Portugal. Math. (N.S.)
Vol. 65, Fasc. 1, 2008, 23–32

Portugaliae Mathematica

A generalisation of Cartwright's Theorem: nonautonomous differential equations case

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(Communicated by Carlos Rocha)

Abstract. In this article we show that the almost-periodic solutions of a large class of non-autonomous delay differential equations are quasi-periodic. This result is a generalisation of a theorem proved by Cartwright for ordinary differential equations.

Mathematics Subject Classification (2000). 34K15.

Keywords. Differential equations, amenable solutions, almost-periodic solutions, quasi-periodic solutions.

1. Introduction

Consider the two differential equations

$$\dot{x}(t) = f\big(x(t)\big) \tag{1.1}$$

and

$$\begin{gathered}\dot{x}(t) = f\big(t, x(t)\big),\\ f\big(t + 2\pi, x(t)\big) = f\big(t, x(t)\big)\end{gathered} \tag{1.2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ (resp. $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$) satisfies the conditions for existence and uniqueness of solutions. M. L. Cartwright [4] proved that the almost-periodic solutions of (1.1) or (1.2) defined on $\mathbb{R}$, when there exist, are quasi-periodic. By another method J. Blot [3] has proved the same result for equation (1.1). J. Mallet-Paret [6] has extended this result to the delayed differential equations with discrete delay of the following form

$$\dot{x}(t) = f\big(x(t), x(t-\tau_1), \ldots, x(t-\tau_N)\big) \tag{1.3}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^{n(N+1)} \to \mathbb{R}^n$ is C^1 and bounded on $\mathbb{R}^{n(N+1)}$, and $\tau_j \in]0,1]$ are constants.

By using a method of reduction due to R. Smith, O. Arino and the author [2] generalized this theorem of Cartwright to a large class of delay differential equations with continuous delay. They are the equations written in the form

$$\dot{x}(t) = Ax_t + B\Phi(Cx_t) \tag{1.4}$$

where B is a constant matrix of type $n \times r$, $A : \mathscr{C} \to \mathbb{R}^n$ and $C : \mathscr{C} \to \mathbb{R}^s$ are bounded linear mappings, and the function $\Phi : C\mathscr{S} \to \mathbb{R}^r$, $\mathscr{S} \subset \mathscr{C}$ an open set, is continuous and satisfies a certain Lipschitz condition. $\mathscr{C}$ is the Banach space of the continuous functions $\varphi[-h, 0] \to \mathbb{R}^n$ and h is a positive constant. In this article we generalize the theorem of Cartwright to nonautonomous differential equations with continuous delay.

Our consideration will be based on the written equations in the feedback control form. In the proofs we use the mapping Π (Smith's projection) that we define in Section 2 and for which we recall some properties proved by R. Smith [7], [8]. In Section 3 results about almost-periodic functions will be recalled and some others will be proved. In Section 4 we state and prove our main result.

2. Summary of Smith's reduction method

Suppose that $0 \le h < \infty$ and let $\mathscr{C}$ be the Banach space of continuous functions $\varphi[-h, 0] \to \mathbb{R}^n$, with $|\varphi| = \sup|\varphi(\theta)|$, $-h \le \theta \le 0$. (Here $|\varphi(\theta)|$ denotes the euclidean norm of $\varphi(\theta)$ in $\mathbb{R}^n$.)

Consider the delayed functional differential equation

$$\dot{x}(t) = f(t, x_t) \tag{2.1}$$

where x_t stands for the function $x(t + \theta)$ and $f : \mathbb{R} \times \mathscr{S} \to \mathbb{R}^n$ is a continuous function satisfying a Lipschitz condition on the open set $\mathscr{S} \subset \mathscr{C}$, that is,

$$\exists k > 0 \ \forall t \in \mathbb{R} \ \forall \varphi_1, \varphi_2 \in \mathscr{S} : |f(t, \varphi_1) - f(t, \varphi_2)| \le k|\varphi_1 - \varphi_2|. \tag{2.2}$$

Throughout this paper we consider a class of retarded functional differential equations expressed in the feedback control form

$$\dot{x}(t) = Ax_t + B\Phi(t, Cx_t) \tag{2.3}$$

where B is a constant $n \times r$ matrix, $A : \mathscr{C} \to \mathbb{R}^n$ and $C : \mathscr{C} \to \mathbb{R}^s$ are bounded linear mappings and the function $\Phi : \mathbb{R} \times C\mathscr{S} \to \mathbb{R}^r$ is continuous and satisfies the Lipschitz condition

$$|\Phi(t, y_1) - \Phi(t, y_2)| \le \Lambda(C\mathscr{S})|y_1 - y_2| \qquad \text{for } t \in \mathbb{R} \text{ and } y_1, y_2 \in C\mathscr{S} \quad (2.4)$$

(since $\mathscr{S} \subset \mathscr{C}$ we have $C\mathscr{S} \subset \mathbb{R}^s$).

We suppose moreover that there exists $T > 0$ such that

$$\Phi(t + T, y) = \Phi(t, y) \qquad \text{for all } (t, y) \in \mathbb{R} \times C\mathscr{S}. \quad (2.5)$$

Equation (2.3) satisfies (2.2) with $k = |A| + |B|\Lambda(C\mathscr{S})|C|$.

The bounded linear mappings A and C can be expressed as (see [7])

$$A\varphi = \int_{-h}^{0} [d\alpha(\theta)]\varphi(\theta), \qquad C\varphi = \int_{-h}^{0} [d\gamma(\theta)]\varphi(\theta), \quad (2.6)$$

where $\alpha(\theta)$ and $\gamma(\theta)$ are matrices of types $n \times n$ and $s \times n$, respectively, whose elements are functions of bounded variation on the interval $-h \le \theta \le 0$. For $z \in \mathbb{C}$ we then define the functions

$$a(z) = \int_{-h}^{0} e^{z\theta}\, d\alpha(\theta), \qquad c(z) = \int_{-h}^{0} e^{z\theta}\, d\gamma(\theta). \quad (2.7)$$

These functions are analytic in $\mathbb{C}$ (see [7]) and the equation

$$\det[zI - a(z)] = 0 \quad (2.8)$$

is called the *characteristic equation* of A. It has only a finite number of roots in the half-plane $\operatorname{Re} z \ge \delta$ for each real δ (see [5], p. 181).

Throughout this paper λ denotes a positive constant which satisfies the following hypothesis:

(H_1) Equation (2.8) has no root z with $\operatorname{Re} z = -\lambda$ and has exactly j roots such that $\operatorname{Re} z > -\lambda$, where j is a positive integer.

Here roots are counted according to their multiplicity.

The matrix

$$\chi(z) = c(z)[zI - a(z)]^{-1}B \quad (2.9)$$

is called the *transfer matrix* of (2.3); it is of the type $s \times r$. When (H_1) holds, we define

$$\mu(\lambda) = \sup_{\omega \in \mathbb{R}} |\chi(-\lambda - i\omega)|. \quad (2.10)$$

Here $|K|$ denotes the spectral norm of the rectangular matrix K ($|K|^2$ is the largest eigenvalue of the symmetric matrix K^*K where K^* is the adjoint matrix of K).

From the bounded linear mapping $A : \mathscr{C} \to \mathbb{R}^n$ we derive a linear mapping $\Pi : \mathscr{C} \to \mathbb{R}^j$ where j is the integer in (H_1). If $j > 0$ then the roots $\zeta_1, \dots, \zeta_j$ of (2.8) in the half-plane $\mathrm{Re}\, z > -\lambda$ give rise to a j-dimensional subspace $\mathscr{P}$ of $\mathscr{C}$, which has a basis $\phi_1, \phi_2, \dots, \phi_j$ consisting of certain generalized eigenfunctions associated with these roots (see [5], [7]). The space $\mathscr{P}$ has a complementary subspace $\mathscr{Q}$ in $\mathscr{C}$ such that $\mathscr{C} = \mathscr{P} \oplus \mathscr{Q}$, and so each element ϕ in $\mathscr{C}$ can be expressed uniquely as

$$\phi = r_1\phi_1 + r_2\phi_2 + \cdots + r_j\phi_j + \phi_q \tag{2.11}$$

where $r_1, r_2, \dots, r_j$ are real constants and $\phi_q \in \mathscr{Q}$.

Then

$$\Pi\phi = \mathrm{col}(r_1, r_2, \dots, r_j) \tag{2.12}$$

defines a linear mapping $\Pi : \mathscr{C} \to \mathbb{R}^j$. For $v = 1, 2, \dots, j$ the numbers r_v are given by

$$r_v = \psi_v(0)\phi(0) - \int_{-h}^{0}\int_{0}^{\theta} \psi_v(\xi - \theta)[d\alpha(\theta)]\phi(\xi)\, d\xi \tag{2.13}$$

where $\alpha(\theta)$ is the $n \times n$ matrix in (2.6) and the continuous row vectors $\psi_1(\theta)$, $\psi_2(\theta), \dots, \psi_j(\theta)$ are certain generalized eigenfunctions of the formal adjoint of $(\dot{x}(t) = Ax_t)$ corresponding to the roots $\zeta_1, \dots, \zeta_j$ (see [7]). It follows from (2.12) and (2.13) that there exists a constant k_1 such that

$$|\Pi\phi| \le k_1|\phi| \qquad \text{for all } \phi \in \mathscr{C}. \tag{2.14}$$

Next we discuss some properties of Π. We assume that $\Lambda(C\mathscr{S}) < \mu(\lambda)^{-1}$ and all symbols $k_1, k_2, \dots$ denote constants that depend only on A, B, C, λ, $\Lambda(C\mathscr{S})$.

Definition 1. A solution x of (2.3) is said to be amenable if $x_t \in \mathscr{S}$ for $-\infty < t \le \tau$ and $\int_{-\infty}^{\tau} e^{2\lambda t}|x(t)|^2\, dt$ converges.

In particular x_t (also called solution) in $\mathscr{S}$ is amenable if it is bounded in $]-\infty, \tau]$ because $\lambda > 0$; thus, every periodic solution x_t in $\mathscr{S}$ is amenable.

Lemma 1. *If x and y are amenable solutions of* (2.3) *in* $]-\infty, \tau]$, *then*

$$e^{\lambda\sigma}|x(\sigma) - y(\sigma)| \to 0 \qquad \text{as } \sigma \to -\infty, \tag{2.15}$$

$$\int_{-\infty}^{\tau} e^{2\lambda t}|x(t) - y(t)|^2\, dt \le k_3^2 e^{2\lambda\tau}|\Pi(x_t - y_t)|^2, \tag{2.16}$$

$$k_4|x_\tau - y_\tau| \le |\Pi(x_\tau - y_\tau)| \le k_2|x_\tau - y_\tau|. \tag{2.17}$$

In particular (2.16) *shows that if* $\Pi x_\tau = \Pi y_\tau$ *then* $x(t) = y(t)$ *for* $-\infty < t \le \tau$.

For any real τ let us denote by $\mathscr{A}_\tau$ the subset of the points x_τ taken along all the solutions of (2.3) that are amenable on $]-\infty, \tau]$. The set $\mathscr{A}_\tau$ is called the *amenable set* of (2.3) in $\mathscr{S}$. From the periodicity hypothesis (2.5) it follows that the solution $x(t)$ is an amenable solution of (2.3) over $]-\infty, \tau]$ if and only if $x(t+T)$ is amenable over $]-\infty, \tau + T]$. It follows that $\mathscr{A}_{\tau+T} = \mathscr{A}_\tau$ for any real τ.

If $p, q \in \mathscr{A}_\tau$, then $p = x_\tau$, $q = y_\tau$ for some amenable solutions x_t, y_t which lie in $\mathscr{S}$ throughout $]-\infty, \tau]$. Hence (2.17) gives

$$k_4|p-q| \le |\Pi p - \Pi q| \le k_2|p-q| \qquad \text{for } p, q \in \mathscr{A}_\tau. \tag{2.18}$$

The restricted mapping $\Pi : \mathscr{A}_\tau \to \Pi\mathscr{A}_\tau$ is therefore bijective. If its inverse mapping is $\Psi : \Pi\mathscr{A}_\tau \to \mathscr{A}_\tau$, then

$$(k_2)^{-1}|\zeta - \xi| \le |\Psi(\zeta) - \Psi(\xi)| \le (k_4)^{-1}|\zeta - \xi| \qquad \text{for } \zeta, \xi \in \Pi\mathscr{A}, \tag{2.19}$$

that is, $\mathscr{A}_\tau$ is homeomorphic to the set $\Pi\mathscr{A}_\tau$.

If $\zeta \in \Pi\mathscr{A}_\tau$ then $\Psi(\zeta) = x_\tau$ for a unique amenable solution x_t which lies in $\mathscr{S}$ throughout $]-\infty, \tau]$. By defining $g(\zeta) = \Pi\dot{x}_\tau$ we obtain a function $g : \Pi\mathscr{A}_\tau \to \mathbb{R}^j$. Since $\Pi\dot{x}_t$ is the derivative of Πx_t, the function Πx_t is a solution of the j-dimensional equation

$$\frac{d\zeta}{dt} = g(\zeta) \tag{2.20}$$

for every amenable solution x_t of (2.3).

The function $g(\zeta)$ satisfies a Lipschitz condition on $\Pi\mathscr{A}_\tau$ (see [8]):

$$|g(\zeta) - g(\xi)| \le k_8|\zeta - \xi| \qquad \text{for all } \zeta, \xi \in \Pi\mathscr{A}_\tau. \tag{2.21}$$

Also we have the following lemma (see [8], p. 221).

Lemma 2. *If $\mathscr{B} \subset \mathbb{R}^v$ and $\mu : \mathscr{B} \to \mathbb{R}$ satisfies*

$$|\mu(x) - \mu(y)| \le k|x-y| \qquad \textit{for all } x, y \in \mathscr{B} \tag{2.22}$$

then there exists $\hat{\mu} : \mathbb{R}^v \to \mathbb{R}$ which satisfies (2.22) *for all x, y in $\mathbb{R}^v$ and $\hat{\mu}(b) = \mu(b)$ for all b in $\mathscr{B}$.*

Now put $g(\zeta) = \big(g_1(\zeta), g_2(\zeta), \ldots, g_j(\zeta)\big)$. Then the functions $g_1, g_2, \ldots, g_j$ satisfy the Lipschitz condition (2.21) in $\Pi\mathscr{A}_\tau$ and Lemma 2 gives functions $\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_j$ which satisfy the same Lipschitz condition throughout $\mathbb{R}^j$ and coincide with $g_1, g_2, \ldots, g_j$ on $\Pi\mathscr{A}_\tau$. If we put

$$\hat{g}(\zeta) = \big(\hat{g}_1(\zeta), \hat{g}_2(\zeta), \dots, \hat{g}_j(\zeta)\big) \tag{2.23}$$

then the differential equation

$$\frac{d\zeta}{dt} = \hat{g}(\zeta) \tag{2.24}$$

is an extension of (2.20), and $\hat{g}(\zeta)$ is Lipschitz on the whole space $\mathbb{R}^j$. It follows that if x is an amenable solution of (2.3) in $]-\infty, \tau]$, then Πx_t is the only solution ζ of (2.20) such that $\zeta(\tau) = \Pi x_\tau$. That is,

$$\zeta(t) = \Pi x_t, \qquad x_t = \Psi\big(\zeta(t)\big) \tag{2.25}$$

provides a one-to-one correspondence between the amenable solutions x of (2.3) and the solutions ζ of (2.20) in $\Pi\mathscr{A}_\tau$.

3. Back to the almost-periodic and quasi-periodic functions

Let X be a Banach space. For $x \in X$ we denote by $|x|$ the norm of x. Let $\mathbb{R}$ be the set of real numbers and f a function defined on $\mathbb{R}$ and with values in X. We write $\mathscr{R}f := \{x = f(t) \mid t \in \mathbb{R}\}$.

A set $E \subset \mathbb{R}$ is said to be relatively dense if there exists a number $l > 0$ (inclusion length) such that every interval $[a, a + l]$, $a \in \mathbb{R}$, contains at least one point of E.

A continuous function $f : \mathbb{R} \to X$ is said to be almost-periodic if to every $\varepsilon > 0$ there corresponds a relatively dense set $\{\sigma\}_\varepsilon$ such that

$$\sup_{\mathbb{R}} |f(t + \sigma) - f(t)| \le \varepsilon \qquad \text{for all } \sigma \in \{\sigma\}_\varepsilon. \tag{3.1}$$

Each element $\sigma \in \{\sigma\}_\varepsilon$ is called an ε-almost-period of f. Thus to the set $\{\sigma\}_\varepsilon$ there corresponds an inclusion length l_ε.

Let us now indicate some properties of almost-periodic functions (see [1], [2]).

The set of almost-periodic functions is closed with respect to the topology of uniform convergence.

Theorem 1. *Let X and Y be two Banach spaces, $f : \mathbb{R} \to X$ an almost-periodic function and $g : X \to Y$ a continuous function on $\overline{\mathscr{R}x}$. Then $g \circ f$ is an almost-periodic function.*

Proof (see [1]). First observe that $g \circ f$ is continuous. Moreover, g is uniformly continuous on the compact set $\overline{\mathscr{R}f} = G$. Hence

$$\forall \varepsilon > 0,\ \exists \delta_\varepsilon > 0,\ \forall x', x'' \in G, \|x'' - x'\| \le \delta_\varepsilon \Rightarrow \|g(x'') - g(x')\| \le \varepsilon.$$

Now let σ be a δ_ε-almost-period of f. Then $\|f(t+\sigma) - f(t)\| \le \delta_\varepsilon$ for all t and consequently (setting $x'' = f(t+\sigma)$, $x' = f(t)$)

$$\|g(f(t+\sigma)) - g(f(t))\| \le \varepsilon.$$

Thus σ is an ε-almost-period for $g \circ f$. □

Observe that the function $ae^{i\beta t}$ is periodic for all $a \in X$ and $\beta \in \mathbb{R}$. It follows that any trigonometric polynomial

$$P(t) = \sum_{k=1}^{n} a_k e^{i\beta_k t}, \qquad a_k \in X,\ \beta_k \in \mathbb{R}, \tag{3.2}$$

is almost-periodic and hence any function f which is the limit, with respect to the uniform convergence on $\mathbb{R}$, of a trigonometric polynomial sequence is almost-periodic.

If a function $f : \mathbb{R} \to X$ is almost-periodic, then for each $\varepsilon > 0$ there exists a trigonometric polynomial

$$P_\varepsilon(t) = \sum_{k=1}^{n} b_k e^{i\beta_k t} \tag{3.3}$$

such that

$$\sup_{\mathbb{R}} \|f(t) - P_\varepsilon(t)\| \le \varepsilon. \tag{3.4}$$

Any almost-periodic function $x = f(t)$ possesses a mean value

$$M(x) = M(f(t)) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\, dt. \tag{3.5}$$

The almost-periodic function defined from $\mathbb{R}$ to the Banach space X can be represented by summable families of exponential complex with Fourier–Bohr vectorial coefficients

$$a(\beta, f(t)) := M(f(t)e^{-i\beta t}) \in X \tag{3.6}$$

where $\beta \in \mathbb{R}$ and $f : \mathbb{R} \to X$ is almost-periodic. We write

$$f(t) \sim \sum_{\beta \in \mathbb{R}} a(\beta, f(t)) e^{i\beta t}. \tag{3.7}$$

The Parseval equality holds:

$$M\left(|f(t)|^2\right) = \sum_{\beta \in \mathbb{R}} \left|a\left(\beta, f(t)\right)\right|^2. \tag{3.8}$$

For an almost-periodic function $x = f(t)$ defined on $\mathbb{R}$ with values in a Banach space X, we put $\Gamma(x) := \{\beta \in \mathbb{R} \mid a(\beta, f(t)) \neq 0\}$. The Parseval equality which ensures the summability of the family $\left(\left|a\left(\beta, f(t)\right)\right|^2\right)_{\beta \in \mathbb{R}}$ implies that $\Gamma(x)$ is at most countable. We call

$$\mathrm{Mod}(x) := \left\{ \sum_{\nu=1}^{n} k_\nu \beta_\nu \mid n \in \mathbb{N}, k_\nu \in \mathbb{Z}, \beta_\nu \in \Gamma(x) \right\}$$

the frequencies module of x; it is the $\mathbb{Z}$-module (or the abelian group) generated by $\Gamma(x)$. When $\mathrm{Mod}(x)$ is a free module of a finite type, we say that x is quasi-periodic.

Proposition 1 ([2]). *If g is an almost-periodic function defined on $\mathbb{R}$ with values in $\mathbb{R}^d$ and $f \in \mathscr{C}^0(\overline{\mathscr{R}x}; \mathbb{R}^m)$, then $f \circ g$ is an almost-periodic function defined on $\mathbb{R}$ with values in $\mathbb{R}^m$ and* $\mathrm{Mod}(f \circ g) \subset \mathrm{Mod}(g)$.

Proposition 2. *If g is an almost-periodic function defined from $\mathbb{R}$ to $\mathbb{R}^n$ and $f \in \mathscr{C}^0(\overline{\mathscr{R}x}, X)$ where X is a Banach space of any dimension, then $f \circ g$ is an almost-periodic function defined from $\mathbb{R}$ to X and* $\mathrm{Mod}(f \circ g) \subset \mathrm{Mod}(g)$.

Proof. The function $f \circ g$ is an almost-periodic function defined from $\mathbb{R}$ to X (see Theorem 1).

It remains to show that $\mathrm{Mod}(f \circ g) \subset \mathrm{Mod}(g)$. Let L be any element of X^* (where X^* designates the dual of X). By replacing f with $L \circ f$ in Proposition 1, we obtain that $L \circ f \circ g$ is an almost-periodic numerical function and $\mathrm{Mod}(L \circ f \circ g) \subset \mathrm{Mod}(g)$,

$$\frac{1}{2T} \int_{-T}^{T} (L \circ f \circ g)(t) e^{-i\beta t}\, dt = L\left(\frac{1}{2T} \int_{-T}^{T} (f \circ g)(t) e^{-i\beta t}\, dt\right). \tag{3.9}$$

Passing to the limit we get

$$a(L \circ f \circ g, \beta) = L\left(a(f \circ g, \beta)\right) \tag{3.10}$$

and

$$a(L \circ f \circ g, \beta) = 0 \qquad \text{for all } L \in X^* \Leftrightarrow a(f \circ g, \beta) = 0. \tag{3.11}$$

If $\beta \in \mathrm{Mod}(f \circ g)$ then $\beta = \sum_{i=1}^{n} k_i \beta i$ where $k_i \in \mathbb{Z}$ and $\beta_i \in \Gamma(f \circ g)$.

If $\beta_i \in \Gamma(f \circ g)$, then $a(f \circ g, \beta_i) \neq 0$ and there exists $L_j \in X^*$ with $L_j(a(f \circ g, \beta_i)) \neq 0$ and so $\beta_i \in \Gamma(L_j \circ f \circ g)$. Consequently $\beta_i \in \mathrm{Mod}(L_j \circ f \circ g) \subset \mathrm{Mod}(g)$. It follows that $\beta_i \in \mathrm{Mod}(g)$ for all $i = 1, 2, \dots n$ and so $\beta = \sum_{i=1}^{n} k_i \beta_i \in \mathrm{Mod}(g)$, hence

$$\mathrm{Mod}(f \circ g) \subset \mathrm{Mod}(g). \tag{3.12}$$

□

4. The main result

In this section we will show, under rather general hypotheses, that the almost-periodic solutions of certain retarded functional differential equations are quasi-periodic.

Theorem 2. *Suppose that for the equation* (2.3) *there exists a real* $\lambda > 0$ *and an integer* $j > 0$ *such that* (H_1) *and* (2.4) *are satisfied with* $\Lambda(C\mathscr{S}) < \mu(\lambda)^{-1}$. *Then every almost-periodic solution of* (2.3), *defined on* $\mathbb{R}$, *is quasi-periodic.*

Proof. The hypotheses imposed on equation (2.3) ensure that there exists a bijective mapping between the amenable solutions of (2.3) over $]-\infty, \tau]$ and the solutions of the ordinary differential equation (2.20) which satisfy (2.25). Let x be an almost-periodic solution of (2.3), defined on $\mathbb{R}$. Then x is bounded and is consequently amenable over $]-\infty, \tau]$ for any real τ. The function ζ defined by $\zeta(t) = \Pi x_t$ for each $t \in \mathbb{R}$ is a solution of the equation (2.20), defined on $\mathbb{R}$. Since $\Pi : \mathscr{A} \to \Pi\mathscr{A}_\tau$ is continuous, it follows that (see Proposition 1) Πx_t is almost-periodic and so quasi-periodic since it is the solution of a finite dimensional ordinary differential equation (see [4]). The inverse mapping Ψ of Π defined from $\Pi\mathscr{A}_\tau \subset \mathbb{R}^j$ to $\mathscr{A}_\tau$ is continuous and associates the almost-periodic solution x of (2.3) to a quasi-periodic solution ζ of (2.20). Also (see Proposition 2) this solution satisfies the relation $x_t = (\Psi \circ \zeta)(t)$ for each $t \in \mathbb{R}$, and is such that $\mathrm{Mod}(x) \subset \mathrm{Mod}(\zeta)$. Since ζ is quasi-periodic, it admits a module of finite type, and it follows that x admits also a module of finite type, so x is quasi-periodic. □

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Received February 21, 2006

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