

Lipschitz continuity for elliptic free boundary problems with Dini mean oscillation coefficients

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Abstract. We establish local interior Lipschitz continuity of the solutions of a class of free boundary elliptic problems assuming the coefficients of the equation of Dini mean oscillation in at least one direction. The novelty in this regularity result lies in the fact that it allows discontinuous coefficients in all but one variable.

1. Introduction

Throughout this paper, we denote by Ω a bounded domain in \mathbb{R}^n and by $\mathbf{A}(x) = (a_{ij}(x))$ an $n \times n$ matrix that satisfies for some positive constant $\lambda \in (0, 1)$

$$\sum_{i,j} |a_{ij}(x)| \leq \lambda^{-1}, \quad \text{for a.e. } x \in \Omega, \quad (1.1)$$

$$\mathbf{A}(x) \cdot \xi \cdot \xi \geq \lambda |\xi|^2, \quad \text{for a.e. } x \in \Omega, \text{ for all } \xi \in \mathbb{R}^n, \quad (1.2)$$

$\mathbf{f} : \Omega \rightarrow \mathbb{R}^n$ is a vector function such that $\mathbf{f}(x) = (f_1(x), \dots, f_n(x))$ and

$$\exists \bar{f} > 0 : \|\mathbf{f}\|_\infty \leq \bar{f}. \quad (1.3)$$

We consider the following problem

$$(P) \begin{cases} \text{Find } (u, \chi) \in H^1(\Omega) \times L^\infty(\Omega) \text{ such that:} \\ \text{(i) } u \geq 0, \quad 0 \leq \chi \leq 1, \quad u(1 - \chi) = 0 \text{ a.e. in } \Omega, \\ \text{(ii) } \operatorname{div}(\mathbf{A}(x)\nabla u + \chi\mathbf{f}(x)) = 0 \quad \text{in } H_0^{-1}(\Omega). \end{cases}$$

This class covers a set of various problems including the heterogeneous dam problem [1, 4, 9, 15], in which case Ω represents a porous medium with permeability matrix $\mathbf{A}(x)$, and $\mathbf{f}(x) = \mathbf{A}(x)\mathbf{e}$, with $\mathbf{e} = (0, \dots, 0, 1)$. A second example is the lubrication

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problem [2] which is obtained when $\mathbf{A}(x) = h^3(x)\mathbf{I}_2$ and $\mathbf{f}(x) = h(x)\mathbf{e}$, where \mathbf{I}_2 is the 2×2 identity matrix, and $h(x)$ is a scalar function related to the Reynolds equation. A third example is the aluminum electrolysis problem [3] which corresponds to $\mathbf{A}(x) = k(x)\mathbf{I}_2$ and $\mathbf{f}(x) = h(x)\mathbf{e}$, with $k(x)$ and $h(x)$ two given scalar functions.

We observe that if $\mathbf{f} \in L_{\text{loc}}^q(\Omega)$ for some $q > n$, then so is $\chi\mathbf{f}$, and by taking into account the assumptions (1.1)–(1.2) and equation (P)(ii), we infer from [14, Theorem 8.24] that $u \in C_{\text{loc}}^{0,\alpha}(\Omega)$ for any $\alpha \in (0, 1)$. In this paper, we will improve this regularity by showing that under suitable assumptions, we actually have $u \in C_{\text{loc}}^{0,1}(\Omega)$. We observe that this regularity is optimal due to the gradient discontinuity across the free boundary which is the interface that separates the sets $\{u = 0\}$ and $\{u > 0\}$ from each other. Moreover, Lipschitz continuity is not only interesting by itself, but is also of particular importance in the analysis of the free boundary (see for example [5, 6]).

Before stating our main result, we need to introduce a definition.

Definition 1.1. (i) We say that a function $\omega : (0, 1] \rightarrow [0, \infty)$ satisfies the Dini condition if

$$\int_0^1 \frac{\omega(r)}{r} dr < \infty.$$

(ii) A function $f \in L^1(\Omega)$ is of partial Dini mean oscillation with respect to $x' = (x_1, \dots, x_{n-1})$ in an open ball $B \subset\subset \Omega$, if the function $\omega_f : (0, 1] \rightarrow [0, \infty)$ defined by

$$\omega_f(r) = \sup_{x \in B} \int_{B_r(x)} \left| f(y) - \int_{B'_r(x')} f(z', y_n) dz' \right| dy,$$

$B'_r(x') = \{y' \in \mathbb{R}^{n-1} : |y' - x'| < r\}$ satisfies the Dini condition.

(iii) For each $\mathbf{f} \in L^1(\Omega)$, we define the following functions:

$$\varpi_{\mathbf{f}}(t) = \int_0^t \frac{\omega_{\mathbf{f}}(s)}{s} ds \quad \text{and} \quad \Phi_{\mathbf{f}}(t) = \varpi_{\mathbf{f}}(t) + \omega_{\mathbf{f}}(t).$$

Remark 1.1. We observe that if a function $f \in L^1(\Omega)$ is such that for each a_n , the function $x' \rightarrow f(x', a_n)$ is Hölder continuous, i.e.,

$$|f(x', a_n) - f(y', a_n)| \leq C|x' - y'|^\alpha$$

for some $\alpha \in (0, 1)$, then it is easy to verify that $\omega_f(r) \leq 2Cr^\alpha$ for any $r \in (0, 1]$, which leads to

$$\int_0^1 \frac{\omega_f(r)}{r} dr \leq \int_0^1 2Cr^{\alpha-1} dr = \frac{2C}{\alpha} < \infty$$

Hence, f is of partial Dini mean oscillation with respect to x' in any open ball $B \subset\subset \Omega$.

Here is the main result of this paper.

Theorem 1.1. *Assume that \mathbf{A} and \mathbf{f} satisfy (1.1)–(1.3) and the following conditions:*

$$\forall i, j = 1, \dots, n, a_{ij} \text{ is of partial Dini mean oscillation with respect to } x^j \text{ in } \Omega, \quad (1.4)$$

$$\forall i = 1, \dots, n, f_i \text{ is of partial Dini mean oscillation with respect to } x^i \text{ in } \Omega. \quad (1.5)$$

Then for any weak solution of (P), we have $u \in C_{\text{loc}}^{0,1}(\Omega)$.

The novelty in Theorem 1.1 lies in the fact that Lipschitz continuity of weak solutions of problem (P) is obtained even when the entries of the matrix $\mathbf{A}(x)$ and the vector function $\mathbf{f}(x)$ are discontinuous provided they satisfy a Dini mean oscillation condition in at least one direction, i.e., if they are regular in at least one variable. Since problem (P) is invariant by rotation in the sense that it is transformed into a similar problem with different coefficients satisfying the same assumptions as the original ones, it is obvious that we only need to have the Dini mean oscillation condition satisfied in any arbitrary space direction.

We recall that interior Lipschitz continuity for problem (P) was established in [5] and the same method was successfully extended to the quasilinear case in [7] and [8]. Interior and boundary Lipschitz continuity were established in [16] for a wide class of linear elliptic equations under some general assumptions. Recently, in [17], Lipschitz continuity was obtained using a different method based on Harnack's inequality. This approach helped relax some of the assumptions required in [5] and [16] and only required that $\mathbf{A}(x) \in C_{\text{loc}}^{0,\alpha}(\Omega)$ and $\text{div}(\mathbf{f}) \in L_{\text{loc}}^p(\Omega)$ for some $\alpha \in (0, 1)$ and $p > n/(1 - \alpha)$.

Lastly, we would like to point out that the assumptions (1.4)–(1.5) were introduced in [13] to obtain C^1 - and C^2 -regularity of solutions to elliptic equations. In this regard, we also refer the reader to the recent work on gradient estimates for elliptic equations in divergence form with partial Dini mean oscillation coefficients [11].

2. Estimates for the equation $\text{div}(\mathbf{A}(x)\nabla u) = -\text{div}(\mathbf{f})$

Under the assumptions of Theorem 1.1, it is known (see [10, Lemma 2.1]) that any weak solution u of the equation $\text{div}(\mathbf{A}(x)\nabla u) = -\text{div}(\mathbf{f})$ is such that $u \in C_{\text{loc}}^{0,1}(\Omega)$. The main result of this section is a local L^∞ -norm estimate of the gradient which will be used in the proof of Theorem 1.1 in Section 3. Needless to say, this estimate is of interest for itself.

Theorem 2.1. Let $\rho > 0$ be such that $B_{3\rho}(x_0) \subset\subset \Omega$ and $\Phi_{\mathbf{A}}(\rho) \leq C_0^*$ for some positive constant C_0^* depending only on n and λ . Assume that $u \in H^1(B_{3\rho}(x_0))$ is a weak solution of the equation $\operatorname{div}(\mathbf{A}(x)\nabla u) = -\operatorname{div}(\mathbf{f})$ in $B_{3\rho}(x_0)$. If moreover, we assume that $f_n \in L^\infty(B_{3\rho}(x_0))$ and \mathbf{A} and \mathbf{f} are of partial Dini mean oscillation with respect to x' in $B_{2\rho}(x_0)$, then $\nabla u \in L^\infty(B_\rho(x_0))$ and we have for some positive constant C_1 depending only on n and λ ,

$$|\nabla u|_{L^\infty(B_\rho(x_0))} \leq 3^{2n} C_1 (\rho^{-n} |\nabla u|_{L^1(B_{3\rho}(x_0))} + \|f_n\|_{L^\infty(B_{3\rho}(x_0))} + 4\Phi_{\mathbf{f}}(\rho)).$$

The proof of Theorem 2.1 requires a few lemmas.

Lemma 2.1. Assume that ω is a Dini function and let $a \in (0, 1)$ and $b > 1$ be two given real numbers. Then the function defined by

$$\tilde{\omega}(t) = \sum_{i=0}^{\infty} a^i (\omega(b^i t) \chi_{\{b^i t \leq 1\}} + \omega(1) \chi_{\{b^i t > 1\}})$$

satisfies

$$\int_0^t \frac{\tilde{\omega}(s)}{s} ds \leq \frac{1}{1-a} \left[\varpi(t) + a\varpi(1) + \frac{\omega(1)}{\gamma} t^\gamma \right] \quad \forall t \in [0, 1]$$

where $\gamma = -\frac{\ln(a)}{\ln(b)}$ and $\varpi(t) = \int_0^t \frac{\omega(s)}{s} ds$. In particular, $\tilde{\omega}$ is also a Dini function.

Proof. First, we recall that the function $\tilde{\omega}$ was introduced in [12, Lemma 3.1], where an estimate was also given. Nevertheless, our estimate is new and more precise.

We start by writing $\tilde{\omega}(t) = \tilde{\omega}_1(t) + \tilde{\omega}_2(t)$ for $t \in (0, 1)$, where

$$\tilde{\omega}_1(t) = \sum_{i=0}^{\infty} a^i \omega(b^i t) \chi_{\{b^i t \leq 1\}} \quad \text{and} \quad \tilde{\omega}_2(t) = \omega(1) \sum_{i=0}^{\infty} a^i \chi_{\{b^i t > 1\}}.$$

Next, let $i_0 = \lfloor -\frac{\ln(t)}{\ln(b)} \rfloor$ and observe that we have $t \leq b^{-i}$ iff $i \leq i_0$. Then we have

$$\begin{aligned} \int_0^t \frac{\tilde{\omega}_1(s)}{s} ds &= \int_0^t a^0 \frac{\omega(b^0 s)}{s} \chi_{\{b^0 s \leq 1\}} ds + \sum_{i=1}^{\infty} a^i \int_0^t \frac{\omega(b^i s)}{s} \chi_{\{b^i s \leq 1\}} ds \\ &= \int_0^t \frac{\omega(s)}{s} ds + \sum_{i=1}^{\infty} a^i \int_0^{t b^i} \frac{\omega(\tau)}{\tau} \chi_{\{\tau \leq 1\}} d\tau \\ &\leq \varpi(t) + \sum_{i=1}^{\infty} a^i \int_0^1 \frac{\omega(\tau)}{\tau} d\tau = \varpi(t) + \frac{a}{1-a} \varpi(1). \end{aligned} \quad (2.1)$$

Given that $i_0 \leq -\frac{\ln(t)}{\ln(b)} < i_0 + 1$, we can write

$$\tilde{\omega}_2(s) = \omega(1) \sum_{i=0}^{\infty} a^i \chi_{\{i > -\frac{\ln(t)}{\ln(b)}\}} = \omega(1) \sum_{i=i_0+1}^{\infty} a^i = \frac{a^{i_0+1}}{1-a} \omega(1).$$

Moreover, since $a < 1$, we have $a^{i_0+1} \leq a^{-\frac{\ln(t)}{\ln(b)}} = t^\gamma$, which leads to $\tilde{\omega}_2(s) \leq \frac{\omega(1)}{1-a} t^\gamma$ and

$$\int_0^t \frac{\tilde{\omega}_2(s)}{s} ds \leq \frac{\omega(1)}{1-a} \int_0^t \frac{s^\gamma}{s} ds \leq \frac{\omega(1)}{\gamma(1-a)} t^\gamma. \quad (2.2)$$

Now, combining (2.1) and (2.2), we obtain

$$\begin{aligned} \int_0^t \frac{\tilde{\omega}(s)}{s} ds &\leq \varpi(t) + \frac{a}{1-a} \cdot \varpi(1) + \frac{\omega(1)}{\gamma(1-a)} t^\gamma \\ &\leq \frac{1}{1-a} \left[(1-a)\varpi(t) + a\varpi(1) + \frac{\omega(1)}{\gamma} t^\gamma \right] \\ &\leq \frac{1}{1-a} \left[\varpi(t) + a\varpi(1) + \frac{\omega(1)}{\gamma} t^\gamma \right]. \quad \blacksquare \end{aligned}$$

The following lemma is a slight improvement of [11, Theorem 1.2] in the sense that it provides a more precise L^∞ -estimate of the gradient.

Lemma 2.2. *Let $u \in H^1(B_3)$ be a weak solution of the equation $\operatorname{div}(\mathbf{A}(x)\nabla u) = -\operatorname{div}(\mathbf{f})$ in B_3 , with \mathbf{A} and \mathbf{f} satisfying (1.1)–(1.3) in B_3 and both \mathbf{A} and \mathbf{f} are of partial Dini mean oscillation with respect to x' in B_2 . Then we have $\nabla u \in L^\infty(B_1)$ with*

$$|\nabla u|_{L^\infty(B_1)} \leq 3^{nk_0} C_1 (|\nabla u|_{L^1(B_3)} + \bar{f} + 4\Phi_{\mathbf{f}}(1))$$

where $C_1 = C_1(n, \lambda)$ is a positive constant depending only on n and λ , and k_0 is an integer greater than 1 satisfying

$$\varpi_{\mathbf{A}}(2^{-k_0}) + 2\omega_{\mathbf{A}}(1)\sqrt{2^{-k_0}} \leq 3^{-n-1}C_0^{-1} = C_0^*.$$

Proof. First, we observe that by scaling, we may replace the ball B_3 by B_6 as in [11]. Next, we denote by $C_0 = C_0(n, \lambda)$ the positive constant depending only on n and λ that was introduced in [11, Proof of Theorem 1.2, p. 1520]. Following this reference, we choose $\gamma = \frac{1}{2}$, $0 < \kappa < \min(2^{-1}, C_0^{-2})$ and we denote by $\tilde{\omega}_{\mathbf{A}}(t)$ the function defined in Lemma 2.1 with $a = \sqrt{\kappa}$ and $b = \frac{1}{\kappa}$. Let now k_0 be a positive integer greater than 1 that satisfies

$$C_0 \int_0^{2^{k_0}} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt \leq 3^{-n}$$

which, by taking into account the estimate of Lemma 2.1, is true if

$$\frac{C_0}{1-\sqrt{\kappa}} (\varpi_{\mathbf{A}}(2^{-k_0}) + \sqrt{\kappa}\varpi_{\mathbf{A}}(1) + 2\omega_{\mathbf{A}}(1)\sqrt{2^{-k_0}}) \leq 3^{-n}.$$

At this step, we further assume that $\kappa \leq 2^{-2}$, which leads to $1 - \sqrt{\kappa} \geq 2^{-1}$, and makes the above inequality hold if

$$2(\varpi_{\mathbf{A}}(2^{-k_0}) + \sqrt{\kappa}\varpi_{\mathbf{A}}(1) + 2\omega_{\mathbf{A}}(1)\sqrt{2^{-k_0}}) \leq 3^{-n}C_0^{-1}.$$

This in turn remains true if κ and k_0 are chosen such that

$$2\sqrt{\kappa}\varpi_{\mathbf{A}}(1), \varpi_{\mathbf{A}}(2^{-k_0}) + 2\omega_{\mathbf{A}}(1)\sqrt{2^{-k_0}} \leq 3^{-n-1}C_0^{-1} = C_0^*.$$

If we replace f_1 by f_n , we get the estimate (see [11, p. 1522]) with a positive constant $C_1(n, \lambda)$ depending only on n and λ ,

$$3^{-nk_0}|\nabla u|_{L^\infty(B_1)} \leq C_1(n, \lambda) \left(|\nabla u|_{L^1(B_3)} + |f_n|_{L^\infty(B_3)} + \int_0^1 \frac{\tilde{\omega}_{\mathbf{f}}(t)}{t} dt \right)$$

which can be written by using Lemma 2.1 again as

$$\begin{aligned} |\nabla u|_{L^\infty(B_1)} &\leq 3^{nk_0}C_1 \left(|\nabla u|_{L^1(B_3)} + \bar{f} + \frac{1}{1 - \sqrt{\kappa}}(\varpi(1) + a\varpi(1) + 2\omega(1)1^\gamma) \right) \\ &\leq 3^{nk_0}C_1 \left(|\nabla u|_{L^1(B_3)} + \bar{f} + 4(\omega_{\mathbf{f}}(1) + \varpi_{\mathbf{f}}(1)) \right) \\ &= 3^{nk_0}C_1 \left(|\nabla u|_{L^1(B_3)} + \bar{f} + 4\Phi_{\mathbf{f}}(1) \right). \quad \blacksquare \end{aligned}$$

The following lemma is a slight improvement of [17, Lemma 2.2].

Lemma 2.3. *Assume that u is a nonnegative weak solution of the equation*

$$\operatorname{div}(\mathbf{A}(x)\nabla u) = -\operatorname{div}(\mathbf{f})$$

in Ω and let $x_0 \in \Omega$ and $r > 0$ such that $B_{5r}(x_0) \subset\subset \Omega$ and $\overline{B_r}(x_0) \cap \{u = 0\} \neq \emptyset$. Then we have for some positive constant C_2 depending only on n , λ and \bar{f} : $\max_{\overline{B_r}(x_0)} u \leq C_2 r$.

Proof. Let $x_1 \in \overline{B_r}(x_0) \cap \{u = 0\}$ and let $\omega_n = |B_1|$ be the measure of the unit ball in \mathbb{R}^n . First since $B_{5r}(x_0) \subset\subset \Omega$, it is easy to verify that $B_{4r}(x_1) \subset\subset \Omega$. Next, we observe that since $\mathbf{f} \in L^\infty(\Omega)$, we can apply Harnack's inequality [14, Theorem 8.17–Theorem 8.18, p. 194] to the equation $\operatorname{div}(\mathbf{A}(x)\nabla u) = -\operatorname{div}(\mathbf{f})$ with $p = n + 1$. Therefore, we get for a positive constant C depending only on n

$$\begin{aligned} \max_{\overline{B_{2r}}(x_1)} u &\leq C \left(\min_{\overline{B_{2r}}(x_1)} u + \frac{1}{\lambda} r^{1 - \frac{n}{n+1}} \|\mathbf{f}\|_{n+1, \overline{B_{2r}}(x_1)} \right) \\ &\leq C \left(0 + \frac{\bar{f}}{\lambda} r^{\frac{1}{n+1}} \cdot |B_{2r}(x_1)|^{\frac{1}{n+1}} \right) \\ &= \frac{C \bar{f} 2^{\frac{n}{n+1}} \omega_n^{\frac{1}{n+1}}}{\lambda} \cdot r^{\frac{1}{n+1}} \cdot r^{\frac{n}{n+1}} = \frac{C \bar{f} 2^{\frac{n}{n+1}} \omega_n^{\frac{1}{n+1}}}{\lambda} r = C_2 r. \end{aligned}$$

Given that $\overline{B_r}(x_0) \subset \overline{B_{2r}}(x_1)$, the lemma follows. \blacksquare

The following lemma is a Cacciopoli type lemma.

Lemma 2.4. *Assume that u is a weak solution of the equation $\operatorname{div}(\mathbf{A}(x)\nabla u) = -\operatorname{div}(\mathbf{f})$ in Ω . For each open ball $B_r(x_0)$ such that $B_{2r}(x_0) \subset \Omega$, we have*

$$\int_{B_r(x_0)} |\nabla u|^2 dx \leq \frac{32}{\lambda^4 r^2} \int_{B_{2r}(x_0)} u^2 dx + \frac{8\bar{f}}{\lambda r} \int_{B_{2r}(x_0)} |u| dx + \frac{2^{n+1}\bar{f}^2 \omega_n}{\lambda^2} r^n.$$

Proof. Let $B_r(x_0)$ be an open ball such that $B_{2r}(x_0) \subset \Omega$, and let $\eta \in C_0^\infty(B_{2r}(x_0))$ be a cut-off function such that

$$\eta = 1 \quad \text{in } B_r(x_0), \quad 0 \leq \eta \leq 1 \quad \text{and} \quad |\nabla \eta| \leq \frac{2}{r} \quad \text{in } B_{2r}(x_0).$$

Using $\eta^2 u$ as a test function for equation $\operatorname{div}(\mathbf{A}(x)\nabla u) = -\operatorname{div}(\mathbf{f})$, we get

$$\int_{B_{2r}(x_0)} a(x)\nabla u \cdot \nabla(\eta^2 u) dx = - \int_{B_{2r}(x_0)} \mathbf{f}(x) \cdot \nabla(\eta^2 u) dx$$

which can be written as

$$\begin{aligned} \int_{B_{2r}(x_0)} \eta^2 a(x)\nabla u \cdot \nabla u dx &= - \int_{B_{2r}(x_0)} 2\eta \cdot u \cdot a(x)\nabla u \cdot \nabla \eta dx \\ &\quad - \int_{B_{2r}(x_0)} 2\eta u \cdot \mathbf{f}(x) \cdot \nabla \eta dx - \int_{B_{2r}(x_0)} \eta^2 \cdot \mathbf{f}(x) \cdot \nabla u dx \end{aligned}$$

or by using (1.1)–(1.2)

$$\begin{aligned} \lambda \int_{B_{2r}(x_0)} \eta^2 |\nabla u|^2 dx &\leq \frac{2}{\lambda} \int_{B_{2r}(x_0)} \eta \cdot |\nabla u| \cdot |u| \cdot |\nabla \eta| dx + \int_{B_{2r}(x_0)} 2\eta |u| \cdot |\mathbf{f}| \cdot |\nabla \eta| dx \\ &\quad + \int_{B_{2r}(x_0)} \eta \cdot |\nabla u| \cdot \eta \cdot |\mathbf{f}|_\infty dx. \end{aligned}$$

By taking into account (1.3) and the fact that $|\nabla \eta| \leq 2/r$, we obtain

$$\begin{aligned} \int_{B_{2r}(x_0)} \eta^2 |\nabla u|^2 dx &\leq \int_{B_{2r}(x_0)} (\eta |\nabla u|) \cdot \left(\frac{4|u|}{\lambda^2 r} \right) dx + \frac{4\bar{f}}{\lambda r} \int_{B_{2r}(x_0)} |u| dx \\ &\quad + \int_{B_{2r}(x_0)} (\eta |\nabla u|) \cdot \left(\frac{\bar{f}}{\lambda} \right) dx \end{aligned}$$

Now, we apply the following Young's type inequality $ab \leq \frac{1}{4}a^2 + b^2$ to the first and third integrals of the right-hand side of the previous inequality

$$\begin{aligned} \int_{B_{2r}(x_0)} \eta^2 |\nabla u|^2 dx &\leq \frac{1}{4} \int_{B_{2r}(x_0)} \eta^2 |\nabla u|^2 dx + \frac{16}{\lambda^4 r^2} \int_{B_{2r}(x_0)} u^2 dx \\ &\quad + \frac{4\bar{f}}{\lambda r} \int_{B_{2r}(x_0)} |u| dx + \frac{1}{4} \int_{B_{2r}(x_0)} \eta^2 |\nabla u|^2 dx \\ &\quad + \int_{B_{2r}(x_0)} \frac{\bar{f}^2}{\lambda^2} dx \end{aligned}$$

which leads, since $\eta = 1$ in $B_r(x_0)$, to

$$\int_{B_r(x_0)} |\nabla u|^2 dx \leq \frac{32}{\lambda^4 r^2} \int_{B_{2r}(x_0)} u^2 dx + \frac{8\bar{f}}{\lambda r} \int_{B_{2r}(x_0)} |u| dx + \frac{2^{n+1} \omega_n \bar{f}^2}{\lambda^2} r^n. \quad \blacksquare$$

Combining Lemmas 2.3 and 2.4, we obtain the following lemma.

Lemma 2.5. *Assume that u is a nonnegative weak solution of the equation*

$$\operatorname{div}(\mathbf{A}(x)\nabla u) = -\operatorname{div}(\mathbf{f})$$

in Ω and let $x_0 \in \Omega$ and $r > 0$ such that $B_{10r}(x_0) \subset\subset \Omega$ and $\bar{B}_{2r}(x_0) \cap \{u = 0\} \neq \emptyset$. Then we have

$$\int_{B_r(x_0)} |\nabla u| dx \leq C_3 r^n$$

where

$$C_3 = \frac{\omega_n 2^{\frac{n+1}{2}}}{\lambda^2} \sqrt{16C_2^2 + 4C_2 \bar{f} \lambda^3 + \bar{f}^2 \lambda^2}$$

and C_2 is the constant in Lemma 2.3.

Proof. From Lemmas 2.3 and 2.4, we have

$$\max_{B_{2r}(x_0)} u \leq C_2(2r) \quad (2.3)$$

and

$$\int_{B_r(x_0)} |\nabla u|^2 dx \leq \frac{32}{\lambda^4 r^2} \int_{B_{2r}(x_0)} u^2 dx + \frac{8\bar{f}}{\lambda r} \int_{B_{2r}(x_0)} u dx + \frac{2^{n+1} \bar{f}^2 \omega_n}{\lambda^2} r^n. \quad (2.4)$$

Combining (2.3) and (2.4), we obtain

$$\begin{aligned}
 \int_{B_r(x_0)} |\nabla u|^2 dx &\leq \frac{32}{\lambda^4 r^2} \int_{B_{2r}(x_0)} C_2^2(2r)^2 dx \\
 &\quad + \frac{8\bar{f}}{\lambda r} \int_{B_{2r}(x_0)} C_2(2r) dx + \frac{2^{n+1} \omega_n \bar{f}^2}{\lambda^2} r^n \\
 &\leq \frac{128}{\lambda^4} \omega_n C_2^2(2r)^n + \frac{16\bar{f}}{\lambda} \omega_n C_2(2r)^n + \frac{2^{n+1} \bar{f}^2 \omega_n}{\lambda^2} r^n \\
 &= \frac{2^{n+1} \omega_n}{\lambda^4} [64C_2^2 + 8C_2 \bar{f} \lambda^3 + \bar{f}^2 \lambda^2] r^n. \tag{2.5}
 \end{aligned}$$

We conclude by using the Cauchy–Schwarz inequality and taking into account (2.5),

$$\int_{B_r(x_0)} |\nabla u| dx \leq \left(\int_{B_r(x_0)} |\nabla u|^2 dx \right)^{\frac{1}{2}} \cdot |B_r(x_0)|^{\frac{1}{2}} \leq C_3 r^n. \quad \blacksquare$$

Proof of Theorem 2.1. We observe that the function $v(y) = \frac{u(x_0 + \rho y)}{\rho}$ satisfies the equation $\operatorname{div}(\mathbf{A}_\rho(x) \nabla v) = -\operatorname{div}(\mathbf{f}_\rho)$ in B_3 , where $\mathbf{F}_\rho(y) = \mathbf{F}(x_0 + \rho y)$. Moreover, it is obvious that \mathbf{A}_ρ and \mathbf{f}_ρ satisfy the assumption of Lemma 2.2 in B_3 . Therefore, we get the estimate

$$|\nabla v|_{L^\infty(B_1)} \leq 3^{nk_0} C_1 (|\nabla v|_{L^1(B_3)} + |f_{\rho n}|_{L^\infty(B_3)} + 4\Phi_{\mathbf{f}_\rho}(1)) \tag{2.6}$$

where C_1 is a positive constant depending only on n and λ , k_0 is an integer satisfying

$$\varpi_{\mathbf{A}_\rho}(2^{-k_0}) + 2\omega_{\mathbf{A}_\rho}(1) \sqrt{2^{-k_0}} \leq C_0^*$$

and C_0 is the positive constant depending only on n and λ from [11, p. 1520].

If we observe that $\omega_{\mathbf{f}_\rho}(s) = \omega_{\mathbf{f}}(\rho s)$, $\varpi_{\mathbf{f}_\rho}(s) = \varpi_{\mathbf{f}}(\rho s)$, and we choose $k_0 = 2$, this condition reduces to $\varpi_{\mathbf{A}}(2^{-2}\rho) + \omega_{\mathbf{A}}(\rho) \leq C_0^*$, which is in particular true if $\Phi_{\mathbf{A}}(\rho) \leq C_0^*$.

Finally, since $\Phi_{\mathbf{f}_\rho}(s) = \Phi_{\mathbf{f}}(\rho s)$, the estimate of Theorem 2.1 follows from (2.6). \blacksquare

3. Proof of Theorem 1.1

Let $\varepsilon > 0$, $\Omega_\varepsilon = \{x \in \Omega : d(x, \partial\Omega) > \varepsilon\}$, and let C_0^* be the positive constant depending only on n and λ that was introduced in Theorem 2.1. Since $\lim_{t \rightarrow 0} \Phi_{\mathbf{F}}(t) = 0$, there exists $t_0 \in (0, 1)$ such that

$$\Phi_{\mathbf{A}}(t) \leq C_0^* \quad \forall t \in (0, t_0). \tag{3.1}$$

We fix $\varepsilon_0 = \frac{3}{4}t_0$ and assume that $\varepsilon < \varepsilon_0$. We shall prove that ∇u is bounded in $\Omega_{41\varepsilon}$ by a constant depending only on $n, \lambda, \bar{f}, \Phi_{\mathbf{f}}(1), |\nabla u|_{L^1(\Omega)}$ and ε , provided $\varepsilon < \varepsilon_0$.

Let $x_0 \in \Omega_{41\varepsilon}$. We distinguish two cases:

(i) $B_{3\varepsilon}(x_0) \subset \{u > 0\}$:

Since $\chi = 1$ a.e. in $B_{3\varepsilon}(x_0)$, u satisfies the equation $\operatorname{div}(\mathbf{A}(x)\nabla u) = -\operatorname{div}(\mathbf{f})$ in $B_{3\varepsilon}(x_0)$. We also have by (1.3)–(1.5) that $\|f_n\|_\infty \leq \bar{f}$ and \mathbf{A} and \mathbf{f} are of partial Dini mean oscillation with respect to x' in $B_{2\varepsilon}(x_0)$. Moreover, we have from (3.1) $\Phi_{\mathbf{A}}(\varepsilon) \leq C_0^*$. Therefore, by Theorem 2.1 applied with $\rho = \varepsilon$, we get for a positive constant $C_1 = C_1(n, \lambda)$

$$|\nabla u|_{L^\infty(B_\varepsilon(x_0))} \leq 3^{2n} C_1 (\varepsilon^{-n} |\nabla u|_{L^1(\Omega)} + |f_n|_{L^\infty(B_{3\varepsilon}(x_0))} + 4\Phi_{\mathbf{f}}(\varepsilon)).$$

Since $\Phi_{\mathbf{f}}(t)$ is nondecreasing and $\varepsilon < \varepsilon_0 < 1$, we obtain

$$|\nabla u|_{L^\infty(B_\varepsilon(x_0))} \leq 3^{2n} C_1 (\varepsilon^{-n} |\nabla u|_{L^1(\Omega)} + \bar{f} + 4\Phi_{\mathbf{f}}(1)). \quad (3.2)$$

(ii) $B_{3\varepsilon}(x_0) \cap \{u = 0\} \neq \emptyset$:

Let $x \in B_\varepsilon(x_0)$ such that $u(x) > 0$ and let $r(x) = \operatorname{dist}(x, \{u = 0\})$ be the distance function to the set $\{u = 0\}$. Our objective is to estimate $|\nabla u|_{L^\infty(B_{r(x)/3}(x))}$. To do that, we will again use Theorem 2.1.

We claim that $r(x) < 4\varepsilon$ and $\overline{B_{10r(x)}(x)} \subset B_{41\varepsilon}(x_0)$. Indeed, by assumption (ii), there exists $z \in B_{3\varepsilon}(x_0) \cap \{u = 0\}$. So, we get $r(x) \leq |x - z| \leq |x - x_0| + |x_0 - z| < \varepsilon + 3\varepsilon = 4\varepsilon$.

Then we have for each $y \in \overline{B_{10r(x)}(x)}$

$$|x_0 - y| \leq |x_0 - x| + |x - y| < \varepsilon + 10r(x) < \varepsilon + 10(4\varepsilon) = 41\varepsilon.$$

This means that $y \in B_{41\varepsilon}(x_0)$, and therefore $\overline{B_{10r(x)}(x)} \subset B_{41\varepsilon}(x_0)$. In particular, we have $\overline{B_{10r(x)}(x)} \subset \Omega_{41\varepsilon} \subset \Omega$.

Now, we obtain from the previous step that $\overline{B_{r(x)}(x)} \subset \Omega$. Moreover, since $B_{r(x)}(x) \subset \{u > 0\}$, we have by (P)(i) that $\chi = 1$ a.e. in $B_{r(x)}(x)$, which leads by (P)(ii) to

$$\operatorname{div}(\mathbf{A}(x)\nabla u) = -\operatorname{div}(\mathbf{f}) \quad \text{in } B_{r(x)}(x).$$

By (1.3)–(1.5), we also know that $\|f_n\|_\infty \leq \bar{f}$ and \mathbf{A} and \mathbf{f} are of partial Dini mean oscillation with respect to x' in $B_{2r(x)/3}(x)$. On the other hand, we have $r(x)/3 \leq 4\varepsilon/3 < t_0$, which ensures by (3.1) that $\Phi_{\mathbf{A}}(r(x)/3) \leq C_0^*$. Hence, we infer from Theorem 2.1 applied with $\rho = r(x)/3$, that we have for a positive constant $C_1 = C_1(n, \lambda)$

$$|\nabla u|_{L^\infty(B_{r(x)/3}(x))} \leq 3^{2n} C_1 \left(\left(\frac{r(x)}{3} \right)^{-n} |\nabla u|_{L^1(B_{r(x)})} + \bar{f} + 4\Phi_{\mathbf{f}}(r(x)/3) \right). \quad (3.3)$$

Since $r(x)/3 < t_0 < 1$ and $\Phi_f(t)$ is nondecreasing, we have $\Phi_f(r(x)/3) \leq \Phi_f(1)$. Furthermore, we have $\overline{B_{10r(x)}(x)} \subset \Omega$ and $\overline{B_{2r(x)}(x)} \cap \{u = 0\} \neq \emptyset$, whence we can use Lemma 2.5 to improve (3.3) as follows:

$$|\nabla u|_{L^\infty(B_{r(x)/3}(x))} \leq 3^{2n} C_1 (3^n C_3 + \overline{f} + 4\Phi_f(1)).$$

Given that x is arbitrary in $B_\varepsilon(x_0) \cap \{u > 0\}$ and $\nabla u(x) = 0$ a.e. in $B_\varepsilon(x_0) \cap \{u = 0\}$, it turns out that ∇u is uniformly bounded in $B_\varepsilon(x_0)$, with

$$|\nabla u|_{L^\infty(B_\varepsilon(x_0))} \leq 3^{2n} C_1 (3^n C_3 + \overline{f} + 4\Phi_f(1)). \quad (3.4)$$

Finally, by taking into account the fact that $\Omega_{41\varepsilon} \subset \bigsqcup_{0 < \varepsilon < \varepsilon_0, x_0 \in \Omega_{41\varepsilon}} B_\varepsilon(x_0)$, we conclude from (3.2) and (3.4) that

$$|\nabla u|_{L^\infty(\Omega_{41\varepsilon})} \leq 3^{2n} C_1 (\varepsilon^{-n} |\nabla u|_{L^1(\Omega)} + 3^n C_3 + \overline{f} + 4\Phi_f(1))$$

which means that $|\nabla u(x)|$ is uniformly bounded in $\Omega_{41\varepsilon}$ by a constant depending only on $n, \lambda, \overline{f}, \Phi_f(1), |\nabla u|_{L^1(\Omega)}$ and $\varepsilon < d(x, \partial\Omega)/41$, for any small enough $\varepsilon > 0$. ■

Remark 3.1. With a slight modification of the proof, it is not difficult to extend Theorem 1.1 to the following problem:

$$\left\{ \begin{array}{l} \text{Find } (u, \chi) \in H^1(\Omega) \times L^\infty(\Omega) \text{ such that:} \\ \text{(i) } u \geq 0, \quad 0 \leq \chi \leq 1, \quad u(1 - \chi) = 0 \text{ a.e. in } \Omega, \\ \text{(ii) } \operatorname{div}(\mathbf{A}(x)\nabla u + \chi\mathbf{f}(x)) = -\chi\mathbf{g}(x) \quad \text{in } \Omega, \end{array} \right.$$

provided that $\mathbf{A}(x)$ satisfies (1.1)–(1.2), $\mathbf{g}(x)$ and $\mathbf{f}(x)$ satisfy (1.3), and the three functions are of partial Dini mean oscillation with respect to x' in Ω .

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