Portugal. Math. (N.S.)
Vol. 65, Fasc. 1, 2008, 1–22

Portugaliae Mathematica

A revision of J.-L. Lions' notion of sentinels

Ousseynou Nakoulima

(Communicated by Enrique Zuazua)

Dedicated to the memory of Jacques-Louis Lions

Abstract. We revisit the notion of sentinel introduced by J.-L. Lions [12] considering the case where the observation of the system and control function of the sentinel have their supports in two different open sets. This point of view leads to problems of null-controllability with or without constraints on the control.

This article focuses on the case of parabolic equations although similar developments can be done for other PDE's.

The main tool used is an observability inequality of Carleman type which is "adapted" to the constraints.

Mathematics Subject Classification (2000). 35K05, 35K15, 35K20, 49J20, 93B05.

Keywords. Heat equation, controllability, Carleman inequalities, sentinels.

1. Statement of the problem

1.1. Problem formulation. For $d \in \mathbb{N}^*$, let Ω be a bounded open subset of $\mathbb{R}^d$ with boundary Γ of class C^2, $T > 0$, and let ω be an open non empty subset of Ω. Set $Q = \Omega \times (0, T)$, $\Sigma = \Gamma \times (0, T)$, $U = \omega \times (0, T)$. We consider the parabolic evolution equation

$$\begin{cases} -q' - \Delta q + a_0 q = h + k\chi_\omega & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(T) = 0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $(\cdot)'$ is the partial derivative with respect to time t, $a_0 \in L^\infty(Q)$, $h \in L^2(Q)$, $k \in L^2(U)$ and χ_ω denotes the characteristic function of ω. It is well known that problem (1) admits a unique solution q in the following Hilbert space (see for instance [11], [13]):

$$H^{2,1}(Q) = \left\{ \varphi \,\middle|\, \varphi, \frac{\partial \varphi}{\partial x_i}, \frac{\partial^2 \varphi}{\partial x_i \partial x_j}, \frac{\partial \varphi}{\partial t} \in L^2(Q) \right\}$$

endowed with the natural norm

$$\|\varphi\|_{H^{2,1}(Q)} = \left\{ \int_Q \left[|\varphi|^2 + \sum_{1 \le i,j \le d} \left| \frac{\partial \varphi}{\partial x_i} \right|^2 + \sum_{1 \le i,j \le d} \left| \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right|^2 + \left| \frac{\partial \varphi}{\partial t} \right|^2 \right] dx\,dt \right\}^{1/2}.$$

This space coincides with the space of φ's (with equality of norms of graphs) such that

$$\varphi \in L^2(0, T; H^2(\Omega)), \qquad \frac{\partial \varphi}{\partial t} \in L^2(Q).$$

Remark 1. System (1) is a backward parabolic problem. It appears under this form in the sentinels theory of J.-L. Lions as the associated adjoint state (cf. [12], p. 22; see also Section 4 below).

We use the notation

$$q = q(x, t; k)$$

to mean that the solution q of (1) depends on the control k which plays a particular role. More precisely, we would like to choose k in order to achieve the following objective: let h be a given function in $L^2(Q)$ and

$$\mathscr{M} \text{ a real closed vector subspace of } L^2(U). \tag{2}$$

Denoting by $\mathscr{M}^\perp$ the orthogonal subspace of $\mathscr{M}$ in $L^2(U)$ we look for a control variable $k \in L^2(U)$ with

$$k \in \mathscr{M}^\perp \tag{3}$$

and such that if $q = q(x, t; k)$ is the unique solution of (1), then

$$q(\cdot, 0; k) = 0 \quad \text{in } \Omega, \tag{4}$$

and

$$\|k\|_{L^2(U)} = \text{minimum} \tag{5}$$

to mean that k is the control of minimal norm in $L^2(U)$.

The role of k is to guarantee the null-controllability property (4) in the presence of the forcing term h and under the restriction (3). The null-controllability problem (1), (3) and (4) is by now well understood in the case $\mathscr{M} = \{0\}$. It has been studied by several authors using different methods. We refer to D. Russell [15], G. Lebeau and L. Robbiano [10], A. Fursikov and O. Imanuvilov [9]. We also refer to V. Barbu [1], A. Doubova et al. [2], C. Fabre et al. [5], E. Fernández-Cara [6], E. Fernández-Cara and S. Guerrero [7], E. Zuazua [16], [18], and the bibliography in these papers for related controllability problems.

This article seems to be the first one dealing with the case $\mathscr{M} \neq \{0\}$. We study the case when $\mathscr{M}$ is of finite dimension. In this case, some compatibility conditions are required for controllability to hold. We shall return to this matter later on.

1.2. The main result. In order to state the main result we introduce a suitable non-negative weight function θ which will be precisely defined below in Section 2 and consider the space

$$L^2_\theta(Q) = \{h \mid h \in L^2(Q), \theta h \in L^2(Q)\}, \tag{6}$$

a Hilbert space for the scalar product and norm

$$(h, \ell)_\theta = \int_Q \theta^2 h\ell \, dx \, dt, \qquad \|h\|_\theta = \|\theta h\|_{L^2(Q)}.$$

We assume that

$$\mathscr{M} \text{ is finite dimensional} \tag{7}$$

and

$$(\forall k \in \mathscr{M}) \qquad (k' - \Delta k + a_0 k = 0 \text{ in } U \Rightarrow k = 0 \text{ in } U). \tag{8}$$

The main result is the following

Theorem 1.1. *Assume that* (7) *and* (8) *hold. Then for any* $h \in L^2_\theta(Q)$ *there exists some control* k *and some state* q *such that* (1), (3) *and* (4) *hold. Moreover, we can get a unique pair* $(\hat{k}_\theta, \hat{q}_\theta)$ *with* $\hat{k}_\theta$ *of minimal norm in* $L^2(U)$, *i.e. such that* (1), (3), (4) *and* (5) *hold.*

The proof of Theorem 1.1 requires several steps which will be carried out in Section 2.

Remark 2. The assumption (8) has been already introduced by J.-L. Lions in [12], p. 33. Here is some case where this assumption is satisfied. As an example, assume that $\mathscr{M}$ be the vector subspace generated in $L^2(U)$ by M independent functions m_i. Consider the case in which each m_i has its support in domains such as $\omega_i \times (0, T)$ with $\omega_i \subset \omega$ and $\omega_i \cap \omega_j = \emptyset$ for $i \neq j$. Assuming that

$m_i' - \Delta m_i + a_0 m_i \neq 0$, then if $k \in \mathscr{M}$ and $k' - \Delta k + a_0 k = 0$ in $\omega \times (0,T)$, we have $k = 0$ in $\omega \times (0,T)$, and assumption (8) is satisfied.

Here is the optimality system satisfied by $(\hat{k}_\theta, \hat{q}_\theta)$. Set

$$P = \text{the orthogonal projection operator from } L^2(U) \text{ onto } \mathscr{M}, \tag{9}$$

and for $\rho \in L^2(Q)$ define

$$P\rho = \text{the orthogonal projection of } \rho\chi_\omega. \tag{10}$$

Theorem 1.2. *The pair* $(\hat{k}_\theta, \hat{q}_\theta)$ *is the optimal solution of problem* (1), (3)–(5) *if and only if there is a function* $\hat{\rho}_\theta$ *such that the triplet* $(\hat{k}_\theta, \hat{q}_\theta, \hat{\rho}_\theta)$ *is the solution of the following optimality system:*

$$\hat{k}_\theta \in \mathscr{M}^\perp, \qquad \hat{q}_\theta \in H^{2,1}(Q), \qquad \hat{\rho}_\theta \in V, \tag{11}$$

$$\begin{cases} -\hat{q}_\theta' - \Delta\hat{q}_\theta + a_0\hat{q}_\theta = h + \hat{k}_\theta\chi_\omega & \text{in } Q, \\ \hat{q}_\theta = 0 & \text{on } \Sigma, \\ \hat{q}_\theta(T) = 0 & \text{in } \Omega, \end{cases} \tag{12}$$

$$\hat{q}_\theta(0) = 0 \qquad \text{in } \Omega; \tag{13}$$

$$\begin{cases} \hat{\rho}_\theta' - \Delta\hat{\rho}_\theta + a_0\hat{\rho}_\theta = 0 & \text{in } Q, \\ \hat{\rho}_\theta = 0 & \text{on } \Sigma, \end{cases} \tag{14}$$

$$\hat{k}_\theta = -(\hat{\rho}_\theta\chi_\omega - P\hat{\rho}_\theta). \tag{15}$$

We proof this theorem in Section 3.

Remark 3. $\hat{\rho}_\theta$ belongs to a Hilbert space V which will be define below in Section 2.

The paper is organized as follows. Section 2 is devoted to show Theorem 1.1. The main tool is a constraint-adapted observability inequality given by Lemma 2.1. In Section 3, we prove Theorem 1.2 using the penalization method. In Section 4, we give an application of the above results to the sentinels theory of J.-L. Lions. More precisely, we propose a notion of sentinel which revisits the one introduced by J.-L. Lions.

2. Null-controllability with constraints on the control

2.1. Preliminaries. It is now well known that the null controllability analysis of parabolic equations is equivalent to the *observability inequality* of the associated adjoint state which is obtained by appropriate Carleman estimates. The main contributions in this area are due to O. Yu. Emanuilov, who developed the use of Carleman estimates in the context of null controllability [4].

In order to state Carleman's inequality, we introduce now some objects and notations. Choose first some auxiliary function $\psi \in C^2(\overline{\Omega})$ which satisfies the following conditions:

$$\psi(x) > 0 \quad \forall x \in \Omega, \qquad \psi(x) = 0 \quad \forall x \in \Gamma, \qquad |\nabla\psi(x)| \neq 0 \quad \forall x \in \overline{\Omega - \omega},$$

Such a function ψ exists according to A. Fursikov and O. Imanuvilov [9].

For any positive parameter λ define then the following weight functions

$$\varphi(x,t) = \frac{e^{\lambda\psi(x)}}{t(T-t)}, \qquad \eta(x,t) = \frac{e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi(x)}}{t(T-t)}.$$

and adopt the following notations

$$\begin{cases} L_0 = \frac{\partial}{\partial t} - \Delta, \\ L = \frac{\partial}{\partial t} - \Delta + a_0 I, \\ \mathscr{V} = \{\rho \in C^\infty(\overline{Q}) \,|\, \rho = 0 \text{ on } \Sigma\} \end{cases} \tag{16}$$

where $a_0 \in L^\infty(Q)$. Now the inequality can be formulated as follows. There exist three constants $\lambda_0 = \lambda_0(\Omega, \omega) > 1$, $s_0 = s_0(\Omega, \omega, T) > 1$ and $C = C(\Omega, \omega) > 0$ such that for any $\lambda \geq \lambda_0$, any $s \geq s_0$ and any $\rho \in \mathscr{V}$ the following inequality holds:

$$\begin{aligned} &\int_Q \frac{e^{-2s\eta}}{s\varphi}(|\rho'|^2 + |\Delta\rho|^2)\,dx\,dt + \int_Q s\lambda^2\varphi e^{-2s\eta}|\nabla\rho|^2\,dx\,dt \\ &\quad + \int_Q s^3\lambda^4\varphi^3 e^{-2s\eta}|\rho|^2\,dx\,dt \\ &\quad \leq C\Big(\int_Q e^{-2s\eta}|L_0|^2\,dx\,dt + \int_U s^3\lambda^4\varphi^3 e^{-2s\eta}|\rho|^2\,dx\,dt\Big). \end{aligned} \tag{17}$$

The above inequality is referred to as the global Carleman inequality (see [9] and [5]). As $L_0 = L - a_0 I$, from the previous inequality (17) we deduce another inequality for the operator L by direct substitution in (17). We conclude the existence of three constants $\lambda_1 = \lambda_1(\Omega, \omega, a_0) > 1$, $s_1 = s_1(\Omega, \omega, T, a_0) > 1$ and $C = C(\Omega, \omega) > 0$ such that for any $\lambda \geq \lambda_1$, any $s \geq s_1$ and any $\rho \in \mathscr{V}$ the following holds:

$$\int_Q \frac{e^{-2s\eta}}{s\varphi}(|\rho'|^2 + |\Delta\rho|^2)\,dx\,dt + \int_Q s\lambda^2\varphi e^{-2s\eta}|\nabla\rho|^2\,dx\,dt$$
$$+ \int_Q s^3\lambda^4\varphi^3 e^{-2s\eta}|\rho|^2\,dx\,dt$$
$$\leq C\Big(\int_Q e^{-2s\eta}|L\rho|^2\,dx\,dt + \int_U s^3\lambda^4\varphi^3 e^{-2s\eta}|\rho|^2\,dx\,dt\Big). \tag{18}$$

Since φ does not vanish, we may set

$$\theta = \frac{e^{s\eta}}{\varphi\sqrt{\varphi}}, \qquad \text{so } \frac{1}{\theta} = \varphi\sqrt{\varphi}e^{-s\eta}.$$

Then $\theta \in C^2(Q)$ and $1/\theta$ is bounded. By substitution in (17) the following inequality holds

$$\int_Q \frac{1}{\theta^2}|\rho|^2\,dx\,dt \leq C\Big(\int_Q \frac{1}{\theta^2\varphi^3 s^3\lambda^4}|L\rho|^2\,dx\,dt + \int_U \frac{1}{\theta^2}|\rho|^2\,dx\,dt\Big).$$

As a consequence of the boundedness of $1/\theta$ and $1/\varphi^3 s^3\lambda^4$, the following inequality holds too:

$$\int_Q \frac{1}{\theta^2}|\rho|^2\,dx\,dt \leq C\Big(\int_Q |L\rho|^2\,dx\,dt + \int_U |\rho|^2\,dx\,dt\Big). \tag{19}$$

All these results are by now well understood. We refer, for instance, to E. Fernández-Cara and E. Zuazua [8].

2.2. Carleman's inequality adapted to linear constraints. We are now concerned with a new observability inequality needed to address the problem that motivates this article. Indeed, for the null controllability problem with constraints, we need another observability inequality with partial measurements. More precisely, in order to deal with the constraint (3) we have to derive a more precise observability inequality adapted to the subspace $\mathscr{M}$ in (2). The following lemma is the key ingredient for our results.

Lemma 2.1. *Assume that* (7) *and* (8) *hold. Then there exists a positive constant* $C = C(\Omega, \omega)$ *such that for any* $\rho \in \mathscr{V}$:

$$\int_Q \frac{1}{\theta^2}|\rho|^2\,dx\,dt \leq C\Big(\int_Q |L\rho|^2\,dx\,dt + \int_U |\rho - P\rho|^2\,dx\,dt\Big). \tag{20}$$

Proof. The proof uses a well known compactness-uniqueness argument. Indeed, suppose that (20) does not hold. Then

$$\begin{cases} \forall n \in N^* \exists \rho_n \in \mathscr{V} \int_Q \frac{1}{\theta^2} |\rho_n|^2 \, dx \, dt = 1, \\ \int_Q |L\rho_n|^2 \, dx \, dt \leq \frac{1}{n} \text{ and } \int_U |\rho_n - P\rho_n|^2 \, dx \, dt \leq \frac{1}{n}. \end{cases} \tag{21}$$

The proof consists in showing that (21) yields a contradiction. We proceed in four steps.

1) We have

$$\int_U \frac{1}{\theta^2} |P\rho_n|^2 \, dx \, dt \leq \int_U \frac{1}{\theta^2} |\rho_n|^2 \, dx \, dt + \int_U \frac{1}{\theta^2} |\rho_n - P\rho_n|^2 \, dx \, dt.$$

Since $1/\theta^2$ is bounded, it follows from (21) that

$$\int_U \frac{1}{\theta^2} |P\rho_n|^2 \, dx \, dt \leq C. \tag{22}$$

Since $P\rho_n \in \mathscr{M}$ and $\mathscr{M}$ is finite dimensional, $P\rho_n$ (and so ρ_n) is bounded in $L^2(U)$.

2) We can extract a subsequence, still denoted $(\rho_n)_n$, such that on the one hand

$$\rho_n \rightharpoonup g \quad \text{weakly in } L^2(U), \tag{23}$$

and on the other hand

$$\rho_n - P\rho_n \to 0 \quad \text{strongly in } L^2(U). \tag{24}$$

Next we deduce from the compactness of P (because $\mathscr{M}$ is of finite dimension) that there exists $\sigma \in \mathscr{M}$ such that

$$P\rho_n \to \sigma \quad \text{strongly in } L^2(U). \tag{25}$$

We deduce from (24) and (25) that $\rho_n \to g = \sigma$ strongly in $L^2(U)$. Due to the continuity of P, we have $P\rho_n \to Pg$ strongly in $L^2(U)$. Therefore, $Pg = g$ and $g \in \mathscr{M}$.

3) In fact, we have $g = 0$. Indeed, from (21), we also have $L\rho_n \to 0$ strongly in $L^2(Q)$. Thus $L\rho_n \to 0$ strongly in $L^2(U)$. We deduce that $L\rho_n \rightharpoonup 0$ weakly in $\mathscr{D}'(U)$ and so $Lg = 0$. The assumption (8) implies that $g = 0$ on U. Finally, $\rho_n \to 0$ strongly in $L^2(U)$.

4) Since $\rho_n \in \mathscr{V}$ it follows from the observability inequality (19) that

$$\int_Q \frac{1}{\theta^2} |\rho_n|^2 \, dx \, dt \leq C\Big(\int_Q |L\rho_n|^2 \, dx \, dt + \int_U |\rho_n|^2 \, dx \, dt\Big).$$

Then, from the conclusions in the third step, we deduce that $\int_Q \frac{1}{\theta^2} |\rho_n|^2 \, dx \, dt \to 0$ when $n \to +\infty$. The contradiction occurs because of the first condition in (21), where $\int_Q \frac{1}{\theta^2} |\rho_n|^2 \, dx \, dt = 1$. The proof of (20) is complete. □

2.3. Proof of Theorem 1.1. Consider now the following symmetric bilinear form

$$a(\rho, \hat{\rho}) = \int_Q L\rho L\hat{\rho} \, dx \, dt + \int_U (\rho - P\rho)(\hat{\rho} - P\hat{\rho}) \, dx \, dt. \tag{26}$$

Due to Lemma 2.1, this bilinear form is a scalar product on $\mathscr{V}$. Let V be the Hilbert space obtained upon taking the closure of $\mathscr{V}$ under the norm:

$$\rho \mapsto \|\rho\|_V = \sqrt{a(\rho, \rho)}. \tag{27}$$

Observe that the norm $\|\cdot\|_V$ is related to the right-hand side of inequality (20). Similarly, the left-hand side of (20) leads to the norm

$$\|\rho\|_\theta = \Big(\int_Q \frac{1}{\theta^2} |\rho|^2 \, dx \, dt \Big)^{1/2}. \tag{28}$$

The completion of $\mathscr{V}$ is the weighted Hilbert space usually denoted by $L^2_{1/\theta}$.

The inequality (20) shows that

$$\|\rho\|_\theta \leq C \|\rho\|_V. \tag{29}$$

This inequality extends to $\rho \in V$. This shows that V is continuously imbedded in $L^2_{1/\theta}$.

Let us now consider $h \in L^2_\theta(Q)$ defined in (6). Then, due to (20) and the Cauchy–Schwarz inequality, we deduce that the linear form defined on V by

$$\rho \to \int_Q h\rho \, dx \, dt$$

is continuous. By the Lax–Milgram theorem, for any $h \in L^2_\theta(Q)$, there exits one and only one solution ρ_θ to the variational problem:

$$\rho_\theta \in V, \qquad \forall \rho \in V: \qquad a(\rho_\theta, \rho) = \int_Q h\rho \, dx \, dt. \tag{30}$$

Proposition 2.1. *Assume that* (7) *and* (8) *hold. Let* ρ_θ *be the unique solution of* (30) *and let* $P\rho_\theta$ *be the projection of* $\rho_\theta \chi_\omega$. *Set*

$$k_\theta = -(\rho_\theta \chi_\omega - P\rho_\theta) \tag{31}$$

and

$$q_\theta = L\rho_\theta. \tag{32}$$

Then the pair (k_θ, q_θ) *is such that* (1), (3) *and* (4) *hold. Moreover, we have*

$$\|\rho_\theta\|_V \leq C\|\theta h\|_{L^2(Q)}, \tag{33}$$

$$\|k_\theta\|_{L^2(U)} \leq C\|\theta h\|_{L^2(Q)}, \tag{34}$$

$$\|q_\theta\|_{H^{2,1}(Q)} \leq C\|\theta h\|_{L^2(Q)}, \tag{35}$$

where C *is a positive constant depending only on* Ω, ω, a_0, T *and* $\mathscr{M}$.

Proof. Since $\rho_\theta \in V$ it follows that $k_\theta = -(\rho_\theta\chi_\omega - P\rho_\theta) \in L^2(U)$ and $q_\theta \in L^2(Q)$. Since $P\rho_\theta \in \mathscr{M}$ we have $k_\theta = -(\rho_\theta\chi_\omega - P\rho_\theta) \in \mathscr{M}^\perp$. By direct substitution in the formulas (26), (30) and (32) it follows that

$$\int_Q q_\theta L\rho\, dx\, dt + \int_U (\rho_\theta - P\rho_\theta)(\rho - P\rho)\, dx\, dt = \int_Q h\rho\, dx\, dt \qquad \text{for all } \rho \in V.$$

Taking into account that $P\rho \in \mathscr{M}$, the above identity reduces to

$$\int_Q q_\theta L\rho\, dx\, dt = \int_Q h\rho\, dx\, dt - \int_U (\rho_\theta - P\rho_\theta)\rho\, dx\, dt \qquad \text{for all } \rho \in V,$$

i.e.,

$$\int_Q q_\theta L\rho\, dx\, dt = \int_Q h\rho\, dx\, dt + \int_U k_\theta\rho\, dx\, dt \qquad \text{for all } \rho \in V. \tag{36}$$

We show now that q_θ is in fact the weak solution by transposition of a backward heat problem. More precisely, if $\phi \in L^2(Q)$, let p be the solution of

$$\begin{cases} p' - \Delta p + a_0 p = \phi & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(0) = 0 & \text{in } \Omega. \end{cases} \tag{37}$$

Then $p \in V$ and so

$$\int_Q q_\theta \phi\, dx\, dt = \int_Q hp\, dx\, dt + \int_U k_\theta p\, dx\, dt. \tag{38}$$

Therefore q_θ is the weak solution by transposition of problem (1) with $k = k_\theta$ (see [13], p. 177). And we know that the solution of this equation is in $H^{2,1}(Q)$.

Therefore $q_\theta \in C([0,T], L^2(\Omega))$. Then multiplying the first equation of (1) by $\rho \in \mathscr{V}$ and integrating by parts over Q, it follows that

$$-\int_\Omega q_\theta(T)\rho(T)\,dx + \int_\Omega q_\theta(0)\rho(0)\,dx + \int_Q q_\theta L\rho\,dx\,dt = \int_Q h\rho\,dx\,dt + \int_U k_\theta \rho\,dx\,dt. \tag{39}$$

for any $\rho \in \mathscr{V}$. Since $\rho \in \mathscr{V}$ we deduce from (36) that

$$\int_\Omega q_\theta(0)\rho(0)\,dx = 0 \qquad \text{for all } \rho \in \mathscr{V}.$$

Therefore $q_\theta(0) = 0$ in Ω. Hence the first statement of Proposition 2.1 is proved.

It remains to prove the estimates (33)–(35). We set $\rho = \rho_\theta$ in (30). It follows from (20) that

$$\begin{aligned} a(\rho_\theta, \rho_\theta) &= \|q_\theta\|^2_{L^2(Q)} + \|k_\theta\|^2_{L^2(U)} \\ &\le \|\theta h\|_{L^2(Q)} \|\rho_\theta\|_\theta \\ &\le C\|\theta h\|_{L^2(Q)} \|\rho_\theta\|_V. \end{aligned} \tag{40}$$

Then from (27) we obtain (33) and thus (34). Finally, (35) is a consequence of (34) and classical properties of the heat equation. □

The adapted observability inequality (20) shows that the choice of the scalar product on $\mathscr{V}$ is not unique. Thus there exist infinitely many control functions k such that (1), (3) and (4) hold.

Consider the set of *control variables* k such that (1), (3) and (4) hold. By Proposition 2.1 this set is nonempty and it is clearly convex and closed in $L^2(U)$. Therefore, there exists a unique *control variable* $\hat{k}_\theta$ of minimal norm in $L^2(U)$. The proof of Theorem 1.1 is complete.

It remains to compute the optimal solution $(\hat{k}_\theta, \hat{q}_\theta)$. This is done in the forthcoming Section 3.

3. Optimality system for the optimal solution

3.1. Penalization. The optimal solution $(\hat{k}_\theta, \hat{q}_\theta)$ can be approximated considering the penalization method by Lions [11]. Let us now describe now the method. Let $\varepsilon > 0$. Define the functional

$$J_\varepsilon(k,q) = \frac{1}{2}\|k\|^2_{L^2(U)} + \frac{1}{2\varepsilon}\|-q' - \Delta q + a_0 q - h - k\chi_\omega\|^2_{L^2(Q)} \tag{41}$$

for any pair (k,q) such that

$$\begin{cases} k \in \mathscr{M}^\perp, \\ -q' - \Delta q + a_0 q \in L^2(Q), \\ q = 0 \text{ on } \Sigma,\ q(0) = q(T) = 0 \text{ in } \Omega. \end{cases} \tag{42}$$

Consider the minimization problem

$$\min J_\varepsilon(k,q), \qquad (k,q) \text{ subject to (42)}. \tag{43}$$

Proposition 3.1. *Under the assumptions of Theorem* 1.1, *the minimization problem has an optimal solution. There exists* $(k_\varepsilon, q_\varepsilon)$ *such that*

$$J_\varepsilon(k_\varepsilon, q_\varepsilon) = \min\{J_\varepsilon(k,q) \,|\, (k,q) \textit{ subject to } (42)\} \tag{44}$$

Proof. Let (k_n, q_n) be a minimizing sequence satisfying (42). The sequence $\big(J_\varepsilon(k_n, q_n)\big)_n$ is bounded from above

$$J_\varepsilon(k_n, q_n) \le C(\varepsilon).$$

Then

$$\|k_n\|_{L^2(U)} \le C(\varepsilon), \qquad \|-q_n' - \Delta q_n + a_0 q_n - h - k_n\chi_\omega\|_{L^2(Q)} \le C(\varepsilon). \tag{45}$$

There is some subsequence of $(k_n)_n$, still denoted by $(k_n)_n$, such that

$$k_n \rightharpoonup k_\varepsilon \qquad \text{weakly in } L^2(U).$$

Since a consequence of (42) the (sub)sequence q_n is bounded,

$$\|q_n\|_{H^{2,1}(Q)} \le C.$$

There is some subsequence of q_n, still denoted by q_n, such that

$$q_n \rightharpoonup q_\varepsilon \qquad \text{weakly in } H^{2,1}(Q).$$

Hence, by weak lower semicontinuity of the functional J_ε,

$$\liminf J_\varepsilon(k_n, q_n) \ge J_\varepsilon(k_\varepsilon, q_\varepsilon).$$

We deduce from the strict convexity of J_ε that $(k_\varepsilon, q_\varepsilon)$ is a unique optimal control. □

Now we study the convergence of $(k_\varepsilon, q_\varepsilon)_\varepsilon$.

Proposition 3.2. *Let* $((k_\varepsilon, q_\varepsilon))_\varepsilon$ *be the sequence of solutions of* (44). *Then for* $\varepsilon \to 0$, *we have the following limits:*

$$\begin{cases} k_\varepsilon \rightharpoonup \hat{k}_\theta & \textit{weakly in } L^2(U), \\ q_\varepsilon \rightharpoonup \hat{q}_\theta & \textit{weakly in } H^{2,1}(Q). \end{cases} \tag{46}$$

Proof. We prove the proposition in three steps:

1) $(\hat{k}_\theta, \hat{q}_\theta)$ satisfies (1), (3) and (4). Then from the structure (41) of $J_\varepsilon(k, q)$ we have of course

$$\begin{cases} \|k_\varepsilon\|_{L^2(U)} \le C, \\ \|-q'_\varepsilon - \Delta q_\varepsilon + a_0 q_\varepsilon - h - k_\varepsilon \chi_\omega\|_{L^2(Q)} \le C\sqrt{\varepsilon}, \end{cases} \tag{47}$$

where the C's are various constants independent of ε.

2) From (47) and the fact that q_ε satisfies (42) we have that

$$\|q_\varepsilon\|_{H^{2,1}(Q)} \le C.$$

There are a subsequence of $(k_\varepsilon, q_\varepsilon)$, again denoted $(k_\varepsilon, q_\varepsilon)$, and two functions $k_0 \in L^2(U)$ and $q_0 \in H^{2,1}(Q)$ such that

$$k_\varepsilon \rightharpoonup k_0 \quad \text{weakly in } L^2(U), k_0 \in \mathscr{K}^\perp; \quad q_\varepsilon \rightharpoonup q_0 \quad \text{weakly in } H^{2,1}(Q). \tag{48}$$

3) Since the injection from $H^{2,1}(Q)$ into $L^2(Q)$ is compact, the pair (k_0, q_0) is such that

$$\begin{cases} -q'_0 - \Delta q_0 + a_0 q_0 = h + k_0 \chi_\omega & \text{in } Q, \\ q_0 = 0 & \text{on } \Sigma, \\ q_0(T) = 0 & \text{in } \Omega, \end{cases} \tag{49}$$

and

$$q_0(0) = 0 \qquad \text{in } \Omega. \tag{50}$$

From the estimate

$$\frac{1}{2}\|k_\varepsilon\|^2_{L^2(U)} \le J_\varepsilon(k_\varepsilon, q_\varepsilon)$$

we get

$$\frac{1}{2}\|k_0\|^2_{L^2(U)} \le \liminf J_\varepsilon(k_\varepsilon, q_\varepsilon).$$

Since the pair $(\hat{k}_\theta, \hat{q}_\theta)$ satisfies (1), (3) and (4) we obtain that

$$\liminf J_\varepsilon(k_\varepsilon, q_\varepsilon) \le \frac{1}{2}\|\hat{k}_\theta\|^2_{L^2(U)}.$$

Thus

$$\|k_0\|_{L^2(U)} \le \|\hat{k}_\theta\|_{L^2(U)}$$

and so

$$\|k_0\|_{L^2(U)} = \|\hat{k}_\theta\|_{L^2(U)}.$$

Hence $k_0 = \hat{k}_\theta$. Since (49) has a unique solution, it follows that $q_0 = \hat{q}_\theta$. □

3.2. Proof of Theorem 1.2.

Proposition 3.3. *The assumptions are as in Theorem* 1.1. *The pair* $(k_\varepsilon, q_\varepsilon)$ *is the optimal solution of problem* (44) *if and only if there is one function* ρ_ε *such that the triplet* $(k_\varepsilon, q_\varepsilon, \rho_\varepsilon)$ *satisfies the following so-called optimality system:*

$$\begin{cases} -q_\varepsilon' - \Delta q_\varepsilon + a_0 q_\varepsilon = h + k_\varepsilon \chi_\omega + \varepsilon\rho_\varepsilon & \text{in } Q, \\ q_\varepsilon = 0 & \text{on } \Sigma, \\ q_\varepsilon(T) = 0 & \text{in } \Omega, \end{cases} \tag{51}$$

$$q_\varepsilon(0) = 0 \qquad \text{in } \Omega \tag{52}$$

$$\begin{cases} \rho_\varepsilon - \Delta\rho_\varepsilon + a_0\rho_\varepsilon = 0 & \text{in } Q, \\ \rho_\varepsilon = 0 & \text{on } \Sigma, \end{cases} \tag{53}$$

$$k_\varepsilon = -(\rho_\varepsilon\chi_\omega - P\rho_\varepsilon). \tag{54}$$

Proof. Express the Euler–Lagrange optimality conditions which characterize $(k_\varepsilon, q_\varepsilon)$:

$$\frac{d}{d\lambda} J_\varepsilon(k_\varepsilon, q_\varepsilon + \lambda\varphi)|_{\lambda=0} = 0$$

for all $\varphi \in C^\infty(\overline{Q})$ such that $\varphi = 0$ on Σ, $\varphi(0) = \varphi(T) = 0$ in Ω, and

$$\frac{d}{d\lambda} J_\varepsilon(k_\varepsilon + \lambda k, q_\varepsilon)|_{\lambda=0} = 0$$

for all $k \in \mathscr{M}^\perp$. After some calculations, we have

$$\int_Q \frac{1}{\varepsilon}(-q_\varepsilon' - \Delta q_\varepsilon + a_0 q_\varepsilon - h - k_\varepsilon\chi_\omega)(-\varphi' - \Delta\varphi + a_0\varphi)\, dx\, dt = 0 \tag{55}$$

for all $\varphi \in C^\infty(\bar{Q})$ such that $\varphi = 0$ on Σ, $\varphi(0) = 0$, $\varphi(T) = 0$ in Ω, and

$$\int_U k_\varepsilon k \, dx \, dt - \int_Q \frac{1}{\varepsilon}(-q_\varepsilon' - \Delta q_\varepsilon + a_0 z_\varepsilon - h - k_\varepsilon \chi_\omega) k \, dx \, dt = 0 \tag{56}$$

for all $k \in \mathscr{M}^\perp$. Define the adjoint state

$$\rho_\varepsilon = -\frac{1}{\varepsilon}(-q_\varepsilon' - \Delta q_\varepsilon + a_0 q_\varepsilon - h - v_\varepsilon \chi_\omega).$$

Then (55) and (56) become respectively

$$\int_Q \rho_\varepsilon(-\varphi' - \Delta\varphi + a_0\varphi) \, dx \, dt = 0 \tag{57}$$

for all $\varphi \in C^\infty(\bar{Q})$ such that $\varphi = 0$ on Σ, $\varphi(0) = \varphi(T) = 0$ in Ω and

$$\int_U k_\varepsilon k \, dx \, dt + \int_Q \rho_\varepsilon k \, dx \, dt = 0 \tag{58}$$

for all $k \in \mathscr{M}^\perp$. Considering the first part (57), we deduce that

$$\rho_\varepsilon' - \Delta\rho_\varepsilon + a_0\rho_\varepsilon = 0$$

in Q. So $\rho_\varepsilon \in L^2(Q)$ with $L\rho_\varepsilon \in L^2(Q)$. Then we can define on the one hand ρ_ε on Γ and on the other hand prove that $\rho_\varepsilon = 0$ on Σ.

Now we consider (58):

$$\int_U (k_\varepsilon + \rho_\varepsilon) k \, dx \, dt = 0$$

for all $k \in \mathscr{M}^\perp$. Hence $k_\varepsilon + \rho_\varepsilon \chi_\omega \in \mathscr{M}$. Since $k_\varepsilon \in \mathscr{M}^\perp$ we have $k_\varepsilon + \rho_\varepsilon \chi_\omega = P(k_\varepsilon + \rho_\varepsilon \chi_\omega) = P\rho_\varepsilon$ and so

$$k_\varepsilon = -(\rho_\varepsilon \chi_\omega - P\rho_\varepsilon).$$

Hence the assertion follows. □

Remark 4. There is no information available concerning $\rho_\varepsilon(0)$ and $\rho_\varepsilon(T)$.

We now look for a priori estimates for the approximate adjoint state ρ_ε. This is the essential point.

From (54) and (47) it follows that

$$\|\rho_\varepsilon \chi_\omega - P\rho_\varepsilon\|_{L^2(U)} \leq C. \tag{59}$$

Now $L\rho_\varepsilon = 0$ yields

$$\|\rho_\varepsilon\|_V \leq C. \tag{60}$$

Therefore there are a subsequence $(\rho_\varepsilon)_\varepsilon$, still denoted by $(\rho_\varepsilon)_\varepsilon$, and $\hat{\rho}_\theta \in V$ such that

$$\rho_\varepsilon \rightharpoonup \hat{\rho}_\theta \quad \text{weakly in } V. \tag{61}$$

Following the lines of the proof of Lemma 2.1, we deduce the weak convergence $\rho_\varepsilon \rightharpoonup \hat{\rho}_\theta$ in $L^2(U)$. Thus

$$P\rho_\varepsilon \to \chi_0 \quad \text{strongly in } L^2(U)$$

so that $\chi_0 \in \mathscr{K}$. By (59) and (61) there is some $\chi_1 \in \mathscr{M}$ with

$$\rho_\varepsilon - P\rho_\varepsilon \rightharpoonup \chi_1 \quad \text{weakly in } L^2(U)$$

so that $\hat{\rho}_\theta = \chi_0 + \chi_1$. Thus $\chi_0 = P\hat{\rho}_\theta$ and

$$\rho_\varepsilon - P\rho_\varepsilon \rightharpoonup \hat{\rho}_\theta - P\hat{\rho}_\theta \quad \text{weakly in } L^2(U).$$

This proves Theorem 1.2.

4. Discriminating sentinels

4.1. Definition. The notion of sentinel was introduced by J.-L. Lions to study systems of incomplete data [12]. The notion permits us to distinguish and to analyse two types of incomplete data: the so-called pollution terms at which we look for information, independently of the other type of incomplete data which is the missing terms and that we do not want to identify.

Typically, the Lions'sentinel is a functional defined on an open set O where we consider three functions: the "observation" y_{obs} corresponding to measurements, a given "mean" function h_0, and a control function w to be determined.

Here we propose a notion of sentinel which revisits the one by Lions.

Let us remind that Lions' sentinel theory [12] relies on the following three features: the state equation y which is governed by a system of PDE, the observation system and some particular evaluation function: the sentinel itself.

More precisely, we consider in the first step the semilinear parabolic equation

$$\begin{cases} y' - \Delta y + f(y) = \xi + \lambda\hat{\xi} & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 + \tau\hat{y}^0 & \text{in } \Omega. \end{cases} \tag{62}$$

We are interested in systems with data that are not completely known. In the present situation $f : \mathbb{R} \to \mathbb{R}$ is a given map, the functions ξ and y^0 are known with ξ in $L^2(Q)$ and y^0 in $L^2(\Omega)$. However, the terms $\lambda\hat{\xi}$ and $\tau\hat{y}^0$ are unknown, but are such that

$$\|\hat{\xi}\|_{L^2(Q)} \leq 1, \qquad \|\hat{y}^0\|_{L^2(\Omega)} \leq 1 \qquad \text{with } \lambda, \tau \in \mathbb{R} \text{ small enough.} \tag{63}$$

In addition to (63), we assume that f verifies

$$f(0) = 0 \tag{64}$$

and the non-linearity of f satisfies the following growth condition:

$$|f(s_1) - f(s_2) - f'(0)(s_1 - s_2)| \leq C(|s_1|^{p-1} + |s_2|^{p-1})|s_1 - s_2|, \tag{65}$$

for all $s_1, s_2 \in \mathbb{R}$ and some $C > 0$ and $p > 1$ such that

$$p \leq (d+4)/d. \tag{66}$$

This growth condition is classical (see for instance [17]). Under this condition, it is proved in [3], p. 63, that there exists $\alpha > 0$ such that when

$$\|\xi + \lambda\hat{\xi}\|_{L^2(Q)} + \|y^0 + \tau\hat{y}^0\|_{L^2(\Omega)} \leq \alpha$$

the problem (62) admits a unique solution in $C\big([0,T], L^2(\Omega)\big)$. For the sake of simplicity, we denote

$$y(x, t; \lambda, \tau) = y(\lambda, \tau) \tag{67}$$

the unique solution of (62). Therefore, the map

$$(\lambda, \tau) \mapsto y(\lambda, \tau) \qquad \text{is in } C^1\big(\mathbb{R} \times \mathbb{R}; C\big([0,T], L^2(\Omega)\big)\big). \tag{68}$$

The general question we want to address is:

given some observation of the state of system, can one obtain $\lambda\hat{\xi}$ without any attempt at computing τy^0? (69)

In this context we refer to $\lambda\hat{\xi}$ as pollution term, which we are trying to identify, and the term τy^0 as the missing one we do not want to identify.

To make things more specific, we consider in the second step the observation process. The observation is the knowledge, along some time period, of some function y_{obs} which is defined on the strip $O \times (0, T)$ over some nonempty open subset $O \subset \Omega$ called observatory. The function y_{obs} is assumed to be of the form

$$y_{\text{obs}} = m_0 + \sum_{i=1}^{M} \beta_i m_i, \tag{70}$$

where the functions $m_0, m_1, \dots, m_M$ are given measurements of y in $L^2\big(O \times (0, T)\big)$, but the real coefficients β_i are unknown. We assume that β_i are *small*. We refer to the terms $\beta_i m_i$ as the *interference terms*. We can assume without loss of generality that

$$\text{the functions } m_i \text{ are linearly independent on } O \times (0, T). \tag{71}$$

Finally, we introduce now the notion of *sentinel*. Let h_0 be a given function on $O \times (0, T)$ such that

$$h_0 \geq 0, \qquad \int_0^T \int_O h_0 \, dx \, dt = 1. \tag{72}$$

Moreover let ω be an open and non empty subset of Ω. For any control function $w \in L^2\big(\omega \times (0, T)\big)$, set

$$S(\lambda, \tau) = \int_0^T \int_O h_0 \, y(\lambda, \tau) \, dx \, dt + \int_0^T \int_\omega w y(\lambda, \tau) \, dx \, dt. \tag{73}$$

The role of the function w appears in the following definition. We say that S defines a discriminating sentinel (for the system (62), (70) and (72)) if there exists w such that the functional S satisfies the following conditions:

(i) S is stationary at first order with respect to the missing terms $\tau \hat{y}^0$, that is,

$$\frac{\partial S}{\partial \tau}(0, 0) = 0 \qquad \text{for all } \hat{y}^0. \tag{74}$$

(ii) S is stationary with respect to the interference terms $\beta_i m_i$, that is,

$$\int_0^T \int_O h_0 m_i \, dx \, dt + \int_0^T \int_\omega w m_i \, dx \, dt = 0, \qquad 1 \leq i \leq M. \tag{75}$$

(iii) The set w is of minimal norm in $L^2\big(\omega \times (0, T)\big)$ among control functions in $L^2\big(\omega \times (0, T)\big)$ which satisfy the above conditions, that is,

$$\|w\|_{L^2(\omega \times (0,T))} = \text{minimum}. \tag{76}$$

Remark 5. At this point some comments are in order.

1. According to (76) the function S, if it exists, is unique. We refer to S as the sentinel.

2. If the functions m_i, $1 \leq i \leq M$, are null functions, the sentinel S is defined only by (74) and (76). If $m_i \neq 0$, the sentinel S is defined by (74), (75) and (76) and it is called a discriminating sentinel.

3. Lions' original sentinel S corresponds to the case $\omega = O$. Here, if we choose $w = -h_0$, then (74) and (75) hold true, so that problem (74)–(76) admits a unique solution. Of course this solution may have an interest only if $w \neq -h_0$. Now if $\omega \neq O$ and if the support $\operatorname{supp}(h)$ of h does not lie in ω, we cannot have $w = -h_0$ except for $w = -h_0 = 0$. Therefore, the previous definition introduces a generalization of Lions's discriminating sentinel to the case where the observation and the control have their supports in two different open subsets.

4. The support $\operatorname{supp}(m_i)$ of functions m_i is assumed to be included in O. Suppose that $\omega \cap O = \emptyset$; then $\int_0^T \int_\omega w m_i \, dx \, dt = 0$. Therefore, it suffices to choose h_0 such that h_0 is orthogonal to each m_i and then (75) is readily satisfied. Therefore, for all ω we can neglect the part of ω which is out of O. So, without loss of generality, it may be assumed that

$$\omega \subset O. \tag{77}$$

4.2. Equivalence to the null-controllability. Here it will be shown that the existence of such a control function w satisfying (74) and (75) is equivalent to the null-controllability property for a system with constrained control. First, we denote by $\bar{y}$ the solution of problem (62) for $\lambda = 0$, $\tau = 0$ and we assume that $\bar{y}$ can be computed in practice. Next, we consider the function y_τ defined by

$$y_\tau = \frac{d}{d\tau} y(\lambda, \tau)|_{\lambda=0,\, \tau=0}. \tag{78}$$

The function y_τ is the solution of the linearized problem

$$\begin{cases} y_\tau' - \Delta y_\tau + f'(\bar{y}) y_\tau = 0 & \text{in } Q, \\ y_\tau = 0 & \text{on } \Sigma, \\ y_\tau(0) = \hat{y}^0 & \text{in } \Omega, \end{cases} \tag{79}$$

where $f'(\bar{y})$ denotes the derivative of f on $\bar{y}$. Due to (65), problem (79) admits a unique solution y_τ.

We now consider the stationary condition (74). It holds if and only if

$$\int_0^T \int_O h_0 y_\tau \, dx \, dt + \int_0^T \int_\omega w y_\tau \, dx \, dt = 0 \qquad \text{for all } \hat{y}^0 \text{ with } \|\hat{y}^0\|_{L^2(\Omega)} \leq 1. \tag{80}$$

In order to transform the equation (80), we introduce now the classical adjoint state. More precisely, consider the solution $q = q(x,t)$ of the linear problem

$$\begin{cases} -q' - \Delta q + f'(\bar{y})q = h_0\chi_O + w\chi_\omega & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(T) = 0 & \text{in } \Omega, \end{cases} \tag{81}$$

where χ_O and χ_ω are the characteristic functions for the open sets O and ω, respectively. As for the problem (79), problem (81) admits a unique solution $q \in H^{2,1}(Q)$. The so-called adjoint state q depends on the unknown w and its usefulness comes from the following observation.

First, multiply both members of the differential equation in (81) by y_τ, and integrate by parts over Q:

$$\int_0^T\int_O h_0 y_\tau\,dx\,dt + \int_0^T\int_\omega w y_\tau\,dx\,dt = \int_\Omega q(0)\hat{y}^0\,dx \qquad \text{for all } \hat{y}^0 \text{ with } \|\hat{y}^0\|_{L^2(\Omega)} \le 1.$$

Thus condition (74) (or (80)) holds if and only if

$$q(0) = 0 \qquad \text{in } \Omega. \tag{82}$$

Then consider the constraints (75). Let $\mathscr{M}$ be the vector subspace generated in $L^2\big(\omega \times (0,T)\big)$ by the M independent functions $m_i\chi_\omega$. There is a unique $k_0 \in \mathscr{M}$ such that

$$\int_0^T\int_O h_0 m_i\,dx\,dt + \int_0^T\int_\omega k_0 m_i\,dx\,dt = 0, \qquad 1 \le i \le M. \tag{83}$$

In other words, condition (75) holds if and only if

$$w - k_0 = k \in \mathscr{M}^\perp. \tag{84}$$

The above considerations show that finding the control w such that the functional S satisfies (74) and (75) is equivalent to finding the control k such that the pair (k,q) satisfies the following system:

$$k \in \mathscr{M}^\perp, \qquad q \in H^{2,1}(Q), \tag{85}$$

$$\begin{cases} -q' - \Delta q + f'(\bar{y})q = h_0\chi_O + k_0\chi_\omega + k\chi_\omega & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(T) = 0 & \text{in } \Omega, \end{cases} \tag{86}$$

$$q(0) = 0 \qquad \text{in } \Omega. \tag{87}$$

We see that (85)–(87) is exactly the problem (1), (3) and (4) when $a_0 = f'(\bar{y})$ and $h = h_0\chi_O + k_0\chi_\omega$. So we can apply the results obtained in the previous sections. Indeed, observe first that $\mathscr{M}$ is finite dimensional. Now assume that (8) holds and $h = h_0\chi_O + k_0\chi_\omega$ in $L^2_\theta(Q)$. We assume also that (77) holds. Therefore, let $(\hat{\rho}_\theta, \hat{k}_\theta, \hat{q}_\theta)$ be defined as in Theorem 1.2. Then $\hat{k}_\theta = -(\hat{\rho}_\theta\chi_\omega - P\hat{\rho}_\theta)$, and the sentinel is defined by

$$\hat{S}_\theta(\lambda,\tau) = \int_0^T\int_O h_0\, y(\lambda,\tau)\,dx\,dt + \int_0^T\int_\omega \big(k_0 - (\hat{\rho}_\theta - P\hat{\rho}_\theta)\big)\, y(\lambda,\tau)\,dx\,dt. \tag{88}$$

4.3. Detection of pollution. Let us now return to problem (69) to show how the sentinel defined above permits to detect the pollution $\lambda\hat{\xi}$. Let $\bar{y}$ be again the solution of the problem (62) for $\lambda = 0$, $\tau = 0$. We assume that $\bar{y}$ can be computed in practice. Then, knowing $\hat{k}_\theta$,

$$\hat{S}_\theta(0,0) = \int_0^T\int_O h_0\,\bar{y}\,dx\,dt + \int_0^T\int_\omega (k_0 + \hat{k}_\theta)\,\bar{y}\,dx\,dt \tag{89}$$

is known. Because of (65) we have

$$\hat{S}_\theta(\lambda,\tau) \approx \hat{S}_\theta(0,0) + \lambda\frac{\partial\hat{S}}{\partial\lambda}(0,0) \qquad \text{for } \lambda \text{ and } \tau \text{ small.} \tag{90}$$

If y_{obs} is known, it follows from (70) and (75) that

$$\hat{S}_\theta(\lambda,\tau) = \int_0^T\int_O h_0 m_0\,dx\,dt + \int_0^T\int_\omega (k_0 + \hat{k}_\theta) m_0\,dx\,dt. \tag{91}$$

Thus

$$\lambda\frac{\partial\hat{S}}{\partial\lambda}(0,0) \approx \int_0^T\int_O h_0(m_0 - \bar{y})\,dx\,dt + \int_0^T\int_\omega (k_0 + \hat{k}_\theta)(m_0 - \bar{y})\,dx\,dt. \tag{92}$$

Now

$$\frac{\partial\hat{S}_\theta}{\partial\lambda}(0,0) = \int_0^T\int_O h_0\, y_\lambda\,dx\,dt + \int_0^T\int_\omega (k_0 + \hat{k}_\theta)\, y_\lambda\,dx\,dt \tag{93}$$

where y_λ is given by

$$\begin{cases} y'_\lambda - \Delta y_\lambda + f'(\bar{y})y_\lambda = \hat{\xi} & \text{in } Q, \\ y_\lambda = 0 & \text{on } \Sigma, \\ y_\lambda(0) = 0 & \text{in } \Omega. \end{cases} \tag{94}$$

Let q_θ be the solution of (86). Multiplying (86) by y_λ it follows that

$$\int_0 q_\theta \hat{\xi} = \int_0^T \int_O h_0 y_\lambda \, dx \, dt + \int_0^T \int_\omega (k_0 + \hat{k}_\theta) y_\lambda \, dx \, dt \, dx \, dt. \tag{95}$$

Therefore (92) becomes

$$\int_0^T \int_O q_\theta \lambda \hat{\xi} \, dx \, dt \approx \int_0^T \int_O h_0 (m_0 - \bar{y}) \, dx \, dt + \int_0^T \int_\omega (k_0 + \hat{k}_\theta)(m_0 - \bar{y}) \, dx \, dt. \tag{96}$$

This equation allows to evaluate the left-hand side of (96) and is thus an available information concerning $\lambda \hat{\xi}$.

Remark 6. Of course, we can study and apply other notions developed by J.-L. Lions in [12] (for instance, the notion of furtivity) to the sentinel defined in (88).

References

[1] V. Barbu, Exact controllability of the superlinear heat equation. *Appl. Math. Optim.* **42** (2000), 73–89. Zbl 0964.93046 MR 1751309

[2] A. Doubova, A. Osses, and J.-P. Puel, Exact controllability to trajectories for semilinear heat equations with discontinuous diffusion coefficients. *ESAIM Control Optim. Calc. Var.* **8** (2002), 621–661. Zbl 1092.93006 MR 1932966

[3] Th. Cazenave and A. Haraux, *Introduction aux problèmes d'evolution semi-linéaires*. Mathématiques & Applications 1, Éditions Ellipses, Paris 1990. Zbl 0786.35070 MR 1299976

[4] O. Y. Emanuilov, Controllability of parabolic equations. *Mat. Sb.* **186** (1995), 109–132; English transl. *Russian Acad. Sci. Sb. Math.* **186** (1995), 879–900. Zbl 0845.35040 MR 1349016

[5] C. Fabre, J.-P. Puel, and E. Zuazua, Approximate controllability of the semilinear heat equation. *Proc. Roy. Soc. Edinburgh Sect. A* **125** (1995), 31–61. Zbl 0818.93032 MR 1318622

[6] E. Fernández-Cara, Null controllability of the semilinear heat equation. *ESAIM Control Optim. Calc. Var.* **2** (1997), 87–103. Zbl 0897.93011 MR 1445385

[7] E. Fernández-Cara and S. Guerrero, Global Carleman inequalities for parabolic systems and applications to controllability. *SIAM J. Control Optim.* **45** (2006), 1395–1446. Zbl 05174453 MR 2257228

[8] E. Fernández-Cara and E. Zuazua, The cost of approximate controllability for heat equations: the linear case. *Adv. Differential Equations* **5** (2000), 465–514. Zbl 1007.93034 MR 1750109

[9] A. V. Fursikov and O. Y. Imanuvilov, *Controllability of evolution equations*. Lecture Notes Series 34, Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul 1996. Zbl 0862.49004 MR 1406566

[10] G. Lebeau and L. Robbiano, Contrôle exact de l'équation de la chaleur. *Comm. Partial Differential Equations* **20** (1995), 335–356. Zbl 0819.35071 MR 1312710

[11] J.-L. Lions, *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*. Dunod, Gauthier-Villars, Paris 1968. Zbl 0179.41801 MR 0244606

[12] J.-L. Lions, *Sentinelles pour les systèmes distribués à données incomplètes*. Recherches en Mathématiques Appliquées 21, Masson, Paris 1992. Zbl 0759.93043 MR 1159093

[13] J.-L. Lions and M. Magenes, *Problèmes aux limites non homogènes et applications*. Dunod, Vol. 1 et 2, Paris 1968. Zbl 0165.10801 MR 0247243 MR 0247244

[14] O. Nakoulima, Contrôlabilité à zéro avec contraintes sur le contrôle. *C. R. Math. Acad. Sci. Paris Ser. I* **339** (2004), 405–410. Zbl 1060.93015 MR 2092753

[15] D. L. Russell, A unified boundary controllability theory for hyperbolic and parabolic partial differential equations. *Studies in Appl. Math.* **52** (1973), 189–211. Zbl 0274.35041 MR 0341256

[16] E. Zuazua, Exact boundary controllability for the semilinear wave equation. In *Nonlinear partial differential equations and their applications (Paris, 1987–1988)*. Pitman Res. Notes Math. Ser. 220, Longman, Harlow 1991, 357–391. Zbl 0731.93011 MR 1131832

[17] E. Zuazua, Finite-dimensional null controllability for the semilinear heat equation. *J. Math. Pures Appl.* (9) **76** (1997), 237–264. Zbl 0872.93014 MR 1441986

[18] E. Zuazua, Controllability of partial differential equations and its semi-discrete approximations. *Discrete Contin. Dyn. Syst.* **8** (2002), 469–513. Zbl 1005.35019 MR 1897693

Received May 18, 2005

Ousseynou Nakoulima, Université Antilles-Guyane, Département de Mathématiques et Informatique, 97159 Pointe-à-Pitre, Guadeloupe, France
E-mail: onakouli@univ-ag.fr