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HOMOCLINIC SOLUTIONS OF A FOURTH-ORDER TRAVELLING WAVE ODE

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Recommended by Luís Sanchez

Dedicated to Academician Peter Popivanov
on the occasion of his 60-th birthday

Abstract: In this paper we investigate via the shooting method the existence of homoclinic solutions of a fourth-order differential equation arising in the theory of water waves.

1 – Introduction

In this paper we investigate the existence of homoclinic solutions of the equation

$$\gamma\, u^{\mathrm{iv}} = u'' + \mu\Big(2\,u\,u'' + (u')^2\Big) + u - u^2, \qquad \gamma > 0\ , \tag{1.1}$$

i.e., classical solutions $u = u(x)$ of (1.1), defined on $\mathbb{R}$, which satisfy the condition

$$\big(u, u', u'', u'''\big)\,(x) \,\to\, (1,0,0,0) \quad \text{as} \ \ x \to \pm\infty\ . \tag{1.2}$$

Equations of the form (1.1) or

$$\gamma_1 v^{\mathrm{iv}} = v'' + \mu_1\Big(2\,v\,v'' + (v')^2\Big) - v - v^2\ , \tag{1.3}$$

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appear in the theory of water waves. For instance, the ordinary differential equation

$$\frac{2}{15}\,u^{\mathrm{iv}} - b\,u'' + a\,u + \frac{3}{2}\,u^2 + \mu\Big(\frac{1}{2}\,(u')^2 + (u\,u')'\Big) \;=\; 0$$

was derived by Craig and Groves [CG], when looking for travelling wave solutions $u = u(x - at)$ of the extended fifth-order KdV equation

$$u_t \;=\; \frac{2}{15}\,u_{xxxxx} - b\,u_{xxx} + 3u + \mu\Big(\frac{1}{2}\,(u_x)^2 + (u\,u_x)_x\Big)_x \;=\; 0\;,$$

which describes gravity water waves on a surface with finite depth (see [CG], [ChG], [GMYK], [P]). Our work is inspired by the paper of Peletier, Rotariu–Bruma and Troy [PBT], and Peletier and Troy [PT] where homoclinic solutions are studied for the stationary extended Fisher–Kolmogorov equation

$$\gamma\,u^{\mathrm{iv}} = u'' + f(u)\,, \qquad \gamma > 0\;,$$

by the shooting method. It is mentioned in [PBT] that this method can be applied to equations of the form (1.3). Note that, under the change $u(x) = 1 + v\big(x/\sqrt{1+2\mu}\big)$, (1.1) becomes

$$\frac{\gamma}{(1+2\mu)^2}\,v^{\mathrm{iv}} \;=\; v'' + \frac{\mu}{1+2\mu}\,(2\,v\,v'' + v'^2) - v - v^2\;,$$

which is of the form (1.3) with

$$\gamma_1 = \frac{\gamma}{(1+2\mu)^2} \qquad \text{and} \qquad \mu_1 = \frac{\mu}{1+2\mu}\;.$$

Since (1.1) is invariant to the change of $u(x)$ with $u(-x)$ we are looking for even solutions on $\mathbb{R}$ and consider (1.1) on $\mathbb{R}^+ = \{x \in \mathbb{R}\colon\, x \ge 0\}$, requiring that $u'(0) = u'''(0) = 0$. Our main results concerning even homoclinic solutions of (1.1) are as follows:

Theorem 1. *Let* $0 < \gamma \le (1+2\mu)^2/4$ *if* $-1/2 < \mu \le 1/2$ *or* $0 < \gamma \le 2\mu$ *if* $\mu > 1/2$. *Then (1.1) has an even homoclinic solution* $u = u(x)$ *which satisfies*

$$-1/2 < u(x) < 1 \;\; \textit{for all } x \in \mathbb{R}\,, \qquad u(0) < 0 \;\; \textit{and } \; u'(x) > 0 \;\; \textit{for all } x > 0\;.$$

The upper bound $u(0) < 0$ in Theorem 1 can be improved. Let $m(\gamma, \mu)$ be the greatest negative zero of the polynomial

$$P_3(s) \;:=\; 8\,\mu^2 s^3 + \big(4\mu^2 + 8\mu - 12\,\gamma\big)\,s^2 + 2\,(1+2\mu)\,s + 1\;,$$

which exists since $P_3(-\infty) = -\infty$ and $P_3(0) = 1$.

Theorem 2. *Let γ and μ be as in Theorem 1. Suppose that $u = u(x)$ is an even, nonconstant homoclinic solution of (1.1) for which $u(x) \le 1$, $x \in \mathbb{R}$ and $u''(0) \ge 0$. Then $u(0) < m(\gamma, \mu)$.*

The paper is organized as follows. In Section 2, the shooting method for (1.1) is developed and Theorem 1 is proved. In Section 3, Theorem 2 is proved.

2 – Proof of Theorem 1 via the shooting method

In this section we prove the existence of a homoclinic solution of the equation

$$\gamma u^{\mathrm{iv}} = u'' + \mu\Big(2\,u u'' + (u')^2\Big) + u - u^2 \tag{2.1}$$

converging to the steady state $u=1$ as $x \to \pm\infty$. More precisely, we require that

$$\big(u, u', u'', u'''\big)(x) \to (1,0,0,0) \quad \text{as} \quad x \to \pm\infty \;. \tag{2.2}$$

We use the shooting method to study the solutions of the initial value problem

$$(P)\colon \begin{cases} \gamma u^{\mathrm{iv}} = u'' + \mu\Big(2\,u u'' + (u')^2\Big) + u - u^2 \,, \\ \big(u, u', u'', u'''\big)(0) = (\alpha, 0, \beta, 0) \;. \end{cases}$$

We will seek for a solution of (P) which is increasing on $\mathbb{R}^+$ and require $\beta \ge 0$. Let $f(s) = s - s^2$ and

$$F(s) = \int_s^1 f(t)\,dt \;=\; \frac{1}{6}\,(1-s)^2\,(1+2s) \;.$$

We have $F(s) \ge 0$ iff $s \ge -1/2$.

Equation (2.1) has a prime integral (conservation law). Indeed, if we multiply (2.1) by $2u'$ and integrate over $]-\infty, x[$ and use (2.2) we obtain

$$E(u) := \; 2\,\gamma\, u' u''' - \gamma u''^2 - u'^2 - 2\,\mu u u'^2 + 2\,F(u) \;=\; 0 \;, \tag{2.3}$$

which is known as the conservation law.

We choose $x = 0$ in (2.3) and α in the interval $I := \,]-1/2, 1[$ and obtain $\gamma \beta^2 = 2\,F(\alpha)$. So

$$\beta = \beta(\alpha) = \sqrt{\frac{2}{\gamma}\,F(\alpha)} \;.$$

Problem (P) has a unique local solution $u=u(x,\alpha)$. If $\alpha\in I$, then $\beta(\alpha)>0$ and $u'(x,\alpha)>0$ in a right neighborhood of 0. Then, the number

$$\xi(\alpha) := \sup\Big\{x>0\colon\ u'(t,\alpha)>0,\ t\in\,]0,x[\Big\} \tag{2.4}$$

is well defined for any $\alpha\in I$.

Lemma 3. *Let $\gamma>0$. We have:*

(**a**) $\xi(\alpha)\to 0$ *as* $\alpha\to -1/2^+$,

(**b**) $u\big(\xi(\alpha),\alpha\big)\to -1/2$ *as* $\alpha\to -1/2^+$.

Proof:

(**a**) Let $\alpha=-1/2$. Then

$$u(0)=-1/2\,,\qquad u'(0)=u''(0)=u'''(0)=0$$

and

$$\gamma\, u^{\mathrm{iv}}(0) = -\frac{1}{4} < 0\ .$$

Therefore, there exists an $\varepsilon>0$ such that

$$u(x,-1/2) < -1/2\,,\qquad u^{(k)}(x,-1/2)<0,\quad k=1,2,3,\quad \forall\, x\in\,]0,\varepsilon]\ .$$

Let $\alpha>-1/2$. By the continuous dependence of the solution $u(x,\alpha)$ on α, there exists a $\delta\in\,]0,3/2[$ such that

$$u(\varepsilon,\alpha)<-1/2\,,\qquad -1/2<\alpha<-1/2+\delta\ .$$

Since

$$u(0,\alpha)=\alpha>-1/2\,,\qquad u'(0,\alpha)=0\,,\qquad \beta=u''(0,\alpha)>0\ ,$$

if $-1/2<\alpha<-1/2+\delta$, it follows that

$$0<\xi(\alpha)<\varepsilon\,,\qquad -1/2<\alpha<-1/2+\delta\ .$$

Taking ε arbitrarily small, we conclude that

$$\xi(\alpha)\to 0 \qquad \text{as}\ \ \alpha\to -1/2^+\ .$$

(**b**) By the continuous dependence of the solution $u(x,\alpha)$ on α, we have that $u(x,\alpha)\to u(x,-1/2)$ as $\alpha\to -1/2^+$. Since $u(x,\alpha)$ is uniformly continuous on compact intervals, it follows from (a) that $u\big(\xi(\alpha),\alpha\big)\to u(0,-1/2)=-1/2$ as $\alpha\to -1/2^+$. ∎

Define the shooting set

$$\mathcal{S} := \left\{\widehat{\alpha} > -1/2\colon\ 0 < \xi(\alpha) < \infty,\ u\big(\xi(\alpha),\alpha\big) < 1,\ \forall\alpha \in \left]-\frac{1}{2},\widehat{\alpha}\right[\right\}.$$

Lemma 4. *If* $0 < \gamma \leq \frac{(1+2\mu)^2}{4}$, *then*

(**a**) $u'\big(\xi(\alpha),\alpha\big) = 0$ *for all* $\alpha \in \mathcal{S}$,

(**b**) $\xi \in C^1(\mathcal{S})$,

(**c**) $\mathcal{S}$ *is an open set.* ∎

The proof follows exactly the same arguments as those of Lemma 2.2 in [PBT]. For the next step we need the following technical

Lemma 5. *Let* $u \in C^2([0,a])$ *and suppose that*

$$u'(0) = 0\,, \qquad u(0) \geq 0\,, \qquad u''(x) \geq 0\,, \quad x \in [0,a]\ ,$$

and u'' *is a nondecreasing function. Then*

$$u'^2(x) \leq 2\,u(x)\,u''(x)\,, \qquad x \in [0,a]\ . \tag{2.5}$$

Proof: We know several different proofs, but we prefer the shortest one which is due to Balazs Komuves. From $u'(0) = 0$, $u''(x) \geq 0$ it follows that $u''(x) \geq 0$ in $[0,a]$. Therefore,

$$\int_0^x \Big(u''(x) - u''(t)\Big)\,u'(t)\,dt\ \geq\ 0\ ,$$

which gives (2.5). ∎

Now we can prove

Lemma 6. *Let* $\alpha^* = \sup S$. *Then* $-1/2 < \alpha^* < 0$.

Proof: It is enough to prove that for $\alpha = 0$

$$u''(x) > 0\,, \quad u'(x) > 0 \qquad \text{as long as} \ \ u(x) \leq 1\ .$$

Case 1. $\mu \geq 0$.

By (2.1)

$$\gamma u^{\mathrm{iv}}(0) = u''(0) = \beta = \sqrt{\frac{2}{\gamma}\,F(0)} = \frac{1}{\sqrt{3\,\gamma}} > 0 \ .$$

Then, there exists an $\varepsilon > 0$ such that $u^{\mathrm{iv}}(x) > 0$, $x \in\,]0, \varepsilon[$. Since $u(0) = u'(0) = u'''(0) = 0$, this implies that $u^{(k)}(x) > 0$, $k = 0, 1, 2, 3, 4$, in a right-neighborhood of $x = 0$. Then, by (2.1)

$$\gamma u^{\mathrm{iv}} \;=\; (1 + 2\,\mu u)\,u'' + \mu u'^2 + u - u^2 \;>\; 0 \ , \tag{2.6}$$

and

$$u > 0\,, \qquad u' > 0\,, \qquad u'' > 0\,, \qquad u''' > 0\,, \qquad u^{\mathrm{iv}} > 0$$

as long as $u \leq 1$. Thus, $u^{(k)}(x) > 0$, $k = 0, 1, 2, 3, 4$, as long as $u \leq 1$.

Case 2. $\mu \in \,]-\frac{1}{2}, 0[$.

As in Case 1, there exists an $\varepsilon > 0$ such that

$$u''(x) > 0\,, \qquad u'''(x) > 0\,, \qquad u^{\mathrm{iv}}(x) > 0\,, \quad x \in\,]0, \varepsilon[\ .$$

Claim. *$u'''(x) > 0$ provided that $0 < u(x) < 1$ and $u'(x) > 0$.*

Suppose the contrary, that there exists $x_0 > \varepsilon$, $u(x_0) \in [0, 1[$, $u'''(x_0) = 0$ and x_0 is the smallest number with these properties. By (2.3)

$$2\,F(u) = \gamma u''^2 + u'^2 + 2\,\mu u u'^2 \qquad \text{if} \ \ x = x_0 \ . \tag{2.7}$$

Since $\gamma > 0$, $\mu > -\frac{1}{2}$ and $1 > 1 - u(x_0) > 0$ we obtain by (2.7) that

$$\frac{1}{3}\big(1 - u(x_0)\big)\big(1 + 2\,u(x_0)\big) \;>\; u'^2(x_0) \ . \tag{2.8}$$

We have by (2.1)

$$\gamma u^{\mathrm{iv}} \;=\; (1 + 2\,\mu u)\,u'' + \mu u'^2 + u(1 - u) \;\geq\; (1 - u)\,(u + u'') - \frac{1}{2}\,u'^2 \ . \tag{2.9}$$

Suppose that $u_0 = u(x_0) \geq \frac{1}{4}$. Then, by (2.8) and (2.9),

$$\gamma u^{\mathrm{iv}}(x_0) \;>\; (1 - u_0)\,u_0 - \frac{1}{6}\,(1 - u_0)\,(1 + 2\,u_0) \;=\; \frac{1}{6}\,(1 - u_0)\,(4\,u_0 - 1) \;\geq\; 0 \ . \tag{2.10}$$

Since $u'''(x) > 0$ for all $x \in\,]0, x_0[$, it is impossible to have $u'''(x_0) = 0$, because by (2.10) $u'''(x)$ is increasing in a neighborhood of x_0.

Suppose now that $u_0 \in \,]0, \frac{1}{4}[$. By (2.8)

$$u'^2(x_0) < \frac{1}{2}(1-u_0) < \frac{1}{2}$$

and by (2.9) and Lemma 5:

$$\begin{aligned}
\gamma u^{\mathrm{iv}}(x_0) &\geq (1-u_0)\Big(u_0 + u''(x_0)\Big) - \frac{1}{2}\,u'^2(x_0) \\
&\geq \frac{3}{4}\,2\sqrt{u_0\,u''(x_0)} - \frac{1}{2}\,u'^2(x_0) \\
&\geq \frac{3}{2\sqrt{2}}\,\big|u'(x_0)\big| - \frac{1}{2}\,u'^2(x_0) \\
&> \frac{1}{2}\,\big|u'(x_0)\big|\left(\frac{3}{\sqrt{2}} - \big|u'(x_0)\big|\right) > 0\;.
\end{aligned}$$

As before, it is impossible to have $u'''(x_0)=0$, and then $u'''(x)>0$ as long as $0<u<1$. Thus we have $u'>0$, $u''>0$, as long as $0<u\leq 1$, which proves the lemma. ∎

Below we also need the Maximum principle and so called Boundary Point Lemma [PW, p. 7] which we summarize as:

Proposition 7. *Suppose that $u \in C^2\big(]a,b[\big) \cap C\big([a,b]\big)$ is a nonconstant solution of differential inequality $u''(x) - c\,u(x) \geq 0$, $x \in \,]a,b[$, $c>0$. Then $u(x)<0$, $\forall x \in \,]a,b[$. If u has a nonnegative maximum at a, then $u'(a)<0$. If u has a nonnegative maximum at b, then $u'(b)>0$.* ∎

We assume $\mu \neq 0$ in further considerations, because the case $\mu=0$ is considered in [PBT].

Lemma 8. *Let $\mu > -\frac{1}{2}$ and $0<\gamma \leq \frac{(1+2\mu)^2}{4}$ if $\mu \leq \frac{1}{2}$ and $0<\gamma\leq 2\mu$ if $\mu > \frac{1}{2}$. Then*

$$\xi(\alpha^*) = +\infty \quad \text{and} \quad u(x,\alpha^*) \to 1 \quad \text{as} \;\; x \to +\infty\;.$$

Proof: Suppose for contradiction that $\xi^* := \limsup\big\{\xi(\alpha)\colon\, \alpha \to \alpha^*\big\} < +\infty$ and let $\{\alpha_j\} \subset \mathcal{S}$ be a sequence such that $\alpha_j \to \alpha^*$ and $\xi(\alpha_j) \to \xi^*$ as $j \to +\infty$. We have that

$$u\big(\xi(\alpha_j),\alpha_j\big) \to u(\xi^*,\alpha^*) \;\text{ and }\; u'\big(\xi(\alpha_j),\alpha_j\big) \to u'(\xi^*,\alpha^*) \quad \text{as} \;\; j \to +\infty\;,$$

by the continuous dependence of solutions on x and α on finite intervals.

Claim 1. *We have*

$$u(\xi^*, \alpha^*) = 1 \quad \text{and} \quad u'(\xi^*, \alpha^*) = 0 \; . \tag{2.11}$$

The second assertion follows by $u'\big(\xi(\alpha_j), \alpha_j\big) = 0$ by passing to limit as $j \to +\infty$. As for the first assertion, if $u(\xi^*, \alpha^*) > 1$ for a j sufficiently large, $u\big(\xi(\alpha_j), \alpha_j\big) > 1$ which is impossible because $\alpha_j \in \mathcal{S}$. If $u(\xi^*, \alpha^*) < 1$ by continuity $u(\xi^*, \alpha) < 1$ in a neighborhood of α^* which contradicts the fact that α^* is the supremum of $\mathcal{S}$. Thus, $u(\xi^*, \alpha^*) = 1$ and (2.11) is proved.

Claim 2. $\xi^* = \xi(\alpha^*) = +\infty$.

To show that $\xi^* < \infty$ leads to a contradiction, we use Proposition 7. We set $u = 1 - v$ and rewrite (2.1) as

$$\gamma\, v^{iv} - \Big(1 + 2\,\mu\,(1 - v)\Big)\, v'' + v \;=\; v^2 - \mu v'^2 \; . \tag{2.12}$$

Case 1. $\mu > 0$.

Let $\mu_1 = -\frac{\mu}{\gamma}\, v + \mu_{10}$ and $\mu_2 = \mu_{20}$ where

$$\mu_{10} \;=\; \frac{1 + 2\mu + \sqrt{\Delta}}{2\,\gamma}\,, \qquad \mu_{20} \;=\; \frac{1 + 2\mu - \sqrt{\Delta}}{2\,\gamma}\;,$$
$$\Delta \;=\; (1 + 2\mu)^2 - 4\,\gamma \;\geq\; 0\;,$$

are the roots of the equation $\gamma\, z^2 - (1 + 2\,\mu)\, z + 1 = 0$, which are real and positive if $\mu > -\frac{1}{2}$ and $0 < \gamma \leq \frac{(1+2\,\mu)^2}{4}$. Equation (2.12) can be rewritten as the system

$$(S_1)\colon\; \begin{cases} v'' - \mu_1\, v \;=\; w\;, \\ w'' - \mu_2\, w \;=\; \dfrac{\mu}{\gamma}\, v'^2 + \left(\dfrac{1 - \mu\,\mu_{20}}{\gamma}\right) v^2\;. \end{cases}$$

We have

$$\mu_1 \;=\; -\frac{\mu}{\gamma}\, v + \mu_{10} \;>\; 0\,, \qquad \text{if} \;\; x \in [0, \xi^*]$$

and

$$1 - \mu\,\mu_{20} \;>\; 0\;.$$

Indeed, since $u \in \,]-\frac{1}{2}, 1]$, $v \in [0, \frac{3}{2}[$, we obtain that

$$\mu_{10} \;=\; \frac{1 + 2\mu + \sqrt{(1 + 2\mu)^2 - 4\,\gamma}}{2\,\gamma} \;\geq\; 2\,\frac{\mu}{\gamma} \;>\; \frac{3}{2}\,\frac{\mu}{\gamma} \;>\; \frac{\mu}{\gamma}\, v \;>\; 0$$

and

$$\mu_1 = \mu_{10} - \frac{\mu}{\gamma} v > 0 ,$$

because

$$1 + 2\mu + \sqrt{(1+2\mu)^2 - 4\gamma} \ge 4\mu \iff \sqrt{(1+2\mu)^2 - 4\gamma} \ge 2\mu - 1 .$$

The last inequality holds if either $\mu \in]0, \frac{1}{2}]$ and $0 < \gamma \le \frac{(1+2\mu)^2}{4}$ or $\mu > \frac{1}{2}$ and $0 < \gamma \le 2\mu$. Note that in the last case it follows that $\gamma \le \frac{(1+2\mu)^2}{4}$. Since $\mu_{10} > 0$ the inequality $1 - \mu\,\mu_{20} > 0$ is equivalent to

$$\frac{\mu}{\gamma} = \mu\,\mu_{10}\,\mu_{20} < \mu_{10} = \frac{1 + 2\mu + \sqrt{(1+2\mu)^2 - 4\gamma}}{2\gamma} ,$$

which is satisfied because $\mu > 0$.

Now, we can apply Proposition 7 to system (S_1). We have for $x \in [0, \xi^*[$

$$\frac{\mu}{\gamma} v'^2 + \frac{1 - \mu\,\mu_{20}}{\gamma} v^2 > 0 ,$$

and

$$\begin{aligned} w(0) &= -u''(0) - \mu_1(0)\,(1-\alpha^*) \\ &= -\beta - \Big(\mu_{10} - \frac{\mu}{\gamma}(1-\alpha^*)\Big)(1-\alpha^*) < 0 , \\ w(\xi^*) &= v''(\xi^*) - \mu_1(\xi^*)\,v(\xi^*) = -u''(\xi^*) = 0 , \end{aligned}$$

since $1-\alpha^* > 0$ and $u''^2(\xi^*) = \frac{2}{\gamma} F\big(u(\xi^*)\big) = \frac{2}{\gamma} F(1) = 0$.

Then, by Proposition 7 it follows that $w(x) < 0$, $x \in]0, \xi^*[$. Hence, again by Proposition 7, applied to the first equation of (S_1) and $v(0) = 1-\alpha^* > 0$, $v(\xi^*) = 0$ we obtain that $v'(\xi^*) < 0$. Then $u'(\xi^*) = -v'(\xi^*) > 0$, which contradicts $u'(\xi^*) = 0$. Thus ξ^* cannot be finite, so $\xi^* = +\infty$.

Case 2. $\mu \in]-\frac{1}{2}, 0[$.

In this case, (2.12) is equivalent to the system

$$(S_2)\colon \begin{cases} v'' - \mu_1 v = w , \\ w'' - \mu_2 w = \dfrac{1}{\gamma}\Big((1 - 2\,\mu\,\mu_{10})\,v^2 - \mu\,v'^2\Big) , \end{cases}$$

where $\mu_1 = \mu_{10}$ and $\mu_2 = -\frac{2\mu}{\gamma} v + \mu_{20}$,

$$\mu_{10} = \frac{1 + 2\mu + \sqrt{\Delta}}{2\gamma} , \quad \mu_{20} = \frac{1 + 2\mu - \sqrt{\Delta}}{2\gamma} , \qquad \Delta = (1+2\mu)^2 - 4\gamma \ge 0 ,$$

are the roots of the equation $\gamma z^2 - (1+2\mu)z+1 = 0$, which are real and positive if $\mu > -\frac{1}{2}$ and $0 < \gamma \leq \frac{(1+2\mu)^2}{4}$. Next, we have

$$\mu_1 = \mu_{10} > 0\,, \quad \mu_2 = -\frac{2\mu}{\gamma}\,v + \mu_{20} > 0\,, \quad \text{if } \ x \in [0,\xi^*]$$

and

$$1 - 2\,\mu\,\mu_{10} > 0 \ .$$

Moreover,

$$w(0) \,=\, -\beta - \mu_{10}\,(1-\alpha^*) \,<\, 0 \ ,$$
$$w(\xi^*) \,=\, v''(\xi^*) - \mu_{10}\,v(\xi^*) \,=\, 0 \ ,$$

by $v(\xi^*) = 1 - u(\xi^*) = 0$ and $u''^2(\xi^*) = \frac{2}{\gamma}\,F(u(\xi^*)) = \frac{2}{\gamma}\,F(1) = 0$, $v''(\xi^*) = -u''(\xi^*)$.

Then, by Proposition 7 applied to the second equation of (S_2), it follows that $w(x) < 0$, $x \in \,]0,\xi^*[$. Again by Proposition 7, applied to the first equation of (S_2), and $v(0) = 1-\alpha^* > 0$, $v(\xi^*) = 0$, we obtain that $v'(\xi^*) < 0$. Thus $u'(\xi^*) = -v'(\xi^*) > 0$, which contradicts $u'(\xi^*) = 0$. So, $\xi^* = +\infty$ in the second case as well, which proves Claim 2.

Claim 3. *We have* $u(x,\alpha^*) \to 1$ *as* $x \to +\infty$.

There exists the limit $l = \lim_{x\to+\infty} u(x,\alpha^*) \leq 1$ by $u(x,\alpha^*) < 1$ and $u'(x,\alpha^*) > 0$. We will prove that the cases (i) $l \leq 0$ and (ii) $0 < l < 1$ are impossible, so $l = 1$.

Case (i.1) $l \leq 0$, $\mu \in \,]-\frac{1}{2}, 0[$.

For brevity, by $u(x)$ or u we mean $u(x,\alpha^*)$. We have

$$\mu\,u'^2(x) < 0\,, \quad u(x) < l \leq 0\,, \quad 1+2\,\mu\,u(x) \geq 1\,, \quad u''(0) > 0$$

and there exists a sequence $\xi_n \to +\infty$ such that $u''(\xi_n) \to 0$ as $n \to +\infty$. Suppose that $u''(\xi_n) \geq 0$ for infinitely many ξ_n. Then, by Proposition 7, applied to $v = u''$ in

$$\gamma\,u^{\mathrm{iv}} - (1+2\,\mu\,u)\,u'' \,=\, \mu\,u'^2 + u - u^2 < 0\,, \quad u''(0) > 0\,, \quad u''(\xi_n) \geq 0 \ ,$$

we obtain that $u''(x) > 0$, $x \in \mathbb{R}^+$. Suppose now, by contradiction, that there exists an $\eta > 0$ such that $u''(\eta) = 0$, $u''(x) < 0$, $x > \eta$ and $u''(x) \to 0$ as $x \to +\infty$. Then $u''(x)$ has a minimum point ξ_0 in $[\eta,\infty)$ in which $u''(\xi_0) < 0$, $u^{\mathrm{iv}}(\xi_0) \geq 0$ and hence $\gamma\,u^{\mathrm{iv}}(\xi_0) - \big(1+2\,\mu\,u(\xi_0)\big)\,u''(\xi_0) > \gamma\,u^{\mathrm{iv}}(\xi_0) \geq 0$, which is a contradiction.

So, we have $u''(x) > 0$, $x \in \mathbb{R}^+$ and then $u(x) > u(\xi_n) + u'(\xi_n)(x - \xi_n)$ which implies that $u(x) \to +\infty$ as $x \to +\infty$, a contradiction.

Case (i.2) $l \le 0$, $\mu > 0$.

We obtain integrating (2.1) from 0 to x

$$\gamma u''' - (1 + 2\mu u)u' = \int_0^x \Big(-\mu u'^2(t) + u(t) - u^2(t)\Big)\, dt \, . \tag{2.13}$$

Denote

$$r_1(x) := -\mu u'^2(x) + u(x) - u^2(x) < 0 \, , \qquad r(x) := \int_0^x r_1(t)\, dt \, .$$

We have integrating (2.13) from 0 to x

$$\gamma u''(x) - \gamma u''(0) - u(x) - \mu u^2(x) + u(0) + \mu u^2(0) = \int_0^x r(t)\, dt \, .$$

Hence,

$$\gamma u''(x) = \gamma u''(0) - \alpha^* - \mu \alpha^{*2} + u(x) + \mu u^2(x) + \int_0^x r(t)\, dt \, . \tag{2.14}$$

Since r is negative and strictly decreasing on $\mathbb{R}^+$, $\int_0^x r(t)dt \to -\infty$ as $x \to +\infty$ and because $l \le 0$, the right hand side of (2.14) tends to $-\infty$ as $x \to +\infty$. This contradicts the existence of the sequence $\xi_n \to +\infty$ such that $u''(\xi_n) \to 0$ as $n \to +\infty$.

Case (ii.1) $0 < l < 1$, $\mu \in \,]-\frac{1}{2}, 0[$.

In this case $r_1(x) = -\mu u'^2(x) + u(x) - u^2(x) \ge C > 0$ for sufficiently large x, and

$$r(x) = \int_0^x r_1(t)\, dt \ge Cx - C_1 \, , \qquad \int_0^x r(t)\, dt \ge C\frac{x^2}{2} - C_1 x \, .$$

Then, by (2.14)

$$\gamma u''(x) \ge C\frac{x^2}{2} - C_1 x - C_2 \, ,$$

so $\lim_{x\to+\infty} u''(x) = +\infty$, which as before leads to a contradiction.

Case (ii.2) $0 < l < 1$, $\mu > 0$.

We will show that $\lim_{x\to+\infty} u'(x) = 0$, which gives $r_1(x) = -\mu u'^2(x) + u(x) - u^2(x) \ge C > 0$ for sufficiently large x, and we can proceed as in previous case. We will prove that $u''(x) < 0$ for sufficiently large x. Then, the assertion $\lim_{x\to+\infty} u'(x) = 0$ follows from the fact that there exists a sequence $(\eta_k)_k$: $\eta_k \to +\infty$, $u'(\eta_k) \to 0$.

By (2.1)

$$\gamma u^{iv} - (1+2\mu u)u'' = \mu u'^2 + u - u^2 > 0$$

for sufficiently large x, because $u(x) \to l \in]0,1[$ as $x \to +\infty$ and $l - l^2 > 0$. Suppose that u'' oscillates and has infinitely many zeros tending to $+\infty$. Let η_1 and η_2 be two subsequent zeros. Since $1+2\mu u(x) > 0$ for sufficiently large x, by Proposition 7 it follows that $u''(x) < 0$, $x \in]\eta_1, \eta_2[$. Then, either $u''(x) < 0$ or $u''(x) > 0$ for sufficiently large x. If $u''(x) > 0$, $x > R$, by $u'(x) > 0$ we get a contradiction with $u(x) < l$, $x > R$. Thus there exists $R > 0$, $u''(x) < 0$, $x > R$.

Therefore, the only possible case is $l = 1$, which proves Claim 3 and ends the proof of Lemma 8. ∎

Proof of Theorem 1: We will prove that the solution $u(x) = u(x, \alpha^*)$, constructed in Lemma 8 satisfies as well

$$\big(u', u'', u'''\big)(x) \to (0,0,0) \quad \text{as} \quad x \to +\infty \ .$$

Case 1. $\mu > 0$.

By Claim 3 in the proof of Lemma 8, there exists $R > 0$ such that $u''(x) < 0$, $\forall x > R$ and therefore $\lim_{x\to+\infty} u'(x) = 0$.

Then, by differentiation of $\gamma u^{iv} - (1+2\mu u)u'' = \mu u'^2 + u - u^2$, we have

$$\gamma u^{v} - (1+2\mu u)u''' = u'(1 - 2u + 4\mu u u'') < 0$$

for $x > R_1 > R$, where R_1 is sufficiently large. By Proposition 7, as in Claim 3, $u'''(x)$ is either positive or negative for large x. In fact, the case $u'''(x) < 0$ is impossible because then $u''(x) < 0$ and $u''(x)$ is decreasing then there is no sequence $\xi_n \to +\infty$ such that $u''(\xi_n) \to 0$ as $n \to +\infty$. Thus $u'''(x) > 0$ and hence $u''(x)$ is an increasing function and by $u''(\xi_n) \to 0$ as $n \to +\infty$ it follows $u''(x) \to 0$ as $x \to +\infty$. Then, by (2.1) we infer $u^{iv}(x) \to 0$ as $x \to +\infty$. As for u''', by Taylor's formula

$$h u'''(x) = u''(x+h) - u''(x) - \frac{h^2}{2} u^{iv}(\xi) , \qquad \xi \in]x, x+h[\ ,$$

for a fixed h, letting $x \to +\infty$, we obtain that $u'''(x) \to 0$ as well.

Case 2. $\mu \in]-\frac{1}{2}, 0[$.

Since $r_1(x) = -\mu u'^2(x) + u(x) - u^2(x) > 0$ for large x, $r(x) = \int_0^x r_1(t)\,dt$ is strictly increasing for large x. There exists a sequence $\xi_n \to +\infty$ such that

$u''(\xi_n) \to 0$ as $n \to +\infty$ and by

$$\gamma\, u''(x) - \gamma\, u''(0) - u(x) - \mu\, u^2(x) + u(0) + \mu\, u^2(0) \;=\; \int_0^x r(t)\, dt$$

for $x = \xi_n$, it follows that

$$\lim_{n\to\infty} \int_0^{\xi_n} r(t)\, dt \;<\; +\infty \;.$$

Since $r(x)$ is an increasing function, the integral $\int_0^\infty r(t)\, dt$ is convergent, and then $\lim_{x\to\infty} u''(x)$ exists and $\lim_{x\to\infty} u''(x) = 0$ since $u''(\xi_n) \to 0$. By Taylor's formula and $\lim_{x\to\infty} u(x) = 1$ it follows $\lim_{x\to\infty} u'(x) = 0$, and as in Case 1 $\lim_{x\to\infty} u^{\mathrm{iv}}(x) = \lim_{x\to\infty} u'''(x) = 0$, which ends the proof of Theorem 1. ∎

3 – Proof of Theorem 2

Let

$$h(s,\mu) \;:=\; \frac{f^2(s)}{(1+2\,\mu s)^2\, F(s)} \;=\; \frac{6\, s^2}{(1+2\,\mu s)^2\,(1+2s)}$$

and for $\gamma > 0$, let $m(\gamma,\mu)$ be the greatest negative root of the equation

$$\frac{6\, s^2}{(1+2\,\mu s)^2\,(1+2s)} = \frac{1}{2\,\gamma}\,, \qquad s > -\frac{1}{2}\;,$$

or the greatest negative zero of the polynomial

$$P_3(s) \;:=\; 8\,\mu^2\, s^3 + 4\,(\mu^2 + 2\mu - 3\,\gamma)\, s^2 + 2\,(2\mu+1)\, s + 1\;.$$

Lemma 9. *We have:*

(**a**) $m(\gamma,\mu) \;=\; \inf\Big\{ s_0 < 0\colon\; h(s,\mu) < \frac{1}{2\gamma},\; s_0 < s < 0 \Big\}$.

(**b**) $m(\gamma,\mu) \to -\frac{1}{2}^{+}$ *as* $\gamma \to 0^+$ *if* $\mu \in\,]-\frac{1}{2}, 1]$ *and* $m(\gamma,\mu) \to -\frac{1}{2\mu}^{+}$ *as* $\gamma \to 0^+$ *if* $\mu > 1$.

(**c**)
$$\lim_{\gamma\to 0^+} \frac{1}{\gamma}\left(m(\gamma,\mu) + \frac{1}{2}\right) = \frac{3}{2\,(1-\mu)^2}\,, \qquad \mu \in\,]-\tfrac{1}{2}, 1[\,;$$

$$\lim_{\gamma\to 0^+} \frac{1}{\gamma}\left(m(\gamma,1) + \frac{1}{2}\right)^3 = \frac{3}{8}\,, \qquad \mu = 1\,;$$

$$\lim_{\gamma\to 0^+} \left(\frac{1}{\gamma}\left(m(\gamma,\mu) + \frac{1}{2\mu}\right)^2\right) = \frac{3}{4\,\mu^3\,(\mu-1)}\,, \qquad \mu > 1\;.$$

Proof:

Claim. *The function* $h(s,\mu)$ *is decreasing in* s *for* $\mu \in \,]-\frac{1}{2}, 1]$, $s \in \,]-\frac{1}{2}, 0[$ *and for* $\mu > 1$, $s \in \,]-\frac{1}{2\mu}, 0[$.

Indeed, by

$$h'_s(s,\mu) \;=\; \frac{12\,s\,(s+1-2\,\mu\,s^2)}{(1+2\,\mu\,s)^3\,(1+2\,s)^2}\;,$$

it follows that $h'_s(s,\mu) < 0$ if either $\mu \in \,]-\frac{1}{2}, 0[$, $s \in \,]-\frac{1}{2}, 0[$ or $\mu > 1$, $s \in \,]-\frac{1}{2\mu}, 0[$. Note that, the factor $s+1-2\mu s^2$ is positive if $s \in \,]\frac{1}{4\mu}(1-\sqrt{1+8\mu}),\, \frac{1}{4\mu}(1+\sqrt{1+8\mu})[$ and $\frac{1}{4\mu}\big(1-\sqrt{1+8\,\mu}\big) < -\frac{1}{2}$, $-\frac{1}{2\mu} < -\frac{1}{2}$ for $0<\mu<1$. Hence $h'_s(s,\mu) < 0$ if $\mu \in \,]0,1[$, $s \in \,]-\frac{1}{2}, 0[$.

Some graphs of functions $h(s,\mu)$ are presented on Figure 1.

Obviously, (a) follows from the Claim. To prove (b) and (c) we consider the cases $\mu \in \,]-\frac{1}{2}, 1]$ and $\mu > 1$.

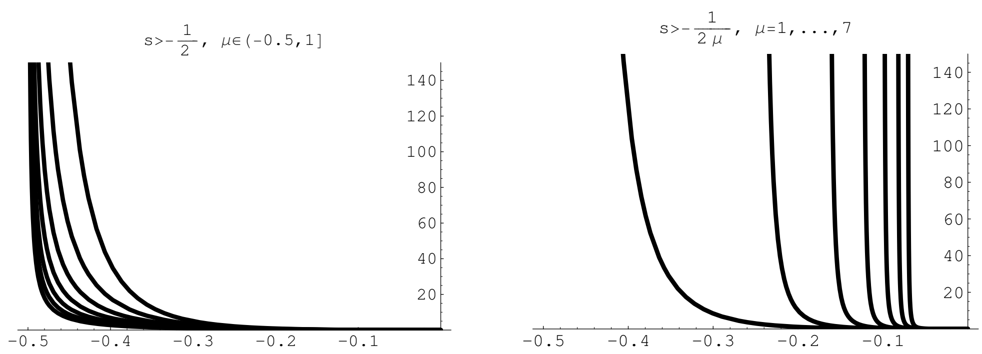

Figure 1 – Graphs of functions $h(s,\mu) = \dfrac{6\,s^2}{(1+2\,\mu\,s)^2\,(1+2\,s)}$.
Left: $\mu = -0.4 + (k-1)\,0.2$, $k = 1, ..., 7$, $-\frac{1}{2} < s < 0$;
Right: $\mu = 1, ..., 7$, $-\frac{1}{2\mu} < s < 0$.

Case 1. $\mu \in \,]-\frac{1}{2}, 1]$.

We have

$$h(s,\mu) \;\to\; \begin{cases} 0\,, & s \to 0^-\,, \\ +\infty\,, & s \to -\frac{1}{2}^+\,. \end{cases}$$

By the Claim, for every $\varepsilon \in \,]0, \frac{1}{2}[$ there exists a number $M_\varepsilon > 0$ such that $h(-\frac{1}{2}+\varepsilon, \mu) = M_\varepsilon$ and $h(s,\mu) < M_\varepsilon$ if $s \in \,]-\frac{1}{2}+\varepsilon, 0[$. Moreover $M_\varepsilon \to \infty$ as $\varepsilon \to 0$.

Then $h(s,\mu) < \frac{1}{2\gamma}$ if $s \in]-\frac{1}{2}+\varepsilon, 0[$ and $0<\gamma<\frac{1}{2M_\varepsilon}$. Hence $m(\gamma,\mu) \to -\frac{1}{2}^+$ as $\gamma \to 0^+$.

We have

$$\lim_{s\to -\frac{1}{2}^+} \left(s+\frac{1}{2}\right) h(s,\mu) \;=\; \lim_{s\to -\frac{1}{2}^+} \frac{3s^2}{(1+2\,\mu s)^2} \;=\; \frac{3}{4(1-\mu)^2}$$

and thus

$$\lim_{\gamma\to 0^+} \left(m(\gamma,\mu)+\frac{1}{2}\right) h\big(m(\gamma,\mu),\mu\big) \;=\; \lim_{\gamma\to 0^+} \left(m(\gamma,\mu)+\frac{1}{2}\right) \frac{1}{2\,\gamma} \;=\; \frac{3}{4(1-\mu)^2} \;\Longrightarrow$$

$$\Longrightarrow\; \lim_{\gamma\to 0^+} \frac{1}{\gamma}\left(m(\gamma,\mu)+\frac{1}{2}\right) \;=\; \frac{3}{2(1-\mu)^2}\;.$$

If $\mu = 1$, a direct calculation shows that

$$\lim_{s\to -\frac{1}{2}^+} \left(s+\frac{1}{2}\right)^3 h(s,\mu) \;=\; \frac{3}{16}$$

and then

$$\lim_{\gamma\to 0^+} \left(m(\gamma,1)+\frac{1}{2}\right)^3 h\big(m(\gamma,1),1\big) \;=\; \lim_{\gamma\to 0^+} \left(m(\gamma,1)+\frac{1}{2}\right)^3 \frac{1}{2\gamma} \;=\; \frac{3}{16} \;\Longrightarrow$$

$$\Longrightarrow\; \lim_{\gamma\to 0^+} \frac{1}{\gamma}\left(m(\gamma,1)+\frac{1}{2}\right)^3 = \frac{3}{8}\;.$$

Case 2. $\mu > 1$.

We have

$$h(s,\mu) \;\to\; \begin{cases} 0\,, & s\to 0^-\,,\\ +\infty\,, & s\to -\frac{1}{2\mu}^+\,, \end{cases}$$

and by the Claim, for every $\varepsilon \in]0, \frac{1}{2\mu}[$ there exists a number $M'_\varepsilon > 0$ such that $h\big(-\frac{1}{2\mu}+\varepsilon,\mu\big) = M'_\varepsilon$ and $h(s,\mu) < M'_\varepsilon$ if $s \in]-\frac{1}{2\mu}+\varepsilon, 0[$. Moreover $M'_\varepsilon \to \infty$ as $\varepsilon \to 0$. Then $h(s,\mu) < \frac{1}{2\gamma}$ if $s \in]-\frac{1}{2\mu}+\varepsilon, 0[$ and $0<\gamma<\frac{1}{2\,M'_\varepsilon}$. Hence $m(\gamma,\mu) \to -\frac{1}{2\mu}^+$ as $\gamma \to 0^+$ and as before we obtain

$$\lim_{\gamma\to 0^+} \left(\frac{1}{\gamma}\left(m(\gamma,\mu)+\frac{1}{2\,\mu}\right)^2\right) \;=\; \frac{3}{4\,\mu^3\,(\mu-1)}\;.\ \blacksquare$$

Remark. Let u_γ be a family of even homoclinic solutions of (2.1). It follows from Theorem 2 and Lemma 9 that

$$u_\gamma(0) \sim \begin{cases} -\dfrac{1}{2} + \dfrac{3\,\gamma}{2\,(1-\mu)^2} & \text{for } \mu \in \,]-\tfrac{1}{2}, 1[\,, \\ -\dfrac{1}{2} + \left(\dfrac{3\,\gamma}{8}\right)^{1/3} & \text{for } \mu = 1\,, \\ -\dfrac{1}{2\mu} + \left(\dfrac{3\,\gamma}{4\,\mu^3\,(\mu-1)}\right)^{1/2} & \text{for } \mu > 1\ . \end{cases}$$

as $\gamma \to 0^+$. □

Define

$$x_1(\alpha) := \sup\Big\{x > 0\colon\ u(t,\alpha) < 1,\ t \in [0,x[\Big\}$$

and

$$\mathcal{A} := \Big\{\widehat{\alpha} < 1\colon\ u'''(x,\alpha) > 0 \ \text{ on }]0, x_1(\alpha)[\ \text{ for all } \widehat{\alpha} < \alpha < 1\Big\}\ .$$

By the proof of Lemma 6 if $u(0) = 0$ for $\mu > -\frac{1}{2}$, then

$$u' > 0, \quad u'' > 0, \quad u''' > 0 \qquad \text{as long as} \quad u \le 1\ . \tag{3.1}$$

The same arguments work for $\alpha \in [0,1[$ and (3.1) holds. Then $\mathcal{A}$ is well defined and $[0,1[\, \subset \mathcal{A}$. It follows by continuity that $\mathcal{A}$ is an open set. Let $\alpha_* := \inf \mathcal{A}$. It is clear that $\mathcal{A} = \,]\alpha_*, 1[$. Let $u(x,\alpha_0)$ be a solution of problem (2.1), (2.2), which is bounded above by $u = 1$. Because $u''(x,\alpha) > 0$ on $]0, x_1(\alpha)[$ for any $\alpha \in \mathcal{A}$ it is clear that $u(x,\alpha)$ can not be bounded above by $u = 1$ if $\alpha \in \mathcal{A}$. Therefore $\alpha_0 \le \alpha_*$. We will prove that

$$\alpha_* < m(\gamma,\mu)\ .$$

Assume on the contrary that $\alpha_* \ge m(\gamma,\mu)$. We have

Claim 1. *$x_1(\alpha_*) < \infty$ and $u'''(x,\alpha_*) \ge 0$ for all $x \in \,]0, x_1(\alpha_*)[$.*

Let $\{\alpha_j\} \subset \mathcal{A}$ be a decreasing sequence such that $\alpha_j \to \alpha_*$. Then, by the continuous dependence on the initial data, $u^{(k)}(x,\alpha_j) \to u^{(k)}(x,\alpha_*)$, $k = 0,1,2,3$. Hence

$$u'''(x,\alpha_*) \ge 0 \qquad \text{for all } \ x, \quad 0 \le x < x_1(\alpha_*) =: x_1\ ,$$

and

$$u''(x,\alpha_*) \ge u''(0,\alpha_*) = \beta(\alpha_*) > 0 \qquad \text{for all } \ x, \quad 0 \le x < x_1\ ,$$

which implies that $x_1 < \infty$ and

$$u'\big(x_1(\alpha), \alpha\big) > 0 \qquad \text{for all} \;\; \alpha \in [\alpha_*, 1[\;. \;\blacksquare$$

Claim 2. *$x_1(\alpha) < \infty$ for all $\alpha \in [\alpha_*, 1[$ and there exists $\widehat{x} \in\,]0, x_1(\alpha_*)[$: $u'''(\widehat{x}, \alpha_*) = 0$.*

Suppose that $u'''(x, \alpha_*) > 0$ for all x, $0 < x \le x_1(\alpha_*)$. By continuity, there exists a sufficiently small $\delta > 0$ such that $u'''(x, \alpha) > 0$ for all $\alpha \in\,]\alpha_* - \delta, \alpha_*[$. This is a consequence of the following facts. Observe that $u'''(0, \alpha_*) = 0$ and $u^{iv}(0,\alpha_*) > 0$ by (2.1). At the other end point $x_1 = x_1(\alpha_*)$ of the interval $[0, x_1(\alpha_*)]$ we have $u'''(x_1, \alpha_*) \ge 0$. In fact, we have $u'''(x_1, \alpha_*) > 0$. Indeed, if $\mu > 0$, by (2.1),

$$\gamma\, u^{iv} = (1 + 2\,\mu u)\, u'' + \mu\, u'^2 > 0 \qquad \text{at} \;\; x = x_1\,,$$

because $u(x_1, \alpha_*) = 1$, $1 + 2\,\mu u(x_1, \alpha_*) = 1 + 2\mu > 0$, $u''(x_1, \alpha_*) > 0$ by Claim 1. Hence $u^{iv}(x_1, \alpha_*) > 0$. If $u'''(x_1, \alpha_*) = 0$, then $u''' < 0$ in a left neighborhood of x_1 which contradicts Claim 1. If $\mu \in\,]-\frac{1}{2}, 0[$, by the conservation law (2.3) we have

$$\gamma\, u''^2 + (1 + 2\,\mu u)\, u'^2 = 0 \qquad \text{at} \;\; x = x_1\,,$$

because $u(x_1, \alpha_*) = 1$, $u'''(x_1, \alpha_*) = 0$ and $1 + 2\mu > 0$. Then $(u', u'', u''')(x_1) = 0$, which by uniqueness property implies that $u \equiv 1$, which is a contradiction. Hence $u'''(x_1, \alpha_*) > 0$. So $u'''(x, \alpha) > 0$ for all $\alpha \in\,]\alpha_* - \delta, \alpha_*[$ and for all x, $0 < x \le x_1(\alpha_*)$, but this contradicts the definition of $\alpha_* = \inf \mathcal{A}$. Thus, there exists $\widehat{x} \in\,]0, x_1(\alpha)[$ such that $u'''(\widehat{x}, \alpha_*) = 0$. $\blacksquare$

Now, we will prove that the assertion of Claim 2, that the function $u'''(x, \alpha_*)$ vanishes at an interior point of the interval $[0, x_1(\alpha)]$ is impossible. Define the function

$$H(x, \alpha) \;:=\; 2\,\gamma\, \frac{u'''(x, \alpha)}{u'(x, \alpha)} - 1 - 2\,\mu\, u(x, \alpha)\;. \tag{3.2}$$

By l'Hôpital's rule it follows that

$$\begin{aligned}
&\lim_{x \to 0^+} \gamma\, \frac{u'''(x, \alpha)}{u'(x, \alpha)} \;= \\
&\quad = \lim_{x \to 0^+} \gamma\, \frac{u^{iv}(x, \alpha)}{u''(x, \alpha)} \\
&\quad = \lim_{x \to 0^+} \gamma\, \frac{1}{u''(x, \alpha)} \Big(\big(1 + 2\,\mu\, u(x, \alpha)\big)\, u''(x, \alpha) + u(x, \alpha) - u^2(x, \alpha) + \mu\, u''^2(x, \alpha) \Big) \\
&\quad = \frac{1}{\beta} \Big((1 + 2\,\mu\, \alpha)\, \beta + \alpha - \alpha^2 \Big)\,,
\end{aligned}$$

and

$$\begin{aligned} H(0,\alpha) &= 2(1+2\mu\alpha) + \frac{2}{\beta}(\alpha-\alpha^2) - 1 - 2\mu\alpha \\ &= 1+2\mu\alpha + \frac{2}{\beta}(\alpha-\alpha^2) \\ &= 1+2\mu\alpha + \frac{2\sqrt{3\gamma}\,\alpha}{\sqrt{1+2\alpha}}\,. \end{aligned}$$

By the assumption $\alpha_* \ge m(\gamma,\mu)$ and Lemma 9 it follows that

$$h(\alpha,\mu) = \frac{6\alpha^2}{(1+2\mu\alpha)^2(1+2\alpha)} < \frac{1}{2\gamma}\,, \qquad \alpha \in \,]\alpha_*, 1[\,.$$

Hence

$$|1+2\mu\alpha| > \frac{2\sqrt{3\gamma}\,|\alpha|}{\sqrt{1+2\alpha}}\,, \qquad \alpha \in \,]\alpha_*, 1[\,.$$

If $|\mu| < \frac{1}{2}$, by $\alpha \in \,]-\frac{1}{2}, 1[$ we have $1+2\mu\alpha > 0$. If $\mu \ge 1$ and $\alpha \in \,]-\frac{1}{2\mu}, 1[$, $m(\gamma,\mu) > -\frac{1}{2\mu}$ and if $0<\mu<1$, $m(\gamma,\mu) > -\frac{1}{2} > -\frac{1}{2\mu}$. So

$$\alpha_* \ge m(\gamma,\mu) > \max\left\{-\frac{1}{2}, -\frac{1}{2\mu}\right\}, \qquad \mu>0\,,$$

and

$$1+2\alpha_* > 0 \quad \text{and} \quad 1+2\mu\alpha_* > 0\,, \quad \mu>0\,. \tag{3.3}$$

Hence

$$1+2\mu\alpha > 0 \qquad \text{for all } \alpha \in [\alpha_*, 1[\text{ and } \mu > -\frac{1}{2}, \quad \mu \ne 0\,,$$

and

$$H(0,\alpha) > 0\,, \qquad \alpha \in [\alpha_*, 1[\,.$$

Claim 3. *$H(x,\alpha) > 0$ for all $x \in [0, x_1(\alpha)[$ and $\alpha \in [0,1[$.*

By the proof of Lemma 6, if $u(0)=\alpha\ge 0$ and $\mu > -\frac{1}{2}$, $\mu \ne 0$, it follows

$$u' > 0\,, \quad u'' > 0\,, \quad u''' > 0 \qquad \text{as long as} \quad u \le 1\,.$$

By (2.1),

$$\begin{aligned} (u'H)' &= 2\gamma u^{\text{iv}} - (1+2\mu u)u'' - 2\mu u'^2 \\ &= (1+2\mu u)u'' + 2(u-u^2) \;>\; 0\,, \end{aligned}$$

because $u(x) \geq u(0) = \alpha \in [0,1)$, $1 + 2\,\mu\, u > 1 - u > 0$, $u - u^2$ and $u'' > 0$. Hence

$$u'(x,\alpha)\, H(x,\alpha) \,>\, u'(0,\alpha)\, H(0,\alpha) \,\geq\, 0 \qquad \text{for all } \; x \in [0, x_1(\alpha)[\;\text{ and }\; \alpha \in [0,1[\;.$$

Since $u'(x,\alpha) > 0$, we have $H(x,\alpha) > 0$ for all $x \in [0, x_1(\alpha))$ and $\alpha \in [0,1[$. ∎

Claim 4. $H\big(x_1(\alpha),\alpha\big) > 0$ *for all* $\alpha \in [\alpha_*, 1[$.

If $\alpha \in \,]\alpha_*, 1[$, then $\alpha \in \mathcal{A}$ and $u'''(x,\alpha) > 0$, $\forall x \in \,]0, x_1(\alpha)[$, so $u''(x,\alpha) > 0$ and $u'(x,\alpha) > 0$ $\forall x \in \,]0, x_1(\alpha)]$. If $\alpha = \alpha_*$, by Claim 1, $u'''(x,\alpha_*) \geq 0$ and $u'\big(x_1(\alpha),\alpha\big) > 0$ if $\alpha \in [\alpha_*, 1[$. Since $u\big(x_1(\alpha),\alpha\big) = 1$ and $F(1) = 0$, it follows by (2.3) that

$$u'^2 H \,=\, 2\,\gamma\, u' u''' - (1 + 2\,\mu\, u)\, u'^2 \,=\, \gamma\, u''^2 \,>\, 0$$

at $x = x_1(\alpha)$, $\alpha \in [\alpha_*, 1[$ and by $u'\big(x_1(\alpha),\alpha\big) > 0$ one gets $H\big(x_1(\alpha),\alpha\big) > 0$. ∎

End of the proof of Theorem 2: Define

$$\mathcal{T} \,:=\, \Big\{ \widehat{\alpha} \in (\alpha_*, 1) \colon\; H(x,\alpha) > 0 \;\text{ for all }\; x \in [0, x_1(\alpha)] \;\text{ and }\; \alpha \in \,]\widehat{\alpha}, 1[\Big\} \;.$$

By Claim 3, we have $[0,1[\, \subset \mathcal{T}$, and by Claim 2 it follows that

$$H(\widehat{x}, \alpha_*) \,=\, -1 - 2\,\mu\, u(\widehat{x}, \alpha_*) \,<\, 0 \;,$$

because since $u(\cdot, \alpha_*)$, $u'(\cdot, \alpha_*)$, $u''(\cdot, \alpha_*)$ are increasing functions, $u(\widehat{x}, \alpha_*) > u(0, \alpha_*) = \alpha_*$, and $1 + 2\,\mu\, u(\widehat{x}, \alpha_*) > 1 + 2\,\mu\, \alpha_* > 0$ by (3.3). Hence $\alpha_* \notin \mathcal{T}$ and $\mathcal{T} \subset \mathcal{A}$. By continuous dependence on parameters, $\mathcal{T}$ is an open subset of $\mathcal{A}$ and let $\widetilde{\alpha} := \inf \mathcal{T}$. Then $\alpha_* < \widetilde{\alpha} < 0$ and $H(x, \widetilde{\alpha}) \geq 0$ for all $x \in [0, x_1(\widetilde{\alpha})]$. By Claims 3 and 4, there exists an interior minimum point $\widetilde{x} \in [0, x_1(\widetilde{\alpha})]$ of the function H and

$$H(\widetilde{x}, \widetilde{\alpha}) \,=\, H_x(\widetilde{x}, \widetilde{\alpha}) \,=\, 0 \;.$$

Next calculations are done for $(x, \alpha) = (\widetilde{x}, \widetilde{\alpha})$. We have

$$\begin{aligned} 0 \,=\, H_x \,&=\, \frac{2\,\gamma}{u'^2} \left(u^{\mathrm{iv}} u' - u''' u'' \right) - 2\,\mu\, u' \\ &=\, \frac{2}{u'^2} \left(\gamma\, u^{\mathrm{iv}} u' - \gamma\, u''' u'' - \mu\, u'^3 \right) \,, \end{aligned}$$

and by (2.1),

$$\begin{aligned} \gamma\, u^{\mathrm{iv}} \,&=\, \frac{\gamma\, u''' u''}{u'} + \mu\, u'^2 \,=\, (1 + 2\,\mu\, u)\, u'' + \mu\, u'^2 + u - u^2 \;, \\ \frac{\gamma\, u''' u''}{u'} \,&=\, (1 + 2\,\mu\, u)\, u'' + u - u^2 \;, \end{aligned}$$

so

$$\gamma\, u''' = (1+2\,\mu\, u)\, u' + (u-u^2)\, \frac{u'}{u''}\ . \tag{3.4}$$

Moreover,

$$0 = H \iff 2\,\gamma\, u''' = (1+2\,\mu\, u)\, u' \tag{3.5}$$

and by the conservation law (2.3) it follows that

$$\gamma\, u''^2 = \frac{1}{3}\,(1-u)^2\,(1+2u)\ .$$

We obtain by (3.4) and (3.5)

$$\frac{1}{2}\,(1+2\,\mu\, u)\, u' = (1+2\,\mu\, u)\, u' + (u-u^2)\, \frac{u'}{u''} \iff$$

$$2\,(u-u^2) = -(1+2\,\mu\, u)\, u'' \tag{3.6}$$

and hence

$$h(u,\mu) = \frac{6\, u^2}{(1+2\,\mu\, u)^2\,(1+2u)} = \frac{1}{2\,\gamma}\ .$$

Since $u<1$, $1+2\,\mu\, u>0$ and $u''>0$ by definition of $\mathcal{A}$, from (3.6) it follows that $u<0$. Then, by the definition of $m(\gamma,\mu)$, it follows $u(\widetilde{x},\widetilde{\alpha}) \le m(\gamma,\mu)$. Since u is increasing on $[0, x_1(\widetilde{\alpha})]$ and $\widetilde{\alpha}\in\mathcal{A}$, we obtain $\alpha_* < \widetilde{\alpha} = u(0,\widetilde{\alpha}) < u(\widetilde{x},\widetilde{\alpha}) \le m(\gamma,\mu)$, which contradicts the original assumption $\alpha_* \ge m(\gamma,\mu)$ and ends the proof of Theorem 2. ∎

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