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THE TUTTE POLYNOMIAL OF A MORPHISM OF MATROIDS 4. COMPUTATIONAL COMPLEXITY

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Abstract: We determine the easy points of the 3-variable Tutte polynomial of a matroid perspective. It turns out that all but one of the sporadic easy points of the 3-variable Tutte polynomial proceed from the 8 sporadic easy points determined in the seminal paper of Jaeger–Vertigan–Welsh on the computational complexity of the Tutte polynomial of a matroid. The exceptional easy point, namely $(-1,-1,-1)$, can be evaluated with polynomial complexity for binary matroid perspectives by a previous result of the author.

1 – Introduction

The Tutte polynomial of a matroid — introduced by W.T. Tutte in 1954 for graphs — is a self-dual form of the generating function for cardinality and rank in the matroid of subsets of elements. This polynomial is relevant in many problems involving numerical invariants of matroids [2]. We have introduced in 1975 the Tutte polynomial of a matroid perspective, as a self-dual form of the generating function for cardinality and ranks in two matroids [8]. The Tutte polynomial of a matroid perspective is a 3-variable polynomial with non negative coefficients. For a general pair of matroids the 3-variable Tutte polynomial is a Laurent polynomial in $Z[x,y,z,z^{-1}]$, equivalent to the linking polynomial recently considered by Welsh and Kayibi [13]. The properties of the 3-variable Tutte polynomial of a matroid perspective generalize and unify properties of the usual 2-variable

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Tutte polynomial of a matroid. For instance, the evaluation at $(0,0,1)$ of the Tutte polynomial of an oriented matroid perspective contains as particular cases both the counting of acyclic orientations of a graph and the counting of orientations with unique source and unique sink, and its generalizations to hyperplane arrangements and oriented matroids [9].

The seminal paper of Jaeger–Vertigan–Welsh on the computational complexity of the Jones polynomial of knot theory and of the Tutte polynomial contains a wide range of results on the general intractability of the evaluation of the Tutte polynomial of a matroid except for a few listed special points and curves [6]. In the present note, we determine the easy points of the Tutte polynomial of a matroid perspective as a corollary of [6, Theorem 2] and of a theorem of G. Etienne and the author ([4, Th. 6.2]). It turns out that all but one of the sporadic easy points of the 3-variable Tutte polynomial proceed from the easy points of the 2-variable Tutte polynomial. The exceptional easy point, namely $(-1,-1,-1)$, has a polynomial evalution for a represented binary matroid.

2 – The Tutte polynomial of a matroid perspective

A matroid perspective, a particular case of matroid strong map or quotient, is the generalization to matroids of linear maps in vector spaces.

Let M, M' be two matroids on a set E. The following properties (i)–(v) are equivalent (see [7] or [12, Section 7.3]) :

- (**i**) every flat of M' is a flat of M;
- (**ii**) every circuit of M is a union of circuits of M';
- (**iii**) for every circuit C of M and cocircuit D' of M' we have $|C \cap D'| \neq 1$;
- (**iv**) $r_{M'}(X) - r_{M'}(Y) \leq r_M(X) - r_M(Y)$ for all $Y \subseteq X \subseteq E$;
- (**v**) there is a matroid N on a set $E \cup A$ such that $M = N \backslash A$ and $M' = N/A$.

We write $M \to M'$ when these equivalent properties hold, and say that $M \to M'$ constitutes a *matroid perspective*. A matroid perspective is the particular case of a strong map of matroids when both matroids are on a same set. Note that no significant generality is lost, since it can easily be shown that any strong map is reducible to a perspective up to a bijection and adding loops and parallel elements. The matroid M' is often called a *quotient* of M in the literature [7], [12]. Standard examples of matroid perspectives are obtained by identification of vertices in graphs or by embeddings of graphs in surfaces, and more generally from linear maps between vector spaces.

A matroid N as in (v) is called a *major* of $M \to M'$. A matroid perspective is said to be *graphic* resp. *binary* if it has a graphic resp. binary major. Let M be a matroid on a set E. We denote by $\mathbf{0}$ the rank zero matroid on E, and by $\mathbf{1}$ the free matroid of rank $|E|$ on E. Then $M \to M$, $M \to \mathbf{0}$ and $\mathbf{1} \to M$ are matroid perspectives on E. As is easily seen, if M is graphic, these matroid perspectives are also graphic. We will use them in Section 3.

The Tutte polynomial of a matroid perspective defined in [8], [10], [11] is a variant of the rank generating function of two matroids. We have

$$t\big(M,M';x,y,z\big) = \sum_{A\subseteq E} (x-1)^{r(M')-r_{M'}(A)}\,(y-1)^{|A|-r_M(A)}\,z^{r(M)-r(M')-\big(r_M(A)-r_{M'}(A)\big)}$$

where $r_M(A)$ denotes the rank of A in M.

The rank generating function of two matroids $R(M,M')$ is defined by

$$R\big(M,M';u,v,w\big) = \sum_{A\subseteq E} u^{|A|}\,v^{r_M(A)}\,w^{r_{M'}(A)}\,.$$

The equivalence of $t(M,M')$ and $R(M,M')$ is given by the formulas

$$t\big(M,M';x,y,z\big) = (x-1)^{r(M')}\,z^{r(M)-r(M')}\,R\left(M,M';y-1,\frac{1}{(y-1)\,z},\frac{z}{x-1}\right),$$

$$R\big(M,M';u,v,w\big) = (uv)^{r(M)}\,w^{r(M')}\,t\left(M,M';\frac{1}{uvw}+1,u+1,\frac{1}{uv}\right).$$

In general, the function $t(M,M')$ is a Laurent polynomial in $Z[x,y,z,z^{-1}]$, and its coefficients may be positive or negative integers. In the case of a matroid perspective $M \to M'$, the function $t(M,M')$ is a polynomial in x,y,z with non negative integer coefficients, and many fundamental properties of the usual Tutte polynomial generalize [8], [10], [11].

Another variant of the rank generating function of two matroids, a 4-variable polynomial called the *linking polynomial*, has been recently considered by D.J.A. Welsh and K.K. Kayibi [13]. We have

$$Q\big(M,M';x,y,u,v\big) = \\ = \sum_{A\subseteq E} (x-1)^{r(M)-r_M(A)}\,(y-1)^{|A|-r_M(A)}\,(u-1)^{r(M')-r_{M'}(A)}\,(v-1)^{|A|-r_{M'}(A)}\,.$$

The linking polynomial $Q(M,M')$ is equivalent to the Tutte polynomial of two matroids $t(M,M')$. We have

$$Q\big(M,M';x,y,u,v\big) = \\ = (v-1)^{r(M)-r(M')}\, t\Big(M,M';(x-1)(u-1)+1,(y-1)(v-1)+1,\frac{x-1}{v-1}\Big),$$

$$t\big(M,M';x,y,z\big) = \Big(\frac{z}{w}\Big)^{r(M)-r(M')} Q\Big(M,M';w+1,\frac{(y-1)\,z}{w}+1,\frac{x-1}{w}+1,\frac{w}{z}+1\Big).$$

In the last formula the variable w can take any value. In particular for $w=z$ we recover the formula in [13, p. 394, (14)]; see also [14].

It follows from this equivalence that properties can be stated in terms of either polynomials. Theorem 6.1 of [8] (see also [9, Th. 8.1]), which gives an expression of $t(M,M')$ in terms of activities when M,M' is a matroid perspective — equivalently, a strong map — on a linearly ordered set of elements, is restated in [13] in terms of $Q(M,M')$ as Theorem 3 (see the acknowledgement [14]).

When $M \to M'$, the polynomial $t(M,M')$ can also be considered as the Tutte polynomial of a matroid pointed by a subset of elements [8], [10], [11]. With notation of (v) in the equivalences defining a matroid perspective, we have $t(M,M';x,y,z) = t(N;A;x,y,z)$ (see details in [11, section 3]). When $r(M) = r(M')+1$, or, equivalently, when A is reduced to one element, say $A=\{e\}$, the polynomial $t(M,M') = t(N;\{e\})$ is equivalent to the 4-variable polynomial introduced by T. Brylawski in [1]. Generalizations of certain results of Brylawski to Tutte polynomials of set-pointed matroids are studied in [3].

3 – Easy points

Two results of the literature will be used in the proof of Theorem 1.

Theorem A (F. Jaeger, D. Vertigan and D.J.A. Welsh [6, Th. 2])**.** *The problem of evaluating the Tutte polynomial of a graph at a point in the (x,y)-plane is #P-hard except when $(x-1)(y-1)=1$ or when (x,y) equals $(1,1)$ $(-1,-1)$ $(0,-1)$ $(-1,0)$ $(i,-i)$ $(-i,i)$ (j,j^2) (j^2,j) where $j=e^{2\pi i}/3$.* ∎

We refer the reader to [6] for the interpretation of the special points in Theorem A (see also [5] for (j,j^2) and (j^2,j)).

Theorem B (G. Etienne, M. Las Vergnas [4, Th. 6.2])**.** *Let $M \to M'$ be a binary matroid perspective, i.e. such that $M = S(V)$ and $M' = S(V')$ for binary subspaces $V \subseteq V' \subseteq GF(2)^E$, where $S(V)$ denotes the matroid on E whose circuits are the inclusion-minimal supports of non zero vectors of V. We have*

$$t\big(M, M'; -1, -1, -1\big) = \begin{cases} 0 & \text{if } \ 1_E \notin V + V'^{\perp}, \\ (-1)^{|E| - \dim(V \cap V^{\perp})}\, 2^{\dim(V \cap V'^{\perp})} & \text{if } \ 1_E \in V + V'^{\perp}. \end{cases} \blacksquare$$

Extending definitions of Jaeger–Vertigan–Welsh, we say that a point (a, b, c) of the complex 3-space is an *easy point* of the 3-variable Tutte polynomial of a matroid perspective if there is a polynomial algorithm to evaluate $t(M, M'; a, b, c)$ on graphic matroid perspectives $M \to M'$.

Theorem 1. *The easy points of the 3-variable Tutte polynomial of a matroid perspective are*

- **(i)** *all points of the curve $(t+1,\, 1/t+1,\, t)$;*
- **(ii)** *15 points obtained from the 8 sporadic easy points of the 2-variable Tutte polynomial of a matroid, namely for each (a, b) in the list of Jaeger–Vertigan–Welsh the points $(a, b, a-1)$ and $(a, b, 1/(b-1))$ if $b \neq 1$;*
- **(iii)** $(-1, -1, -1)$.

Proof: Let M be a graphic matroid. Then $M \to \mathbf{0}$ and $\mathbf{1} \to M$ are graphic matroid perspectives. By straightforward substitutions we have $t(M, \mathbf{0}; x, y, z) = t(M; z+1, y)$ and $t(\mathbf{1}, M; x, y, z) = z^{|M| - r(M)}\, t(M; x, 1/z+1)$ ([11, (5.4), (5.5)]). It follows that if (a, b, c) is an easy point of the 3-variable Tutte polynomial then $(c+1, b)$ and $(a, 1/c+1)$ are easy points of the 2-variable Tutte polynomial. By Theorem A, we have $c(b-1) = 1$ or $(c+1, b) \in \mathcal{L} = \big\{(1,1), (-1,-1), (0,-1), (-1,0), (i,-i), (-i,i), (j,j^2), (j^2,j)\big\}$, and $(a-1)/c = 1$ or $(a, 1/c+1) \in \mathcal{L}$. Therefore either $c(b-1) = 1$ and $(a-1)/c = 1$ — case (i), or $c(b-1) = 1$ and $(a, 1/c+1) \in \mathcal{L}$, equivalently $(a, b) \in \mathcal{L}$ and $c = 1/(b-1)$, or $(a-1)/c = 1$ and $(c+1, b) \in \mathcal{L}$, equivalently $(a, b) \in \mathcal{L}$ and $c = a-1$ — case (ii), or $(a, 1/c+1) \in \mathcal{L}$ and $(c+1, b) \in \mathcal{L}$ — case (iii). If $(a, 1/c+1) \in \mathcal{L}$ then $c \in \big\{-1/2, -1, -1/2 + i/2, -1/2 - i/2, -1/3 + j/3, -1/3 + j^2/3\big\}$ and if $(c+1, b) \in \mathcal{L}$ then $c \in \big\{0, -2, -1, -1+i, -1-i, -1+j, -1+j^2\big\}$. The intersection of the two lists is $c = -1$, and then $a = b = -1$.

We prove that conversely each point in (i)–(ii)–(iii) is easy. Let $M \to M'$ be a matroid perspective on a set E.

(i) We have $t(M,M';t+1,1/t+1,t) = t^{r(M)-r(M')}\sum_{A\subseteq E} t^{-|A|} = t^{r(M)-r(M')}t^{-|E|}$, hence $(t+1, 1/t+1, t)$ is easy for any matroid perspective.

(ii) By straightforward substitutions in the formula defining $t(M,M')$ we have $t(M,M';x,y,x-1) = t(M;x,y)$ and $t\big(M,M';x,y,1/(y-1)\big) = (y-1)^{-\left(r(M)-r(M')\right)}t(M';x,y)$. It follows that the 15 points of case (ii) amount to easy points of 2-variable Tutte polynomials, hence are easy by Theorem A.

(iii) With notation of Theorem B, if V and V' are defined by bases, all necessary computations to evaluate $t(M,M';-1,-1,-1)$ can be made by means of polynomial algorithms. It follows that $(-1,-1,-1)$ can be polynomially evaluated for binary matroid perspectives with a succint presentation in the sense of [6], hence is an easy point for the 3-variable Tutte polynomial of a matroid perspective. ∎

An alternate proof of Theorem 1 is obtained by using the perspective $M \to M$ in place of $M \to \mathbf{0}$ resp. $\mathbf{1} \to M$.

When $(a,b) = (1,1)$, the point $\big(a,b,1/(b-1)\big)$ is not defined. However, in view of the identity $(y-1)^{r(M)-r(M')}\,t\big(M,M';x,y,1/(y-1)\big) = t(M';x,y)$, we may consider that the limit evaluation at $(1,1,\infty)$ is also an easy point, dual to the evaluation at $(1,1,0)$. This limit is equal to the evaluation at $(1,1)$ of the 2-variable polynomial coefficient of $z^{r(M)-r(M')}$ in $t(M,M';x,y,z)$. With this convention, we get 16 easy points in case (ii).

REFERENCES

[**1**] Brylawski, T. – A decomposition for combinatorial geometries, *Trans. Amer. Math. Soc.,* 171 (1972), 235–282.

[**2**] Brylawski, T. and Oxley, J. – *The Tutte polynomial and its applications*, chap. 6 in “Matroid Applications” (N. White, Ed.), Cambridge University Press (1992), 123–225.

[**3**] Chaiken, S. – The Tutte polynomial of a ported matroid, *J. Combin. Theory Ser. B,* 46 (1989), 96–117.

[**4**] Etienne, G. and Las Vergnas, M. – The Tutte polynomial of a morphism of matroids, 3. Vectorial matroids, *Advances in Applied Mathematics,* 32 (2004), 198–211.

[**5**] Gioan, E. and Las Vergnas, M. – On the evaluation at (j, j^2) of the Tutte polynomial of a ternary matroid, *J. Algebraic Combin.,* 25 (2007), 1–6.

[**6**] Jaeger, F.; Vertigan, D.L. and Welsh, D.J.A. – On the computational complexity of the Jones and Tutte polynomial, *Math. Proc. Camb. Phil. Soc.,* 108 (1990), 35–53.

[**7**] Kung, J.P.S. – *Strong maps*, chap. 8 in "Theory of Matroids" (N. White, Ed.), Cambridge Univeristy Press, 1986, pp. 224–253.

[**8**] Las Vergnas, M. – Extensions normales d'un matroïde, polynôme de Tutte d'un morphisme, *C.R. Acad. Sci. Paris sér. A,* 280 (1975), 1479–1482.

[**9**] Las Vergnas, M. – Acyclic and totally cyclic orientations of combinatorial geometries, *Discrete Mathematics,* 20 (1977), 51–61.

[**10**] Las Vergnas, M. – On the Tutte polynomial of a morphism of matroids, *Annals Discrete Mathematics,* 8 (1980), 7–20.

[**11**] Las Vergnas, M. – The Tutte polynomial of a morphism of matroids, 1. Set-pointed matroids and matroid perspectives, *Annales de l'Institut Fourier,* 49 (1999), 973–1015.

[**12**] Oxley, J.G. – *Matroid Theory*, Oxford Science Publications, 1992.

[**13**] Welsh, D.J.A. and Kayibi, K.K. – A linking polynomial of two matroids, *Advances in Applied Mathematics,* 32 (2004), 391–419.

[**14**] Welsh, D.J.A. and Kayibi, K.K. – A linking polynomial of two matroids: acknowledgement, *Advances in Applied Mathematics,* to appear.

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