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LAX EQUATIONS, FACTORIZATION AND RIEMANN–HILBERT PROBLEMS

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Abstract: The paper deals with the problem of existence and calculation of solutions to Lax equations that define finite-dimensional integrable systems. The method presented in the paper is based on Wiener–Hopf factorization and related Riemann–Hilbert problems on Riemann surfaces. The idea behind the method was first proposed by Semenov-Tian-Shansky but, to the authors' knowledge, is here applied, for the first time, in an infinite dimensional setting. The method dealt with in the paper enables one to analyse the global existence of solutions which seems more difficult by other methods. An example of a dynamical system associated with an elliptic curve is completely worked out in the paper.

1 – Introduction

In this paper we investigate the existence and calculation of solutions to Lax equations defining finite-dimensional integrable systems. This is done by means of the method of Wiener–Hopf factorization of a certain matrix-valued function G in appropriate function spaces which, as will be shown in section 4, involves solving a Riemann–Hilbert problem on a Riemann surface defined by the spectral curve associated to the Lax equation.

The method of factorization was proposed by Semenov-Tian-Shansky and Reymann ([7], [9], [10]) and may be seen as a generalization to loop algebras of the AKS theorem (Adler-Konstant-Symes) which applies to finite-dimensional

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Lie algebras ([1], [8]). However, to the authors' knowledge, no example appears in the literature for cases where the factorization is done in an infinite dimensional setting. This is probably due to the fact that certain delicate questions appear both in the proof of the formulas of the solutions of the Lax equations in terms of the factorization and in the calculation of the factors themselves which involve a factorization on a Riemann surface.

The method of solving the Lax equation through a Winer–Hopf factorization of the above-mentioned function G is closely related to the method of the Baker–Akhiezer function as shown in [9] section 2.10. However, as is shown in section 2 of the present paper, the factorization method makes it easier to investigate the global existence of solution which may be important in the investigation of some dynamical systems. This question is answered in Theorem 2.7 and, we believe, is done here for the first time.

The paper is organized as follows.

In section 2 the factorization method for the solution of what we call standard Lax equations is expounded. The presentation avoids the delicate question of whether the factors of the factorization are differentiable with respect to the evolution parameter t (cf. Theorems 2.3 and 2.5). This makes the treatment fully rigorous. For the reduction of more general Lax equations to standard form we follow the exposition in [6], where an important example is worked out through the Baker–Akhiezer method. The question of global existence of solution is treated in this section as mentioned above.

In section 3 we obtain the Baker–Akhiezer function from the factorization of the function G. This result is known (cf. [9, section 2.10]) but is included here for completness. The inverse approach *i.e.* calculating the factorization from the Baker–Akhiezer function seems to us a very difficult one, contrary to the suggestion in [9], section 2.10, and leaves, apparently, unanswered the question of global existence of the solution.

In section 4 we illustrate the application of the factorization method by working out completely the solution of a nonlinear dynamical system that leads to a standard Lax equation. This system has certain analogies to the Euler top when described in terms of Nahm's equations (cf. [8, chapter 2], but the latter does not lead to a standard Lax equation and because of that involves an additional step in the calculations of the solutions (see Theorem 2.9).

The paper ends with two appendices. Appendix A reviews some basic notions on Wiener–Hopf factorization that are needed in sections 2 to 4. Appendix B introduces notation and reviews basic facts on Riemann surfaces. A factorization of the function $\exp(\mu t/\lambda)$ is also calculated there.

2 – Lax equations and factorization

We shall begin this section by considering dynamical systems described by Lax equations of the form

$$\frac{dL_t}{dt} = \left[L_t^+, L_t\right] \tag{2.1}$$

where the dynamical variables L_t^+, L_t depend on a parameter λ (usually called spectral parameter), varying on the unit circle S^1, L_t is a matrix-valued Laurent polynomial in λ and L_t^+ is the part of L_t analytical in the unit disc $\mathbb{D}$. Later on, in this section, we shall generalize equation (2.1) allowing L_t^+ to be more general in a way that the Lax equation corresponds to a larger class of finite-dimensional integrable systems.

In the following definition we make more precise the class of equations corresponding to (2.1).

Definition 2.1. Let $[C_1(I)]^{n\times n}$ be the space of continuously differentiable $n \times n$ matrix functions on the open interval $I \subset \mathbb{R}^+$ (relative to the dynamical variable t) and let $L(\lambda) \in [C_1(I)]^{n\times n}$ be a Laurent polymomial of the form

$$L_t(\lambda) = \sum_{k=-m}^{1} L_t^{(k)}\lambda^k \qquad \left(m \in \mathbb{N},\ \lambda \in S^1\right) .$$

The class of equations (2.1) with

$$L_t^+(\lambda) = \sum_{k=0}^{1} L_t^{(k)}\lambda^k \tag{2.2}$$

will be called the standard Lax class. □

Remark 2.2. Note that equation (2.1), together with the definition of L_t^+ in (2.2), implies that $L_t^{(1)}$ is a constant of the dynamics (*i.e.* is independent of t). We state first a well known result for the form of a solution to (2.1). □

Theorem 2.3. *Let L_t be an $n \times n$ matrix-valued function satisfying the Lax equation* (2.1) *in a neighborhood I_0 of the origin. Then the solution of the equation in I_0 is given by the formulas*

$$L_t = G_+ L_0 G_+^{-1} = G_-^{-1} L_0 G_- \tag{2.3}$$

where $L_0 = L_{t_{|t=0}}$ and G_+, G_- satisfy the linear differential equations.

$$\frac{dG_+}{dt} = L_t^+ G_+ \ , \qquad \frac{dG_-}{dt} = G_- L_t^- \tag{2.4}$$

subject to the initial conditions $G_+|_{t=0} = G_-|_{t=0} = I_n$ where I_n is the identity matrix of order n.

Proof: The first part of our proof follows the lines of Segal's exposition in [8, Chapter 3] for an analogous formula for the KdV equation. Let L_t satisfy (2.1) and define L_H by

$$L_H = H^{-1} L_t H \tag{2.5}$$

where $H \in C_1(I)$ satisfies the equation

$$\frac{dH}{dt} = L_t^+ H \ ,$$

subject to the initial condition $H(0) = I_n$. Differentiating (2.5) yields:

$$\frac{dL_H}{dt} = -H^{-1}\frac{dH}{dt}H^{-1}L_t H + H^{-1}\frac{dL_t}{dt}H + H^{-1}L_t\frac{dH}{dt} \ .$$

Substituting (2.1) and the first equation of (2.4) into the above equation gives

$$\frac{dL_H}{dt} = 0$$

i.e.

$$L_H = \left[H^{-1} L_t \, H\right]_{t=0} = L_0 \ .$$

For the second of formulas (2.4) we note that

$$L_t = L_t^- + L_t^+$$

where $L_t^- = \sum\limits_{k=-m}^{-1} L_t^{(k)} \lambda^k$, from which it follows that (2.1) takes the form

$$\frac{dL_t}{dt} = \left[L_t, L_t^-\right] \ . \tag{2.6}$$

Consider now $L_H = H L_t H^{-1}$ with

$$\frac{dH}{dt} = H L_t^- \tag{2.7}$$

subject to the initial condition $H(0) = I_n$.

Differentiating L_H as above and using (2.6) and (2.7) we get

$$\begin{aligned}\frac{dL_H}{dt} &= HL_-L_tH^{-1} + H\left[L_t, L^-\right] - HL_tH^{-1}\frac{dH}{dt}H^{-1} \\ &= H\left[L_t^-, L_t\right]H^{-1} + H\left[L_t, L_t^-\right] = 0\ .\end{aligned}$$

Thus L_H is constant and equals L_0 taking into account the initial condition for H. This completes the proof of formulas (2.3), (2.4). ∎

Remark 2.4. Formulas (2.3) imply the known result that the spectrum of L_t is invariant with t *i.e.*

$$\det\left(\mu I_n - L_t\left(\lambda\right)\right) = \det\left(\mu I_n - L_0\left(\lambda\right)\right)\ . \tag{2.8}$$

Since $L_0\left(\lambda\right)$ is a Laurent polynomial in λ the above relations define an algebric curve which is characteristic of the dynamics. □

Next we prove a result that is fundamental for the use of the Riemann–Hilbert problem as a tool for solving equation (2.1).

Theorem 2.5. *Let $G = G_-G_+$ where G_-, G_+ satisfy equations* (2.4) *in a neighbourhood of the origin with the normalization $G|_{t=o} = I_n$. Then*

$$G = \exp\left(tL_0\right)\ . \tag{2.9}$$

Proof: Since G_-, G_+ satisfy (2.4), G is differentiable and thus we have

$$\begin{aligned}\frac{dG}{dt} &= \frac{dG_-}{dt}G_+ + G_-\frac{dG_+}{dt} = \\ &= G_-\left[G_-^{-1}\frac{dG_-}{dt} + \frac{dG_+}{dt}G_+^{-1}\right]G_+ \\ &= G_-L_tG_+ = L_0G \qquad (2.10)\end{aligned}$$

where the last equality follows from $L_0 = G_-L_tG_-^{-1}$. Since L_0 is constant with t, the solution to (2.10) is

$$G = \exp\left(tL_0\right)$$

taking into account the normalization $G|_{t=o} = I_n$. ∎

The result of Theorem 2.5

$$\exp(tL_0) = G_- G_+ \tag{2.11}$$

is a canonical bounded Wiener–Hopf factorization of $\exp(tL_0)$ in the interval I_0 (see Appendix A). This follows from the fact that L_t^+, L_t^- are bounded analytic matrix-valued functions, respectively, in $\Omega^+ = \mathbb{D}$ and $\Omega^- = \mathbb{C} \backslash \overline{\mathbb{D}}$. From equations (2.4) it can be seen that the same is true for G_-, G_+ and their inverses (it can be easily seen that the inverses of G_-, G_+ satisfy equations analogous to (2.4)). We can now use (2.11) to prove the important result of global existence of the solution to (2.1).

Remark 2.6. Note that, contrary to what is commonly found in the literature on integrable systems, our proof of equation (2.11) does not require any apriori assumption on the differentiability (with respect to t) of the factors in the factorization of $\exp(tL_0)$. G_-, G_+ are defined as solutions of equations (2.4). Only later in Theorem 2.5 do we prove that they are the factors in the factorization of $\exp(tL_0)$. □

Theorem 2.7. *The solution to equation* (2.1) *exists in a given interval* $[0, T[$ *if and only if* $\exp(tL_0)$ *has a canonical Wiener–Hopf factorization in this interval with factors* G_-, G_+ *differentiable with respect to* t *in the above interval and* G_- *normalized as* $G_-|_{\lambda=\infty} = I_n$.

Proof: Suppose first that equation (2.1) has a solution in the given interval $[0, T[$. Then, by Theorem 2.3, there exist functions G_-, G_+ satisfying equations (2.4) in $[0, T[$, that is, differentiable in this interval. By Theorem 2.5, $G_- G_+ = \exp(tL_0)$ in $[0, T[$ which is equivalent to saying that $\exp(tL_0)$ has a canonical Wiener–Hopf factorization in $[0, T[$ (note that G_-^{-1} and G_+^{-1} exist and are differentiable in the same interval).

For the sufficiency part of the proof assume that $\exp(tL_0) = G_- G_+$ with factors G_-, G_+ differentiable in $[0, T[$. Differentiating the above equality gives:

$$G_+ L_0 G_+^{-1} = G_-^{-1} \frac{dG_-}{dt} + \frac{dG_+}{dt} G_+^{-1} \,. \tag{2.12}$$

Let

$$L_t = G_+ L_0 G_+^{-1} \,. \tag{2.13}$$

Then from (2.12) we get, for the part of L_t analytic in the unit disc $\mathbb{D}$,

$$L_t^+ = \frac{dG_+}{dt} G_+^{-1} \,. \tag{2.14}$$

Now, differentiating (2.13) yields:

$$\frac{dL_t}{dt} = \frac{dG_+}{dt} G_+^{-1} L_t - L_t \frac{dG_+}{dt} G_+^{-1}$$

and using (2.14)

$$\frac{dL_t}{dt} = \left[L_t, L_t^+\right]$$

i.e. L_t satisfies equation (2.1) as we wanted to prove. ∎

Remark 2.8. As t varies, the solution to equation (2.1) must remain a Laurent polynomial of the same degree in λ and λ^{-1} as L_0. Formulas (2.3) appear to hide this fact since $G_\pm$ and their inverses are, in general, transcendental functions of λ. However, noting that

$$L_t = G_+ L_0 G_+^{-1} = G_-^{-1} L_0 G_- \;,$$

Liouville's theorem applied to the second equality immediately shows that the degree of the polynomial dependence of L_t in λ and λ^{-1} is independent of t. □

To end this section we consider in the theorem that follows the reduction to standard form, of a Lax equation more general than (2.1) (cf. [6]).

Theorem 2.9. *Let* $L_t = \sum_{k=-m}^{1} L_t^{(k)} \lambda^k$ *satisfy the Lax equations*

$$\frac{dL_t}{dt} = \left[L_t, \widetilde{L}^+\right] \tag{2.15}$$

where $\widetilde{L}_+ = \gamma L_t^{(0)} + L_t^{(1)} \lambda$, $\gamma \in \mathbb{C}$ *(note that* $\widetilde{L}_+$ *is strictly related but not equal to the analytic part of* L_t*). Then there exists* F *satisfying*

$$\frac{dF}{dt} = (1-\gamma)\, L_0 F \tag{2.16}$$

such that $\widehat{L}_t = F^{-1} L_t F$ *belongs to the standard Lax class i.e. satisfies the equation*

$$\frac{d\widehat{L}_t}{dt} = \left[\widehat{L}_t, \widehat{L}_t^+\right] \tag{2.17}$$

where $\widehat{L}_t^+ = \widehat{L}_t^{(0)} + \widehat{L}_t^{(1)} \lambda$ *as in Definition 2.1.*

Proof: To prove (2.17) we differentiate the expression for $\widehat{L}_t$ to get

$$\frac{d\widehat{L}_t}{dt} = -F^{-1}\frac{dF}{dt}F^{-1}L_tF + F^{-1}\frac{dL_t}{dt}F + F^{-1}L_t\frac{dF}{dt} \ .$$

Substituting (2.15) and(2.16) in this expression gives

$$\frac{d\widehat{L}_t}{dt} = (1-\gamma)\left[\widehat{L}_t, \widehat{L}_0\right] + F^{-1}\left[L_t, \widetilde{L}^+\right]F$$

where $\widehat{L}_o = F^{-1}L_0F$. Noting that $F^{-1}\left[L_t, \widetilde{L}^+\right]F = \left[\widehat{L}_t, F^{-1}\widetilde{L}^+F\right]$ we obtain expression (2.17) where $\widehat{L}_t^+ = F^{-1}\widetilde{L}^+F = \widehat{L}_o + \widehat{L}_1\lambda$, with $\widehat{L}_i = F^{-1}L_iF$ $(i=0,1)$. ∎

3 – Factorization and Baker–Akhiezer functions

Although our approach to solving equation (2.1) is through Wiener–Hopf factorization, since the method based on the Baker–Akhiezer function is widely used in the literature on integrable systems (cf. [5], [6] and [9]), we review below how the Baker–Akhiezer function can be calculated from the factorization of $\exp(tL_0)$. In this exposition we follow, in part, references [6] and [9].

We begin by defining a differential operator associated with the analytic part of L_t (we continue to consider the standard Lax class).

Definition 3.1. Let $C_1(\mathbb{R}^+)$ be the space of continuosly differentiable functions on $\mathbb{R}^+$. Define the operator on $C_1(\mathbb{R}^+)$

$$(D - L_+)\,\varphi = \frac{d}{dt}\varphi - \left(L_t^{(0)} + L_t^{(1)}\lambda\right)\varphi$$

where $L_t^{(0)} \in C_1(\mathbb{R}^+)$ and $L_t^{(1)}$ is constant. □

Proposition 3.2. *Let $L_t \in C_1(\mathbb{R}^+)$ and $(D - L_+)$ be as in Definition 2.1. Then the operators $D - L_+$ and L_t commute i.e.*

$$[D - L_+, L_t] = 0$$

if and only if L_t satisfies equation (2.1).

Proof: For any $\varphi \in C_1(\mathbb{R}^+)$ we have

$$\begin{aligned}\left(D - L^+\right) L_t\varphi &= \frac{dL_t}{dt}\varphi + L_t\frac{d\varphi}{dt} - L^+L_t\varphi \\ L_t\left(D - L^+\right)\varphi &= L_t\frac{d\varphi}{dt} - L_tL^+\varphi\end{aligned}$$

from which it follows that

$$[D - L_+, L_t]\,\varphi = \frac{dL_t}{dt}\varphi - [L_t, L_+]\,\varphi$$

which is equal to zero if and only if L_t satisfies (2.1). ∎

Remark 3.3. Since $L_t^{(1)}$ is constant, with the additional assumption that $L^{(1)}$ is invertible, we may take $L^{(1)} = I_n$. Then the conditions $[D - L^+,\ L_t] = 0$ and $[D - L_0, L_t]$ are equivalent. This means that a common eigenfunction of $D - L^+$ and L_t is also a common eigenfunction of $D - L_0$ and L_t. This fact is at the base of the definition that follows. □

Definition 3.4. A Baker–Akhiezer function for the Lax equation (2.1) is any common eigenfunction of the operators $D - L_0$ and L_t, *i.e.* φ is a common solution to the equations

$$(D - L_0)\varphi = \lambda\varphi \tag{3.1}$$

$$L_t\varphi = \mu\varphi \tag{3.2}$$

i.e. λ and μ are, respectively, the eigenvalues of $D - L_t^{(0)}$ and L_t corresponding to the common eigenfunction φ. □

Remark 3.5. Since L_t depends on λ the above definition implies an algebraic relation between μ and λ. This is precisely the algebraic curve defined by

$$\det\left(\mu I_n - L_0\right) = \det\left(\mu I_n - L_t\right) = 0\ ,$$

which, in turn, means that the common eigenfunctions of the operators $D - L_t^{(0)}$ and L_t are parametrized by the points of the above algebraic curve. □

The method of solution of (2.1) based on the calculation of the Baker–Akhiezer function involves identifying the singularities of φ on the spectral curve:

poles at the poles of μ, essencial singularities at the poles of λ. Since this does not define φ uniquely, a certain degree of intuition is required to complete the calculation which is unnecessary if we use the factorization method. For our next result we follow [9]. In Proposition 3.5, Ω^+ and Ω^- are the inverse images of $\mathbb{D}$ and $\mathbb{C}\backslash\overline{\mathbb{D}}$ under the canonical projection from the spectral curve $\det(\mu I_n - L_t) = 0$ to $\mathbb{C}$.

Proposition 3.6. *Let G_-G_+ be the Wiener–Hopf factorization of* $\exp(tL_0)$ *and Ψ_0 an eigenfunction of L_0 i.e.*

$$\exp(tL_0) = G_-G_+$$
$$L_0\Psi_0 = \mu\Psi_0 \ .$$

Then the function

$$\Psi_t = \begin{cases} G_+\Psi_0 & \text{in } \Omega^+ \\ G_-^{-1}\Psi_0 & \text{in } \Omega^- \end{cases}$$

is an eigenfunction of L_t i.e.

$$L_t\Psi_t = \mu\Psi_t \tag{3.3}$$

on the spectral curve $\det(\mu I_n - L_t) = 0$

Proof: The proof is straightforward. Ψ_0 is a meromorphic function on the spectral curve. From formulas (2.3) we get

$$\begin{array}{llll} \text{in } \Omega^+ & G_+^{-1}L_tG_+\Psi_0 = \mu\Psi_0 & \Leftrightarrow & L_t\,(G_+\Psi_0) = \mu G_+\Psi_0 \\ \text{in } \Omega^- & G_-L_tG_-^{-1}\Psi_0 = \mu\Psi_0 & \Leftrightarrow & L_t\left(G_-^{-1}\Psi_0\right) = \mu G_-^{-1}\Psi_0 \ . \end{array}$$ ∎

Following Reymann and Semenov-Tian-Shansky [9, II, Section 2.10] we note that writing $\Psi_t^+ = G_+\Psi_0$, $\Psi^t = G_-^{-1}\Psi_0$ we conclude that the function Ψ_t satisfies the following Riemann–Hilbert problem on the spectral curve

$$G_+^{-1}\Psi_t^+ = G_-\Psi_t^- \Leftrightarrow \Psi_t^+ = G_+\Psi_t^-$$

i.e.

$$\exp(-tL_0)\,\Psi_t^+ = \Psi_t^- \ .$$

This is, precisely, the Riemann–Hilbert problem associated to the factorization of $\exp(tL_0)$ encountered in section 2, equation (2.11).

Remark 3.7. In [9] the objective of the authors is the opposite of ours: to determine the factorization from the Baker–Akhiezer function, although only in an abstract formulation. □

4 – Example

In this section we solve a standard Lax equation for a concrete integrable system, as an illustration of the results of the preceding sections.

Consider the Lax equation

$$\frac{dL_t}{dt} = \left[L_t^+, L_t\right] \tag{4.1}$$

where L_t is the matrix-valued function given by

$$L_t(\lambda) = \begin{bmatrix} v(\lambda) & u(\lambda) \\ w(\lambda) & -v(\lambda) \end{bmatrix} \tag{4.2}$$

where $\lambda \in S^1$ and

$$\begin{aligned} v\,(\lambda) &= z\lambda^{-1} \\ u\,(\lambda) &= a\lambda + y\lambda^{-1} + x, \quad a \in \mathbb{C} \\ w\,(\lambda) &= a\lambda + y\lambda^{-1} - x\ . \end{aligned} \tag{4.3}$$

Here x, y, z are the dynamical variables and a is a constant. $L_t\,(\lambda)$ is a Laurent polynomial in λ,

$$L_t(\lambda) = L_t^{(-1)}\lambda^{-1} + L_t(0) + L_t^{(1)}\lambda \tag{4.4}$$

where $L^{(1)}$ is constant. With

$$L_t^+\,(\lambda) = L_t^{(0)} + L_t^{(1)}\lambda \tag{4.5}$$

equation (4.1) becomes a standard Lax equation. It can easily be shown that with L_t given by (4.2) and (4.3), (4.1) is equivalent to the following nonlinear system of ordinary differential equations:

$$\begin{aligned} \frac{dx}{dt} &= -2az \\ \frac{dy}{dt} &= -2xz \\ \frac{dz}{dt} &= 2xy\ . \end{aligned} \tag{4.6}$$

As shown in section 2 the solution of equation (4.1) by the factorization method involves solving the following Wiener–Hopf factorization problem:

$$G = \exp\,(tL_0) = G_- G_+ \tag{4.7}$$

where $L_0 = L_t|_{t=0}$ as in Section 2. As will be seen later, this, in turn, involves, a fundamental step consisting of obtaining a factorization on a Riemann surface Σ of the scalar exponential function

$$E_\nu(\lambda) = \exp(t\nu(\lambda)) \tag{4.8}$$

where $\nu(\lambda)$ is defined by the characteristic equation of $L_0(\lambda)$:

$$\det(\nu I_2 - L_0(\lambda)) = 0 \ . \tag{4.9}$$

The Riemann surface Σ is precisely defined by equation (4.9).

Next we proceed to derive a Riemann–Hilbert problem whose solution gives us the factors G_-, G_+ in (4.7). From (4.7) we get

$$\exp(tL_0)\, G_+^{-1} = G_- \ . \tag{4.10}$$

Denoting by ϕ^+ (ϕ^-) the first column in the matrix function G_+^{-1} (respectively G_-) we have the following vector Riemann–Hilbert problem

$$\exp(tL_0)\,\phi^+ = \phi^- \tag{4.11}$$

in the original contour S^1.

This is also true for the second column (in fact it will be seen that the above problem has two linearly-independent solutions corresponding to the two columns of G_+^{-1} and G_-).

We shall deduce next a scalar Riemann–Hilbert problem on the Riemann surface defined by (4.9), which is equivalent to the Riemann–Hilbert problem (4.11). Firstly we note that

$$\det(\nu I_2 - L_0(\lambda)) = \nu^2 - \lambda^{-2} p(\lambda) \tag{4.12}$$

with

$$p(\lambda) = a^2\lambda^4 - \left(x^2 - 2ay\right)\lambda^2 + z^2 + y^2 \ . \tag{4.13}$$

In what follows we shall assume that all the zeros of $p(\lambda)$ are real. This is true, for example, for a sufficiently small. Diagonalization of L_0 gives:

$$L_0 = SD_0S^{-1} \tag{4.14}$$

where $D_0 = \operatorname{diag}\left(\lambda^{-1}\mu, -\lambda^{-1}\mu\right)$ and

$$S = \begin{bmatrix} 1 & -1 \\ \dfrac{\mu - z}{p_2} & \dfrac{\mu + z}{p_2} \end{bmatrix} \tag{4.15}$$

with $\mu = \sqrt{p(\lambda)}$ (here we take $\mathrm{Re}(\mu) > 0$) and

$$p_2(\lambda) = \lambda u = a^2\lambda^2 + x\lambda + y \; . \tag{4.16}$$

From (4.14) we get

$$\exp\left(tL_0\right) = SDS^{-1} \tag{4.17}$$

where $D = \operatorname{diag}\left(\exp\left(t\lambda^{-1}\mu\right), \exp\left(-t\lambda^{-1}\mu\right)\right)$.

From (4.15) and (4.16) it follows that the Riemann–Hilbert problem (4.11) takes the form

$$\begin{aligned} d_1\left(z\phi_1^+ + p_2\phi_2^+ + \mu\phi_2^+\right) &= \phi_1^- + p_2\phi_2^- + \mu\phi_2^- \\ d_2\left(z\phi_1^+ + p_2\phi_2^+ - \mu\phi_2^+\right) &= \phi_1^- + p_2\phi_2^- - \mu\phi_2^- \end{aligned} \tag{4.18}$$

where $\phi^{\pm} = \left(\phi_1^{\pm}, \phi_2^{\pm}\right)$, $d_1 = \exp\left(t\lambda^{-1}\mu\right)$ and $d_2 = \exp\left(-t\lambda^{-1}\mu\right)$.

Consider now the nonsingular algebraic curve Σ_0 defined by the equation

$$\mu^2 = p\left(\lambda\right) \tag{4.19}$$

where $p(\lambda)$ is the polynomial defined in (4.13). It can be completed by adding two "points at infinity". The completion of Σ_0 is an elliptic curve, which we denote by Σ (see Appendix B for more details). Consider now the contour $\Gamma \subset \Sigma_0$ that is the pre-image of the original contour S^1 under the projection $\lambda\colon \Sigma_0 \to \mathbb{C}$ defined by $(\lambda, \mu) \mapsto \lambda$. It is easily seen that the two scalar equations in (4.18) correspond to a single scalar equation on the disconnected contour Γ. This equation may be written

$$d\left(\phi_2^+ + \frac{z+\mu}{p_2}\phi_1^+\right) = \phi_2^- + \frac{z+\mu}{p_2}\phi_1^- \tag{4.20}$$

where $d = d_1$ on one of the two components of Γ and $d = d_2$ on the other component. In simple words: under the projection λ, Σ_0 is a two-sheeted covering of the complex plane. Equation (4.20) corresponds to the first equation of (4.18), in one sheet, and to the second equation, in the other sheet.

In (4.20) $\phi_{1,2}^{\pm}$ are understood as functions analytic and bounded in the region $\Omega^+ = \lambda^{-1}\left(\mathbb{D}\right)$ and $\phi_{1,2}^-$ similarly for the region $\Omega^- = \lambda^{-1}\left(\mathbb{C} \setminus \overline{\mathbb{D}}\right)$. Note that from the original formulation of the Riemann–Hilbert problem d belongs to $C_\mu\left(\Gamma\right)$ (in fact $d \in C_\infty\left(\Gamma\right)$).

Since μ changes into $-\mu$ through the involution $\tau\colon (\lambda, \mu) \to (\lambda, -\mu)$, from the expressions for d_1 and d_2, it follows that

$$d\left(\lambda, \mu\right) = \exp\left(\frac{\mu}{\lambda} t\right), \quad (\lambda, \mu) \in \Gamma \; . \tag{4.21}$$

It is shown in Appendix B that d has a factorization of the form

$$d = d_t^- r_t \, d_t^+ \tag{4.22}$$

with $(d_t^\pm)^{\pm 1} \in H_\infty(\Omega^\pm)$ (see Definition A.1) and r_t a rational function on Σ with one pole and one zero in each region $\Omega^\pm$. From (4.20) and (4.22) we get

$$r_t d_t^+ \psi^+ = \left(d_t^-\right)^{-1} \psi^- = R \tag{4.23}$$

where R is a rational function on Σ and

$$\psi^\pm = \frac{z+\mu}{p_2}\phi_1^\pm + \phi_2^\pm$$

are meromorphic functions in $\Omega^\pm$.

In (4.23) we look for a solution that is zero at a chosen point in Ω^+. Then R is completely determined apart from a multiplicative constant and an explicit expression can be deduced. Since Σ has genus 1, R is as an elliptic function in the variable u related to (λ, μ) by the Abel map:

$$(\lambda, \mu) \mapsto u := \frac{1}{4\mathbf{K}} \int_{\mathbf{0}_1}^{(\lambda,\mu)} \frac{d\lambda}{\mu} \, ,$$

where $\mathbf{K}$ is a constant determined by equation (4.19) and $\mathbf{0}_1$ is the origin of the first sheet (we review this transformation in Appendix B). For simplicity, we write $R(u)$ to denote the result of this change of variable. We also abuse notation and denote the image of the regions $\Omega^\pm$ under this transformation by $\Omega^\pm$ as well. We have the following conditions for $R(u)$:

- **(i)** it has a zero at a point $v_1 \in \Omega^+$ that we fix;
- **(ii)** it has a pole at the pole of r_t in Ω^+;
- **(iii)** it has a zero at the zero of r_t in Ω^+ that we denote by v_0;
- **(iv)** it has two poles at the zeros of p_2 that do not coincide with zeros of $z+\mu$ (it is easily shown that $z+\mu$ and p_2 have two common zeros). These poles will be denoted by u_1, u_2;
- **(v)** Abel's condition for a rational function on Σ now gives for the third zero (denoted v_2)

$$v_2 \equiv u_0 + u_1 + u_2 - (v_0 + v_1)$$

where $\equiv$ denotes equality modulo elements of the lattice Λ of periods of the differential $d\lambda/4\mathbf{K}\mu$ (see Appendix B for more details).

Hence R has the representation

$$R(u) = \gamma \frac{\vartheta_1(u - v_0)\vartheta_1(u - v_1)\vartheta_1(u - v_2)}{\vartheta_1(u - u_0)\vartheta_1(u - u_1)\vartheta_1(u - u_2)} \tag{4.24}$$

where $\gamma \in \mathbb{C}$ and ϑ_1 is the theta function defined in Appendix B.

From (4.23) and (4.24) we now get

$$\phi_1^+ = \frac{p_2}{2\mu} D, \quad \phi_2^+ = \frac{zD - \mu S}{2\mu} \tag{4.25}$$

where

$$D = (r_t d_+)^{-1} R - \left[(r_t d_+)^{-1} R \right]_\tau \tag{4.26}$$

$$S = (r_t d_+)^{-1} R + \left[(r_t d_+)^{-1} R \right]_\tau \tag{4.27}$$

where the subscript τ denotes the image by the involution of the expression in brackets.

We can now state the following proposition:

Proposition 4.1. *The Riemann–Hilbert problem* (4.20) *has two linearly-independent solutions given by formulas* (4.25) *corresponding to two functions R determined by two distinct pairs of zeros* (v_1, v_2) *of* ψ^+.

Proof: The only part of the proposition that remains to be proven is the statement that the dimension of the space of solutions is two. This is a consequence of the fact that, in (4.23), the dimension of the space of rational functions with three fixed poles and one fixed zero is two, in view of the Riemann-Roch theorem. ∎

Since the expressions for ϕ_1^+ and ϕ_2^+ are invariant for the involution τ, formulas (4.25) give the solution of original Riemann–Hilbert problem

$$\exp(tL_0)\phi^+ = \phi^-$$

where $\phi^\pm = \left(\phi_1^\pm, \phi_2^\pm\right)$. The factor G_+ in the statement of Theorem 2.6 is given by

$$G_+^{-1} = \begin{bmatrix} \phi_1^+ & \widehat{\phi}_1^+ \\ \phi_2^+ & \widehat{\phi}_2^+ \end{bmatrix}, \quad G_+ = \frac{1}{\Delta} \begin{bmatrix} \widehat{\phi}_2^+ & -\widehat{\phi}_1^+ \\ -\phi_2^+ & \phi_1^+ \end{bmatrix} \tag{4.28}$$

where $\phi_{1,2}^+$ and $\widehat{\phi}_{1,2}^+$ are two distinct solutions of (4.25) and $\triangle = \phi_1^+\widehat{\phi}_2^+ - \phi_2^+\widehat{\phi}_1^+$. The final formula for L_t is

$$L_t = G_+ L_0 G_+^{-1} \tag{4.29}$$

as indicated in Theorem 2.3.

Proposition 4.2. *The solution* (4.28) *to the Lax equation* (4.1) *is valid for all t in the interval* $[0,1]$.

Proof: The statement is a direct consequence of Proposition B.9 which gives the factorization of $\exp(\mu t/\lambda)$. ∎

Remark 4.3. The factorization of $\exp(\mu t/\lambda)$ allows us only to obtain the solution to (4.1) in the interval $[0,1]$ which, however, goes significantly beyond just knowing that the solution exists in some unspecified neighborhood of the origin.

A more thorough investigation of the conditions imposed by the expressions for $d_t^\pm(\mathbf{p})$ in Definition B.7 might allow one to obtain a solution valid for any specified interval by adjusting the radius of the original contour in which the spectral parameter λ takes values. □

Appendix A – Wiener–Hopf factorization and Riemann–Hilbert problems

Let Γ be a system of oriented piecewise-smooth closed curves in $\mathbb{C}$ such that the index or winding number of Γ relative to any point of $\mathbb{C}\setminus\Gamma$ takes the values 1 or 0. We denote by Ω^+ the component (or union of components) of $\mathbb{C}\setminus\Gamma$ for which points $\operatorname{ind}\Gamma$ takes the value 1 and Ω^- the component for which $\operatorname{ind}\Gamma$ takes only the value 0. A simple example is the unit circle, oriented counterclockwise, for which Ω^+ is the unit disk $\mathbb{D}$ and Ω^- is $\mathbb{C}\setminus\mathbb{D}$. In this case Ω^+ is a simply-connected region. A slightly more complicated example is a system of two circles, the unit circle oriented as above and a second circle centred at the origin with radius less than 1 and oriented clockwise. In this case Ω^+ is the annular region bounded by the two circles (not simply connected) and $\Omega^- = \mathbb{C}\setminus\overline{\Omega^+}$ is the union of two separate open sets.

The above definition can be readily generalized to a contour Γ in a Riemann surface.

We can now define a canonical bounded Wiener–Hopf factorization of a function defined on Γ.

Definition A.1. Let Γ be a countour in $\mathbb{C}$ and G a Hölder continuous $n \times n$ matrix-valued function on Γ $\left(G \in [C_\mu(\Gamma)]^{n\times n}, 0 < \mu \leq 1\right)$. G is said to possess a canonical bounded Wiener–Hopf factorization if it can be represented in the form

$$G = G_- G_+ \tag{A.1}$$

where G_-, G_+ and their inverses belong to the Hardy spaces $H_\infty(\Omega^-)$, $H_\infty(\Omega^+)$, respectively (these are the spaces of analytic and bounded in functions in Ω^-, Ω^+).□

Remark A.2.

(i) In general for a contour Γ in $\mathbb{C}$ the factorization takes the more general form

$$G = G_- D G_+ \tag{A.2}$$

where $D = \operatorname{diag}\left(r^{k1}, ..., r^{kn}\right)$ with $k_1 \geq k_2 \geq \cdots k_n$ and r is a rational function with a zero in Ω^+ and a pole in Ω^-. However, we will only need the canonical form (A.1)

(ii) For functions that are not Hölder continuous or that do not belong to other special spaces of continuous functions (*e.g.*, the Wiener algebra), the factors may be unbounded. For these more general symbols the notion of factorization to be used is generalized factorization (cf. [3], [4]).

(iii) Outside the area of operator theory the factorization (A.2) is often called (perhaps improperly) Birkhoff factorization. Another designation that is found in the literature is Riemann–Hilbert factorization. □

Factorizations (A.1) and (A.2) have close connections with Fredholm properties of Toeplitz operators with symbol G (cf. [4]) and the problem of solvability of Riemann–Hilbert problems with coefficient G, *i.e.*

$$G\phi^+ = \phi^-, \tag{A.3}$$

for ϕ^+, ϕ^- in appropriate spaces. The last question is the one that is relevant for the present paper.

To illustrate this connection we consider the simple example of the unit circle S^1 with G a Hölder continuous function on S^1 (*i.e.* $G \in C_\mu[(S^1)]^{n\times n}$). It is wellknown that $C_\mu(S^1)$ decomposes into a direct sum of closed subspaces,

$$C_\mu\left(S^1\right) = C_\mu^+\left(S^1\right) \oplus C_\mu^-\left(S^1\right)$$

where the functions in $C_\mu^+(S^1)$ have bounded analytic continuations into $\Omega^+ = \mathbb{D}$ and functions in $C_\mu^-(S^1)$ have bounded analytic continuations into $\Omega^- = \mathbb{C} \setminus \overline{\mathbb{D}}$ vanishing at infinity. For vector-valued functions we have a similar decomposition

$$\left[C_\mu\left(S^1\right)\right]^n = \left[C_\mu^+\left(S^1\right)\right]^n \oplus \left[C_\mu^-\left(S^1\right)\right]^n . \tag{A.4}$$

Then if G has a factorization of the form (A.1) (canonical), the Riemann–Hilbert problem

$$G\phi^+ = \phi^- \tag{A.5}$$

with $\phi^\pm \in [C_\mu^\pm(S^1)]^n$ has only the trivial solution. If we consider $\phi^- \in [C_\mu^-(S^1)]^n \oplus \mathbb{C}^n$, *i.e.* if we drop the condition $\phi^-(\infty) = 0$, the Riemann–Hilbert problem (A5) has n linearly independent solutions. This is the setting considered in section 4.

Appendix B – Factorization of $\exp(\mu t/\lambda)$

In this section we compute a factorization of the function $\exp(\mu t/\lambda)$, on the Riemann surface Σ, used in Section 4. We also introduce notation and review some of the basic theory of Riemann surfaces needed for this computation. Our general reference on this subject is [11].

Let Σ denote the compact Riemann surface obtained by completing the algebraic curve

$$\Sigma_0 = \left\{(\xi,\eta) \in \mathbb{C}^2 \mid \eta^2 = (1-\xi^2)(1-\kappa^2\xi^2)\right\}$$

where $0 < \kappa < 1$. Hence Σ is obtained from Σ_0 by adding two points "at infinity" such that $\zeta = \xi^{-1}$ is a local parameter at these points. At the points where $\eta = 0$ (the branch points of the function $(\xi,\eta) \mapsto \xi$) η is a local parameter. At all other points ξ is a local parameter.

Assumption B.1. In order that Σ represents the Riemann surface of section 4, for an appropriate value of κ, we make the assumption that the polynomial $p(\lambda)$ defined in equation (4.13) has real roots. This is true, for example, if the leading coefficient a is small enough. Under this assumption the Riemann surface defined by the polynomial $p(\lambda)$ is isomorphic to Σ with $\kappa = \lambda_1/\lambda_2$, where $\lambda_1 < \lambda_2$ are the positive roots of $p(\lambda)$. The isomorphism is given by

$$(\lambda, \mu) \mapsto (\xi(\lambda, \mu), \eta(\lambda, \mu)) = \left(\lambda\lambda_1^{-1}, \mu(a\lambda_1\lambda_2)^{-1}\right) .$$

Note that, in particular, we have $\mu/\lambda = \mathbf{e}\eta/\xi$, where $\mathbf{e} = \mathbf{e}(a, x, y, z) = a\lambda_2$.

In order to simplify some of the arguments below we will also assume that the initial values a, x, y and z are small enough so that $\mathbf{e} < 1$. □

There are two natural merormorphic functions on Σ: those induced by the projections $(\xi, \eta) \mapsto \xi$ and $(\xi, \eta) \mapsto \eta$. They will be denoted respectively by ξ and η.

As usual, it is convenient to view Σ as a 2-branched cover of $\mathbb{P}(\mathbb{C}^2)$ under the map $\xi\colon \Sigma \to \mathbb{P}(\mathbb{C}^2) = \mathbb{C} \cup \{\infty\}$. We denote by S_i, $i = 1, 2$, the component of $\xi^{-1}(\mathbb{C} \setminus \{t \in \mathbb{R} \mid \ 1 \leq |t| \leq 1/\kappa\})$ such that $(0, (-1)^{i+1}) \in S_i$, and refer to its closure $\overline{S_i}$ as the i-th sheet of Σ. The "point at infinity" in the i-th sheet will be denoted by ∞_i . We also say that $\mathbf{0}_i = \left(0, (-1)^{i+1}\right)$ is the origin of the i-th sheet.

The figure below represents the basis for the homology group $H_1(\Sigma; \mathbb{Z})$ that will be used throughout. The solid lines represent the lift of the curve depicted, under the map ξ, such that it lies in the first sheet. The broken lines represent the lift that lies in the second sheet.

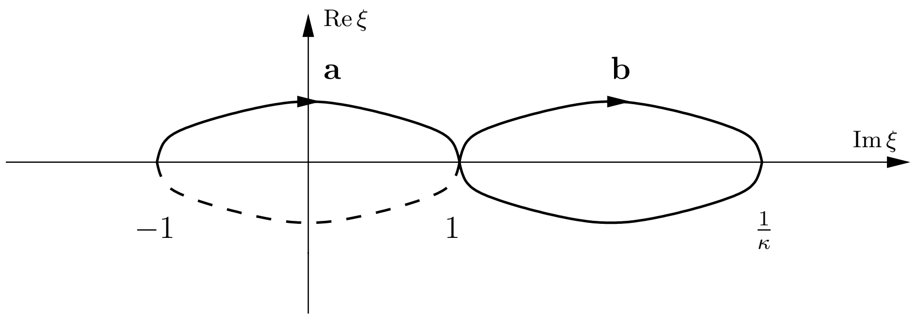

Figure 1 – Basis for $H_1(\Sigma; \mathbb{Z})$.

Once the homology basis $\{\mathbf{a}, \mathbf{b}\}$ is fixed, we can choose a normalized holomorphic differential. That is, a holomorphic 1-form ω such that $\int_{\mathbf{a}} \omega = 1$.

An easy computation shows that $d\xi/\eta$ is holomorphic. Thus $\omega = c^{-1}d\xi/\eta$, where

$$c = \int_{\mathbf{a}} \frac{d\xi}{\eta} = 4\int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-\kappa^2\xi^2)}} \ .$$

The integral above is called a *complete elliptic integral* and it is usually denoted by $\mathbf{K}(\kappa)$ or just $\mathbf{K}$. Using this notation, the normalized holomorphic differential is $\omega = d\xi/4\mathbf{K}\eta$.

Notation B.2. We will use the following notation for elliptic integrals [2], where $\kappa \in]0,1[$ and $\kappa' = \sqrt{1-\kappa^2}$.

- **(i)** $\mathbf{K} = \mathbf{K}(\kappa) = \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-\kappa^2\xi^2)}}$;
- **(ii)** $\mathbf{K}' = \mathbf{K}'(\kappa) = \mathbf{K}(\kappa') = \int_1^{1/\kappa} \frac{d\xi}{\sqrt{(\xi^2-1)(1-\kappa^2\xi^2)}}$;
- **(iii)** $\mathbf{E} = \mathbf{E}(\kappa) = \int_0^1 \sqrt{\frac{1-\kappa^2\xi^2}{1-\xi^2}}\, d\xi$. □

The first two integrals above are called *complete elliptic integrals of the first kind* and the last one is called a *complete elliptic integral of the second kind.*

We will also need the Abel-Jacobi map given by the holomorphic differential ω. It is a biholomorphic map from Σ to its image $\mathrm{Jac}(\Sigma)$ and it is defined as

$$A(\mathbf{p}) = \int_{\mathbf{0}_1}^{\mathbf{p}} \omega \ .$$

Its image $\mathrm{Jac}(\Sigma)$ is the torus $\mathbb{C}/\Lambda$ where Λ is the lattice of periods of ω. We have $\Lambda = \mathbb{Z}\cdot\mathbf{1} + \mathbb{Z}\cdot\tau$ with $\tau = \int_{\mathbf{b}} \omega = i\mathbf{K}'/2\mathbf{K}$. The point in $\mathrm{Jac}(\Sigma)$ determined by $u \in \mathbb{C}$ is denoted by $[u]$.

A convenient way to represent rational functions on Σ is to write them as quotients of theta functions. For this purpose we will use the theta function of characteristic $(1,1)$:

$$\vartheta_1(u) = \vartheta_1(u\,|\,\tau) = ie^{\pi i(u-\tau/4)}\vartheta_3(u + (1-\tau)/2\,|\,\tau) \ ,$$

where $\vartheta_3(u\,|\,\tau) = \sum_{n\in\mathbb{Z}} \exp \pi i(n^2\tau + 2nu)$. We recall that ϑ_1 satisfies:

- **(i)** $\vartheta_1(u + n + m\tau) = e^{-2\pi i\left(\frac{1}{2}m^2\tau + mu\right)}\vartheta_1(u)$;
- **(ii)** $\vartheta_1(u) = 0 \Leftrightarrow [u] \in \Lambda$;

and refer to [2] for more details.

We need another definition in order to state the factorization problem we want to address.

Definition B.3. Let $\mathbf{p} = A^{-1}([\mathbf{e}/4]) \in \Sigma$ (see Assumption B.1 for the definition of the constant $\mathbf{e}$), and let $\mathbb{D} \subset \mathbb{C}$ be the open disk of radius $\xi(\mathbf{p})$, centered at the origin. Define

$$\Omega^+ = \xi^{-1}(\mathbb{D}) \subset \Sigma$$
$$\Omega^- = \Sigma \setminus \overline{\Omega^+} \subset \Sigma \ .$$

In particular $\Sigma = \overline{\Omega^+} \cup \Omega^-$. □

Remark B.4. The reason for the choice of the radius of $\mathbb{D}$ will become clear after Definition B.7 below. □

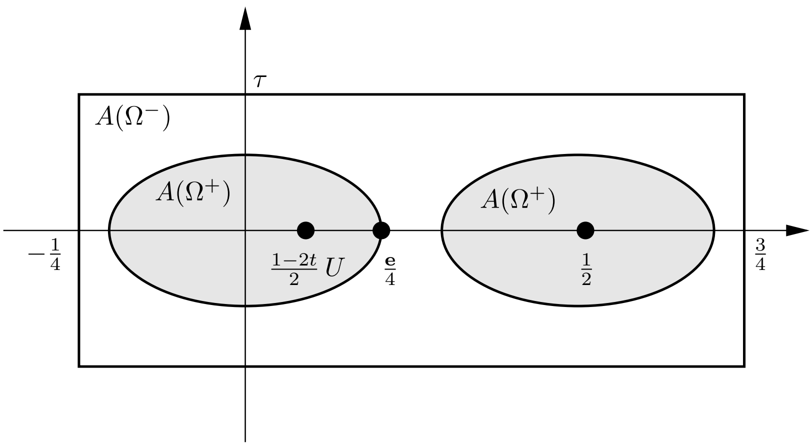

Figure 2 – The image of regions Ω^+ and Ω^+ under the Abel map. (See Definition B.7 for the definition of U.)

The factorization problem that appears in the example of Section 4 is the following: for each $t \in \mathbb{C}$, find functions $d_t^{\pm}$ and a rational function r_t on Σ such that $(d_t^{\pm})^{\pm 1} \in H(\Omega^{\pm})$ and

$$\exp\left(\frac{\mu}{\lambda}\, t\right) = d_t^+ r_t d_t^- \ , \tag{B.1}$$

on $\Omega^+ \cap \Omega^-$.

Since $\mu/\lambda = \mathbf{e}\eta/\xi$, where $\mathbf{e}$ is a constant defined in Assumption B.1, it will be useful to analyse the function η/ξ. The next two results concern its differential.

Lemma B.5. *Consider the meromorphic differentials γ_0, γ_∞ defined by*

$$\gamma_0 = \frac{d\xi}{\eta\xi^2} - \frac{(\mathbf{K}-\mathbf{E})}{\mathbf{K}}\frac{d\xi}{\eta}, \quad \gamma_\infty = \frac{\kappa^2\xi^2 d\xi}{\eta} - \frac{(\mathbf{K}-\mathbf{E})}{\mathbf{K}}\frac{d\xi}{\eta} .$$

Then γ_0 is holomorphic in Ω^-, γ_∞ is holomorphic in Ω^+ and we have

$$d(\eta/\xi) = \gamma_\infty - \gamma_0 .$$

Proof: A direct computation shows γ_0 and γ_∞ are holomorphic respectively in Ω^- and Ω^+, as stated. Thus, both $d(\eta/\xi)$ and $\gamma_\infty - \gamma_0$ are holomorphic in $\Sigma \setminus \{\mathbf{0}_i, \infty_i \mid i = 1,2\}$, and one easily checks that they have the same principal part near the points $\mathbf{0}_i$ and ∞_i. It follows that there is a constant c such that $d(\eta/\xi) = \gamma_\infty - \gamma_0 + c\omega$. Integrating over the cycle $\mathbf{a}$, we get

$$c = \int_{\mathbf{a}} \gamma_0 - \gamma_\infty .$$

Hence the result will follow if $\int_{\mathbf{a}} \gamma_0 = \int_{\mathbf{a}} \gamma_\infty = 0$.

Since

$$\begin{aligned}\int_{\mathbf{a}} \frac{\kappa^2\xi^2 d\xi}{\eta} &= 4\int_0^1 \frac{\kappa^2\xi^2 d\xi}{\eta} \\ &= 4\int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-\kappa^2\xi^2)}} - 4\int_0^1 \frac{1-\kappa^2\xi^2}{\sqrt{(1-\xi^2)(1-\kappa^2\xi^2)}}\, d\xi \\ &= 4(\mathbf{K}-\mathbf{E})\end{aligned}$$

we have $\int_{\mathbf{a}} \gamma_\infty = 4(\mathbf{K}-\mathbf{E}) - (\mathbf{K}-\mathbf{E})\mathbf{K}^{-1}\int_{\mathbf{a}} d\xi/\eta = 0$. A similar computation gives $\int_{\mathbf{a}} \gamma_0 = 0$. ∎

Thus $d(\eta/\xi)$ decomposes as a sum of a differential, γ_∞, holomorphic in Ω^+, and a differential, γ_0, holomorphic in Ω^-. This fact will be used to construct the factors $d_t^\pm$. But first we need information about the periods of γ_0 and γ_∞.

Lemma B.6. *The $\mathbf{b}$-period of the differentials γ_0 and γ_∞ is $\pi i/\mathbf{K}$.*

Proof: Represent Σ as a rectangle $\mathcal{P}$ with the sides identified and such that the singularities of γ_∞ lie in the interior of $\mathcal{P}$. Let $\varphi\colon \mathcal{P} \to \mathbb{C}$ be such that $d\varphi = \omega$. Since $\int_{\mathbf{a}} \omega = 1$ and $\int_{\mathbf{b}} \gamma_\infty = 0$, the bilinear relations (see [11]) give

$$\int_{\partial\mathcal{P}} \varphi\gamma_\infty = \int_{\mathbf{b}} \gamma_\infty = 2\pi i \sum_{\mathbf{p}\in\mathcal{P}} \mathrm{Res}_{\mathbf{p}}(\varphi\gamma_\infty) .$$

Near the points ∞_j, $j = 1, 2$, $\zeta = \xi^{-1}$ is a local parameter and we have

$$\gamma_\infty = (-1)^j \frac{\kappa}{\zeta^2}\, d\zeta + \mathbf{O}(1) d\zeta$$
$$\varphi = \text{const.} + \frac{(-1)^j}{4\kappa \mathbf{K}}\, \zeta + \mathbf{O}(\zeta^2)\ .$$

Hence $\mathrm{Res}_{\infty_j}(\varphi\gamma_\infty) = 1/4\mathbf{K}$, from which it follows that $\int_{\mathbf{b}} \gamma_\infty = \pi i/\mathbf{K}$.

The computation of $\int_{\mathbf{b}} \gamma_0$ is similar. ∎

Definition B.7. Let $U = \mathbf{e}/2\mathbf{K}$. Following [5, Chapter 3] we use the differentials γ_0, γ_∞ to define two functions on Σ as follows:

$$d_t^+(\mathbf{p}) = \exp\left(\int_{\mathbf{0}_1'}^{\mathbf{p}} t\mathbf{e}\gamma_\infty\right) \frac{\vartheta_1\left(A(\mathbf{p}) - \tau + tU\right)}{\vartheta_1\left(A(\mathbf{p}) - \tau\right)}$$
$$d_t^-(\mathbf{p}) = \exp\left(-\int_{\mathbf{0}_1'}^{\mathbf{p}} t\mathbf{e}\gamma_0\right) \frac{\vartheta_1\left(A(\mathbf{p}) - \frac{1}{2}U\right)}{\vartheta_1\left(A(\mathbf{p}) - \frac{1-2t}{2}U\right)}$$

where $\mathbf{0}_1' \in \Sigma \setminus \{\mathbf{0}_1\}$ is a point close to $\mathbf{0}_1$, and $A(\mathbf{p})$ denotes the value of the integral $\int_{\mathbf{0}_1}^{\mathbf{p}} \omega$ obtained using the same path of integration as that for computing the integral in the argument of the exponential function. □

It is easy to check that these functions are well defined. Indeed, adding a cycle homotopic to $n\mathbf{a} + m\mathbf{b}$ to the integration path, $g_t^+(\mathbf{p})$ transforms as follows

$$d_t^+(\mathbf{p}) \mapsto e^{\left(2\pi i t U + \int_{\mathbf{0}_1'}^{\mathbf{p}} t\mathbf{e}\gamma_\infty\right)} \frac{e^{-2\pi i\left(\frac{1}{2}m^2\tau + m(A(\mathbf{p}) - \tau + tU)\right)}\vartheta_1\left(A(\mathbf{p}) - \tau + tU\right)}{e^{-2\pi i\left(\frac{1}{2}m^2\tau + m(A(\mathbf{p}) - \tau)\right)}\vartheta_1\left(A(\mathbf{p}) - \tau\right)}$$
$$= d_t^+(\mathbf{p})\ ,$$

where we have used the equality $2\pi i t U = \int_{\mathbf{b}} t\mathbf{e}\gamma_\infty$. Similarly one checks that $d_t^-(\mathbf{p})$ is independent of the integration path.

Remark B.8. Since $\mathbf{K} > \pi/2$ we have $|(1-2t)U/2|$, $|U/2| < \mathbf{e}/4$ for $t \in [0, 1]$. Hence the zeros and poles of d_t^- lie in Ω^+. Similarly, it follows that the zeros and poles of d_t^+ lie in Ω^-. This is the reason for the choice of the radius of $\mathbb{D}$ in Definition B.3. □

Proposition B.9. *For $t \in [0,1]$ we have*

(**a**) $(d_t^+)^{\pm 1} \in H(\Omega^+)$;

(**b**) $(d_t^-)^{\pm} \in H(\Omega^-)$;

(**c**) $\exp\left(\frac{\mu}{\lambda}\, t\right) = d_t^+ r_t d_t^-$ *where r_t is the rational function*

$$r_t = k_t\, \frac{\vartheta_1\left(A(\mathbf{p}) - \frac{1-2t}{2}\, U\right) \vartheta_1\left(A(\mathbf{p}) - \tau\right)}{\vartheta_1\left(A(\mathbf{p}) - \frac{1}{2}\, U\right) \vartheta_1\left(A(\mathbf{p}) - \tau + tU\right)}\ ;$$

where k_t is the value of $\exp(t\mu/\lambda)$ at the point $\mathbf{0}_1'$ (see Definition B.7).

(**d**) *r_t has one exactly zero and one pole in Ω^+ (respectively Ω^-).*

Proof: (**a**) It is clear from the definition that d_t^+ has essential singularities at ∞_1, ∞_2 and is holomorphic on $\Sigma \setminus \{\infty_1, \infty_2\}$, where it has only one zero located at $A^{-1}(\tau + tU)$. Since $A^{-1}(\tau + tU) \in \Omega^-$, (a) follows.

(**b**) Again, it is easy to see from the definition that d_t^+ has essentials singularities at $\mathbf{0}_1, \mathbf{0}_2$ and is meromorphic on $\Sigma \setminus \{\mathbf{0}_1, \mathbf{0}_2\}$, where its divisor is $A^{-1}\left(\frac{1-2t}{2}\, U\right) - A^{-1}\left(\frac{t}{2}\, U\right)$. Since these points lie in Ω^+, (b) follows.

(**c**) Since $d\left(\frac{\mu}{\lambda}\right) = \mathbf{e}(\gamma_\infty - \gamma_0)$, it follows that

$$\int_{\mathbf{0}_1'}^{\mathbf{p}} t\mathbf{e}(\gamma_\infty - \gamma_0) + t\left(\frac{\mu}{\lambda}\right)_{|\mathbf{p}=\mathbf{0}_1'} = t\,\frac{\mu}{\lambda}\ .$$

Hence $\exp\left(\frac{\mu}{\lambda}\, t\right)\left(d_t^+ d_t^-\right)^{-1} = r_t$ where r_t is the rational function given in (c).

(**d**) This is clear from the expression for r_t given in (c). ∎

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