v) - f(t, u_1, v) \leq C(u_2 - u_1)$$

for a.e. $t \in [0, 2\pi]$*, all* $u_1, u_2 \in [\alpha(t), \beta(t)]$ *with* $u_2 \geq u_1$*, and all* $v \in \mathbb{R}$*. Let us assume further*

(A-1) $A < 0 < B$ *and* $C > 0$,

and write

(A-2) $\kappa = \dfrac{B + \sqrt{B^2 + 4C}}{2}$ *and* $\lambda = \kappa - A$.

Then the sequences $(\alpha_n)_n$ *and* $(\beta_n)_n$ *defined from*

$$\alpha_0 = \alpha\,, \quad \mu_0 = \alpha' - \kappa\,\alpha\;,$$
$$\alpha_n = R(\mu_{n-1})\,, \quad \mu_n = S(\alpha_{n-1}, \mu_{n-1})\,, \qquad n \in \mathbb{N}^*\,,$$

$$\beta_0 = \beta\,, \quad \nu_0 = \beta' - \kappa\,\beta\;,$$
$$\beta_n = R(\nu_{n-1})\,, \quad \nu_n = S(\beta_{n-1}, \nu_{n-1})\,, \qquad n \in \mathbb{N}^*\,,$$

converge uniformly and monotonically to solutions $u_{\min}$ *and* $u_{\max}$ *of (5.1) such that*

$$\alpha \leq u_{\min} \leq u_{\max} \leq \beta\;.$$

Further, any solution u *of (5.1), such that* $\alpha \leq u \leq \beta$*, verifies*

$$u_{\min} \leq u \leq u_{\max}\;.$$

Proof: Let $W^{1,1}_{\text{per}}(0,2\pi) = \{u \in W^{1,1}(0,2\pi) \mid u(0) = u(2\pi)\}$, $X = W^{1,1}_{\text{per}}(0,2\pi) \times W^{1,1}_{\text{per}}(0, 2\pi)$, $Z = \mathcal{C}([0, 2\pi]) \times \mathcal{C}([0, 2\pi])$ and define the order cone $K = \{(u, v) \in Z \mid u \geq 0,\ v \leq 0\}$. Next, we consider $\hat{\alpha} = (\alpha_0, \mu_0)$ and $\hat{\beta} = (\beta_0, \nu_0)$.

Given that $g(t, u, v)$ defined by (5.4) is non-increasing in u and non-decreasing in v and Lipschitz in v with Lipschitz constant $L = B + \lambda - \kappa < \lambda$ we can check with the aid of Proposition 2.6 that $\hat{\alpha} \leq \hat{\beta}$.

Next we consider the space $\mathcal{E} = \{(u, v) \in X \mid \alpha_0 \leq u \leq \beta_0,\ \nu_0 \leq v \leq \mu_0\}$ and the operator

$$T\colon\ \mathcal{E} \to X,\ (u, v) \mapsto T(u, v) = \big(R(v), S(u, v)\big)\,.$$

It is easy to see, using Propositions 2.5 and 2.6, that T is monotone increasing and $\hat{\alpha} \leq T\hat{\alpha}$ and $\hat{\beta} \leq T\hat{\beta}$, i.e. $\alpha_0 \leq R(\mu_0)$, $\mu_0 \geq S(\alpha_0, \mu_0)$, $\beta_0 \geq R(\nu_0)$ and $\nu_0 \leq S(\beta_0, \nu_0)$.

Finally, since $T(\mathcal{E})$ is relatively compact in X, the proof follows from Theorem 2.1, the final assertion being proved as in the theorems of the previous sections. ∎

To study the case where the lower and upper solutions are in the reversed order, we write problem (5.1) as

$$u' - \kappa\, u = v\,, \qquad u(0) = u(2\pi)\ , \tag{5.5}$$

$$v' - \lambda v = g(t,u,v)\,, \qquad v(0) = v(2\pi)\ , \tag{5.6}$$

where $\kappa > 0$, $\lambda > 0$ and

$$g(t,u,v) \;=\; f(t,u,v+\kappa\, u) - (\lambda+\kappa)v - \kappa^2 u\ .$$

Define as above

$$R\colon\ \mathcal{C}([0,2\pi]) \to \mathcal{C}([0,2\pi]) \quad \text{ and } \quad S\colon\ \mathcal{C}([0,2\pi])\times\mathcal{C}([0,2\pi]) \to \mathcal{C}([0,2\pi])\ ,$$

so that solutions of (5.5) and (5.6) read

$$u = R(v) \quad \text{ and } \quad v = S(u,v)\ .$$

A version of Theorem 5.1 for the case $\alpha \geq \beta$ still holds within the above framework provided (A-1), (A-2) are replaced by

(B-1) $0 < A < B$ and $C = -\frac{A^2}{4}$,
(B-2) $\kappa = \frac{A}{2}$ and $\lambda = B - \kappa$.

To prove such a result, the argument follows the lines of the proof of Theorem 5.1, but uses the order cone $K = \big\{(u,v) \in Z \mid u \leq 0,\ v \geq 0\big\}$. The function

$$g(t,u,v) \;=\; f(t,u,v+\kappa\, u) - (\lambda+\kappa)v - \kappa^2 u\ ,$$

which is the equivalent of (5.4), turns out to be non-decreasing in u, non-increasing in v and lipschitzian in v with Lipschitz constant $L = -A + \lambda + \kappa$.

In a similar way, we can replace (A-1), (A-2) by

(C-1) $A < B < 0$ and $C = -\frac{B^2}{4}$,
(C-2) $\kappa = -\frac{B}{2}$ and $\lambda = -\kappa - A$.

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