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Null controllability and Stackelberg–Nash strategy for a 2 × 2 system of parabolic equations

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Abstract. This paper is dedicated to solving a multi-objective control problem for a 2×2 system of parabolic equations. Here, we have many objectives, possibly conflictive, and we adopt a concept of hierarchy for the controls. We have the *leader* control, responsible for objectives of controllability type, and the *follower* controls which intend to be a Nash-equilibrium with respect to some given cost functionals. The novelty here is that we formulate this problem for systems of parabolic equations, meaning that we have many variables and naturally much more objectives to accomplish.

1. Introduction

In a standard mono-objective control problem for PDEs, the task is to see whether it is possible or not to find an external force so that its solutions achieve a single target. In this case, the objectives are usually of the controllability type, where one wants to drive the state to the target exactly (or approximately) in a finite time, or also, it may appear in the form of an optimal control problem, where minimizing a cost for the state is desirable. In real life, many control problems are more complex, especially when considering many objectives which are in conflict. For instance, one may want to control the temperature of a region (or object) in a finite time while we maintain it at desired levels in sub-regions. Another application is that one may want to control the concentration of a chemical product in a lake, keeping desirable levels in specific regions of the lake. By these examples, we see that achieving the best scenario for each objective might be impossible in the sense that approaching the state to one target may turn it very distant to the other one. To overcome that, concepts of equilibrium are usually adopted.

There are several concepts of equilibria for multi-objective control problems. They have their origin in game theory and are mainly motivated by problems in economic sciences. We mention the non-cooperative optimization strategy proposed by

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J. Nash in [16], the Pareto cooperative strategy in [17], and the Stackelberg hierarchic-cooperative strategy in [18]. Concerning multi-objective control problems for systems of PDEs, a relevant question is whether one can exactly (or approximately) control it to a given state in a given time and also make the state accomplish other goals, such as minimizing a cost functional.

Being more precise, let $A : D(A) \subset H \to H$ be a differential operator where its domain $D(A)$ is a subspace of a Hilbert space H. Consider the differential equation

$$\begin{cases} Y_t - AY = F(t) + BU(t), \quad t \in (0,T), \\ \quad Y(0) = Y_0 \in H, \end{cases} \tag{1}$$

where $Y : [0,T] \to H$ is the state, $F : [0,T] \to H$ is an external force, U is the control and B is an operator which represents the way the control acts. We assume that system (1) is well posed with solutions in $C([0,T],H)$. A standard controllability problem is the following: for each $Y_0, Y_1 \in H$, is there a control U such that the solution of (1) satisfies $Y(T) = Y_1$? In general, this question is not easy to answer and depends on the nature of the operator A. For instance, there are positive results for hyperbolic equations (see [6]), while for some parabolic equations it might be impossible due to a regularizing effect (see [10]). Controlling an equation becomes even more complicated when more objectives are in sight. Indeed, in addition to controllability, assume we want the state to be close to some other targets. We can represent these secondary objectives by minimizing a functional $J(Y,U)$ that measures the distance between the state and the other target. Then, the control U has at least two objectives to be fulfilled simultaneously. As mentioned previously, this might be impossible to do with one single control, and what we can do to overcome that is to adopt the *Stackelberg's optimization* strategy (see [15]), which consists in splitting the control U into two (or more) parts $U = U^0 + U^1$. In this case, the objective of U^0 is of controllability, while U^1 wants to minimize J. When we aim for more than one secondary objective, we might have n_0 functionals $\{J^1, \ldots, J^{n_0}\}$ to minimize, and the strategy follows similarly by splitting the control U into the form $U = U^0 + U^1 + \cdots + U^{n_0}$, assigning a role to each of them. It is important to mention that minimizing one functional J_k for some k may turn the values of J_l $(k \neq l)$ as worst as possible, that is why we have to define concepts of solution in equilibrium. In this paper, we adopt the concept of the *Nash-equilibrium* (see [7]) and more details about it will be given in the text.

There are several classical references concerning multi-objective control problems for PDEs. We refer to the works of J. L. Lions in [14, 15], where some Pareto and Nash strategies are applied for control problems in PDEs, and we also cite the book of J. I. Diaz and J. L. Lions in [7] as a complementary reading. It is important to remark that in [14, 15], the controllability type objective is to control the state approximately.

A more difficult question is how one can solve the problem in an exact controllability level. In this framework, we can cite the works of Araruna et al., in [4, 5], where for the heat equation some positive null controllability results are proved. The main motivation of the present work is to extend the results of [4, 5] to 2×2 systems of parabolic equations.

Let us give more details about the particular problem in sight. Let $\Omega \subset \mathbb{R}^N$ be a bounded, connected open set, whose boundary $\partial\Omega$ is regular enough. Let $T > 0$, $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$. We consider a system of coupled heat equations of the form

$$\begin{cases} y^1_t - \Delta y^1 = \displaystyle\sum_{p=1}^{2} A_{1p} y^p + F^1 & \text{in } Q, \\ y^2_t - \Delta y^2 = \displaystyle\sum_{p=1}^{2} A_{2p} y^p + F^2 & \text{in } Q, \\ y^1 = y^2 = 0 & \text{in } \Sigma, \\ y^1(0) = y^1_0, \quad y^2(0) = y^2_0 & \text{in } \Omega, \end{cases} \tag{2}$$

where (y^1, y^2) represents the *state*, (F^1, F^2) are *controls* and $\{A_{ij}\}_{i,j=1,2} \in [L^\infty(Q)]^4$ is what we call the *coupling matrix*. Each control F^i wants the state (y^1, y^2) to accomplish three objectives, one of controllability type and the other two of optimal control type.

In general, the problem of finding one control solving more than one objective is ill-posed, and that is why we adopt Stackelberg's strategy. As mentioned before, the strategy consists in dividing each control F^i into three others, each one responsible for its objective. So we write,

$$F^i = f^i \mathbb{1}_{\mathcal{O}} + v^{i1} \mathbb{1}_{\mathcal{O}_{i1}} + v^{i\,2} \mathbb{1}_{\mathcal{O}_{i2}}, \quad i = 1, 2, \tag{3}$$

where for each $i = 1, 2$ the sets $\mathcal{O}$, $\mathcal{O}_{i1}$ and $\mathcal{O}_{i2}$ are open sets, pairwise disjoint. We call the controls $\{f^i\}_{i=1}^2$ *the leaders*, having objectives of controllability type, in a sense that they want to drive the state to an exact value, and we call $\{v^{ij}\}_{i,j=1}^2$ *the followers*, with optimal control type objectives, having to be optimal concerning some specific functionals $\{J^{ij}\}_{i,j=1}^2$ that will be defined in the next section.

The problem we consider here is the following:

- The leaders want the state to reach a given target in time T. Due to the fact in (2) we are dealing with a parabolic system, the relevant target in this null trajectory. In other words, we must find (f^1, f^2) so that $(y^1, y^2)(T) = (0, 0)$.
- Taking into account the leaders policy, the followers have to work to make the costs $\{J^{ij}\}_{i,j=1}^2$ as small as possible. As mentioned before, minimizing one functional may turn bad the values of the others. Due to that, we will select them

according to equilibrium criteria. In this work, we will adopt the non-cooperative Nash equilibrium.

We will see in the following section that the accomplishment of the secondary objectives (minimize J^{ij}) turns out to be equivalent to the existence of solutions of a corresponding 10×10 optimality system, two forward and eight backward in time, with the leader acting in two of these equations. We also consider the case where the controls are present in only one equation. In this case, on the one hand, the optimality system becomes 6×6, which is better to deal with from the control point of view. But, on the other hand, the leader acts in only one equation, being the additional difficulty of this case. In this way, in both situations, dealing with this forward-backward structure and the role of the coupling coefficients to control are the main difficulties faced here.

To the best of our knowledge, few results concerning multi-objective control problems for systems of differential equations are known. In the spirit of [5], Hernández-Santamaría et al. (see [11]) considered a 2×2 system of parabolic equations assuming some particular conditions over their functionals $\{J^{ij}\}$. Essentially, we can interpret these conditions by saying that all the secondary objectives have to be achieved in the same sub-region of the domain Ω. However, in real applications, more general targets may be of interest. That is what motivates the present paper. The strategy is to follow some of the ideas in [4] where, for the case of one single heat equation, these conditions over the secondary objective are relaxed compared to [5]. The idea is to use duality arguments, with Carleman estimates as the primary tool. By performing very careful computations and using the coupling coefficients to see how information passes from one equation to another, we prove an observability estimate to an adjoint system.

2. Statement of the problem

As mentioned in the previous section, here we consider a control problem with many simultaneous objectives, the primary ones of controllability type, where the leader is responsible for it, and the secondary one, of optimal control type, managed by secondary actuators called the followers. In this section, we will see more precisely the secondary objectives, and the strategy we will follow to accomplish them, giving a characterization of the controls. This will allow us to see more clearly which controllability problem we are dealing with. Also, we state in this section the main results of the paper.

For $k, l = 1, 2$, let y_d^{kl} be regular functions defined on a region $\mathcal{O}_d^{kl} \times (0, T)$, $\{\mu_{kl}\}_{k,l=1}^2$ some positive real numbers, and define further the functional J^{kl} :

$L^2(\mathcal{O}_{kl} \times (0,T)) \to \mathbb{R}$ by

$$J^{kl}(\{v^{ij}\}_{i,j=1}^2) = \frac{1}{2}\iint_{\mathcal{O}_d^{kl}\times(0,T)} |y^l - y_d^{kl}|^2\,dxdt + \frac{\mu_{kl}}{2}\iint_{\mathcal{O}_{kl}\times(0,T)} |v^{kl}|^2\,dxdt, \quad k,l = 1,2, \tag{4}$$

where (y^1, y^2) is the solution of (2) and (3). For each $f := \{f^i\}_{i=1}^2$ fixed, the followers intend to be a *Nash equilibrium* for the functionals $\{J^{kl}\}_{k,l=1}^2$, meaning that $\{v^{ij}\}_{i,j=1}^2$ satisfies

$$\begin{aligned} J^{11}(\{v^{ij}\}) &= \min_{\hat{v}^{11}\in L^2(\mathcal{O}_{11}\times(0,T))} J^{11}(\hat{v}^{11}, v^{12}, v^{21}, v^{22}),\\ J^{12}(\{v^{ij}\}) &= \min_{\hat{v}^{12}\in L^2(\mathcal{O}_{12}\times(0,T))} J^{12}(v^{11}, \hat{v}^{12}, v^{21}, v^{22}),\\ J^{21}(\{v^{ij}\}) &= \min_{\hat{v}^{21}\in L^2(\mathcal{O}_{21}\times(0,T))} J^{21}(v^{11}, v^{12}, \hat{v}^{21}, v^{22}),\\ J^{22}(\{v^{ij}\}) &= \min_{\hat{v}^{22}\in L^2(\mathcal{O}_{22}\times(0,T))} J^{22}(v^{11}, v^{12}, v^{21}, \hat{v}^{22}), \end{aligned} \tag{5}$$

which corresponds to a non-cooperative optimization strategy. For each $k,l = 1,2$, the follower v^{kl} wants to minimize the objective J^{kl} even if this represents the worst scenario for the other objectives.

Since the problem is linear and the parameters $\{\mu_{kl}\}$ are positive real numbers, the functionals in (4) are differentiable and convex in each direction. Then, it becomes clear that the four conditions in (5) are completely equivalent to find $\{v^{ij}\}_{i,j=1}^2$ such that

$$\frac{\partial J^{kl}}{\partial \hat{v}^{kl}}\big(\{v^{ij}\}\big) = 0 \quad \text{for every } k,l = 1,2. \tag{6}$$

Using (6), we can prove that (see [5, Section 2.1.2]) the Nash equilibrium can be characterized by

$$v^{kl} = -\frac{1}{\mu_{kl}}\varphi^{l,kl}\mathbb{1}_{\mathcal{O}_{kl}}, \quad k,l = 1,2, \tag{7}$$

where the function $\varphi^{j,kl}$ satisfies the following system:

$$\begin{cases} -\varphi_t^{l,kl} - \Delta\varphi^{l,kl} = \displaystyle\sum_{p=1}^{2} A_{pl}\varphi^{p,kl} + (y^l - y_d^{kl})\mathbb{1}_{\mathcal{O}_d^{kl}}, & \text{in } Q,\\ -\varphi_t^{j,kl} - \Delta\varphi^{j,kl} = \displaystyle\sum_{p=1}^{2} A_{pj}\varphi^{p,kl}, \quad j \neq l, & \text{in } Q,\\ \varphi^{j,kl} = 0, & \text{over } \Sigma,\\ \varphi^{j,kl}(\cdot, T) = 0, & \text{in } \Omega, \end{cases} \tag{8}$$

for $j,k,l = 1,2$.

Once the Nash equilibrium is characterized by formula (7), we search for leader controls which drives the state (y^1, y^2) to zero. In this way, we are led to prove the null controllability for the following system:

$$\begin{cases} y^j_t - \Delta y^j = \displaystyle\sum_{p=1}^{2} A_{jp} y^p + f^j \mathbb{1}_{\mathcal{O}} - \sum_{p=1}^{2} \frac{1}{\mu_{jp}} \varphi^{p,jp} \mathbb{1}_{\mathcal{O}_{jp}}, & \text{in } Q, \\ y^j = 0, & \text{in } \Sigma, \\ y^j(0) = y^j_0, & \text{in } \Omega, \end{cases} \tag{9}$$

for $j = 1, 2$.

To deal with the control problems proposed in here, some geometric assumptions over the sets $\mathcal{O}^{kl}_d$ appearing in (4) are needed, for $k, l = 1, 2$. Here, we deal with two possible situations,

$$\mathcal{O}^{1l}_d = \mathcal{O}^{2l}_d, \tag{10}$$

or

$$\mathcal{O}^{1l}_d \cap \mathcal{O} \neq \mathcal{O}^{2l}_d \cap \mathcal{O}. \tag{11}$$

Also, we have to ask that these sets touch the control domain and that some of the coupling coefficients are bounded from bellow in there, that is, there exists $C_0 > 0$ such that

$$\text{int}\big\{x \in \mathcal{O}^{kl}_d \cap \mathcal{O};\ A_{lp}(x) \geq C_0\big\} \neq \emptyset, \tag{12}$$

for every $k, l, p \in \{1, 2\}$, with $p \neq l$.

In the case we have (11) for some value of $l \in \{1, 2\}$, we find that

$$[\mathcal{O}^{\tilde{k}l}_d \setminus \mathcal{O}^{kl}_d] \cap \mathcal{O} \neq \emptyset, \tag{13}$$

for some $(k, \tilde{k}) \in \{(1, 2), (2, 1)\}$. If that is the case, then we also assume the existence of a constant $C_0 > 0$ such that

$$\text{int}\big\{x \in [\mathcal{O}^{\tilde{k}l}_d \setminus \mathcal{O}^{kl}_d] \cap \mathcal{O};\ A_{lp}(x) \geq C_0\big\} \neq \emptyset, \tag{14}$$

for $p \neq l$.

It is relevant to mention that similar conditions to (10) and (11) also appear when one single heat equation is considered (see [4,5]), so these conditions are also expectable here. Also, it is important to say that conditions (12) and (14) are associated with the fact that we are considering a 2×2 system of equations. Very similar conditions also appear in many results about the controllability of systems of PDEs, see [1], for instance. Finally, we remark that for different values of $l \in \{1, 2\}$, we may have combinations of properties (10) and (11), this represents an additional difficulty to be dealing with systems of equations.

The first problem we want to solve is the following.

Theorem 2.1. *Let* $(y_0^1, y_0^2) \in [L^2(Q)]^2$, $\{A_{ij}\} \in [L^\infty(Q)]^4$, *and* C_0 *a positive constant. Assume that the sets* $\mathcal{O}_d^{kl}$ *satisfy* (10) *or* (11) *and we also assume* (12) *for every* $k, l, p \in \{1,2\}$, $p \neq l$. *Moreover, we assume* (14) *for the values of* $l, k, \tilde{k} \in \{1,2\}$ *such that* (13) *holds. For* $\mu = \min\{\mu_{kl}\}$ *sufficiently large, let* $(v_1, v_2) \in \prod_{i=1}^2 L^2(\mathcal{O}_i \times (0,T))$ *be the Nash equilibrium for the functionals* (4). *Under these assumptions, there exists a leader control* $(f^1, f^2) \in [L^2(\mathcal{O} \times (0,T))]^2$ *such that the solution of* (2), *with* $\{F^i\}_{i=1}^2$ *given by* (3), *satisfies* $(y^1(T), y^2(T)) = (0,0)$.

The main difficulty in proving Theorem 2.1 is that, once we characterize the Nash equilibrium, we reduce the multi-objective control problem to a partial null controllability problem for a system of several equations (ten precisely) acting only over two equations. We remark that for the case of one single equation (see [4, 5]), the corresponding optimality system is composed of three equations only.

Another interesting problem arises when one tries to control the system with fewer controls. Indeed, let us consider in (2) the control $F^2 = 0$, and let us follow a similar strategy. Now, for $k = 1,2$, we consider y_d^k sufficiently regular functions, defined over a region $\mathcal{O}_d^k \times (0,T)$, and $\{\mu_k\}_{k=1}^2$ sufficiently large real numbers. In this case, we define the functionals

$$J^k(\{v^j\}_{j=1}^2) = \frac{1}{2} \iint_{\mathcal{O}_d^k \times (0,T)} |y^k - y_d^k|^2 \, dx dt + \frac{\mu_k}{2} \iint_{\mathcal{O}_k \times (0,T)} |v^k|^2 \, dx dt, \quad (15)$$

$k = 1,2$, and we search for followers satisfying

$$J^1(\{v^j\}) = \min_{\hat{v}^1 \in L^2(\mathcal{O}_1 \times (0,T))} J^1(\hat{v}^1, v^2),$$
$$J^2(\{v^j\}) = \min_{\hat{v}^2 \in L^2(\mathcal{O}_2 \times (0,T))} J^1(v^1, \hat{v}^2).$$

Here, we also have the following characterization for the Nash equilibrium (see [5, Section 2.1.2]):

$$v^j = -\frac{1}{\mu_j} \varphi^{j,j} \mathbb{1}_{\mathcal{O}_j}, \quad j = 1,2, \quad (16)$$

where, for $i, j = 1,2$

$$\begin{cases} -\varphi_t^{j,j} - \Delta \varphi^{j,j} = \sum_{p=1}^2 A_{pj} \varphi^{p,j} + (y^j - y_d^j) \mathbb{1}_{\mathcal{O}_d^j} & \text{in } Q, \\ -\varphi_t^{i,j} - \Delta \varphi^{i,j} = \sum_{p=1}^2 A_{pi} \varphi^{p,j}, \quad i \neq j & \text{in } Q, \\ \varphi^{i,j} = 0, & \text{over } \Sigma, \\ \varphi^{i,j}(\cdot, T) = 0, & \text{in } \Omega. \end{cases} \quad (17)$$

In this case, we also assume some geometric conditions over the sets $\mathcal{O}_d^i$. Precisely, we ask them to touch the control domain and also, we suppose the existence of a constant $C_0 > 0$ such that

$$\operatorname{int}\left\{x \in (\mathcal{O}_d^i \cap \mathcal{O}) \setminus \operatorname{supp} A_{ji};\ A_{ij}(x) \geq C_0\right\} \neq \emptyset, \tag{18}$$

and

$$\operatorname{int}\left\{x \in [\mathcal{O}_d^j \setminus \mathcal{O}_d^i] \cap \mathcal{O};\ A_{ji}(x) \geq C_0\right\} \neq \emptyset, \tag{19}$$

for some $(i, j) \in \{(1, 2), (2, 1)\}$.

In this context, the second problem under view is the following.

Theorem 2.2. *Let $(y_0^1, y_0^2) \in [L^2(Q)]^2$, $\{A_{ij}\} \in [L^\infty(Q)]^4$, and C_0 a positive constant. Assume that the sets $\mathcal{O}_d^k$ and the coupling coefficients satisfy properties (18) and (19) for some $(i, j) \in \{(1, 2), (2, 1)\}$. For $\tilde{\mu} = \min\{\mu_k\}$ sufficiently large, let $\{v^j\}_{j=1}^2$ be the Nash equilibrium for the functionals (15). Under these assumptions, there exists a leader control $f^i \in L^2(\mathcal{O} \times (0, T))$ such that the solution of (2) with F^i given by (3), and $F^j = 0$ for $j \neq i$, satisfies $(y^1(T), y^2(T)) = (0, 0)$.*

We have the following remarks.

Remark 2.3. In assumptions (12), (14), (18) and (19), we can replace the conditions $A_{ij}(x) \geq C_0$ by $-A_{ij}(x) \geq C_0$ such that all results presented in this paper still hold.

Remark 2.4. It is not clear that the existence of a Nash equilibrium, as well as the existence and uniqueness of solutions to the optimality system (8)–(9) holds for any choice of the parameters $\{\mu_{kl}\}$. A sufficient condition to overcome that is to assume that $\mu = \min\{\mu_{kl}\}$ is positive, and sufficiently large, greater than a constant $C = C(T, \Omega, \{\mathcal{O}_{kl}\}, \{\mathcal{O}_d^{kl}\}, \{A_{ij}\})$. The proof of that is standard and will be omitted, we can cite [5] for a proof in the case where one single heat equation is controlled under similar strategies to the ones presented here. The same remark holds for the parameter $\{\mu_k\}$ appearing in the functionals (15), that is, they are also considered in such a way that $\tilde{\mu} = \min\{\mu_k\}$ is sufficiently large. This is essentially why these assumptions appear in Theorems 2.1 and 2.2.

Note that the proof of Theorem 2.1 cannot be interpreted as a natural step to follow to prove Theorem 2.2. Indeed, in Theorem 2.1 we have to deal with an optimality system composed of ten equations and two leaders, while in Theorem 2.2, we have to deal with an optimality system with six equations and only one leader. In this way, in terms of difficulty to solve, the problems generated in Theorems 2.2 and 2.1 are not comparable.

By the Hilbert uniqueness method, controllability properties for linear systems of PDEs are equivalent to some suitable observability inequalities to the solutions of an adjoint state. That is the approach we are going to follow, and to prove such inequalities, we are going to make use of some appropriated Carleman estimates.

3. Carleman estimates

We dedicate this section to proving some Carleman type estimates to the solutions of an adjoint equation. Before we start, we introduce some sets and notation very important for what follows.

Under assumption (12), there exist $C_0 > 0$ and open subsets

$$\omega^{kl} \subset\subset \widehat{\omega}^{kl} \subset\subset \widetilde{\omega}^{kl} \subset\subset \mathcal{O}_d^{kl} \cap \mathcal{O}, \quad \text{for every } k, l \in \{1, 2\}, \tag{20}$$

such that

$$A_{lp} \geq C_0 \quad \text{in } \widetilde{\omega}^{kl}, \tag{21}$$

for every $k, l, p \in \{1, 2\}$, $p \neq l$.

It is not difficult to see that if (13) holds, then we can assume that

$$\widetilde{\omega}^{\tilde{k}l} \cap \mathcal{O}_d^{kl} = \emptyset. \tag{22}$$

These conditions will be crucial in forthcoming computations.

If conditions (18) and (19) hold for some (i, j), then there exist C_0 and open subsets

$$\omega^k \subset\subset \widehat{\omega}^k \subset\subset \widetilde{\omega}^k \subset\subset \omega_*^k \subset\subset \mathcal{O}_d^k \cap \mathcal{O}, \quad \text{for every } k = 1, 2, \tag{23}$$

such that

$$A_{ij}(x) \geq C_0 \quad \text{and} \quad A_{ji}(x) = 0, \quad \text{for every } x \in \omega_*^i, \tag{24}$$

and that

$$\widetilde{\omega}^j \cap \mathcal{O}_d^i = \emptyset \quad \text{and} \quad A_{ji}(x) \geq C_0, \quad \text{for every } x \in \omega_*^j. \tag{25}$$

The resolution of Theorems 2.1 or 2.2 is entirely equivalent to the following observability estimates to the solutions of an adjoint system. In what follows, we prove such inequalities, which correspond to the main result of the present paper.

Theorem 3.1. (i) *For $j,k,l \in \{1,2\}$, let $(\psi^j, \gamma^{j,kl})$ the solution of the following adjoint system:*

$$\begin{cases} -\psi^j_t - \Delta\psi^j = \sum_{p=1}^{2} A_{pj}\psi^p + \sum_{p=1}^{2} \gamma^{j,pj} \mathbb{1}_{O_d^{pj}} & \text{in } Q, \\ \gamma^{j,kl}_t - \Delta\gamma^{j,kl} = \sum_{p=1}^{2} A_{jp}\gamma^{p,kl}, \quad j \neq l, & \text{in } Q, \\ \gamma^{l,kl}_t - \Delta\gamma^{l,kl} = \sum_{p=1}^{2} A_{lp}\gamma^{p,kl} - \frac{1}{\mu_{kl}}\psi^k \mathbb{1}_{\mathcal{O}_{kl}} & \text{in } Q, \\ \psi^j = \gamma^{j,kl} = 0, & \text{in } \Sigma, \\ \psi^j(T) = \psi^T_j, \quad \gamma^{j,kl}(0) = 0, & \text{in } \Omega, \end{cases} \tag{26}$$

where $\psi_j^T \in L^2(\Omega)$. Under the same assumptions as in Theorem 2.1, there exist $C > 0$ and a weight function $\rho(t)$ with $\lim_{t\to T} \rho(t) = 0$, such that the following observability inequality holds:

$$\sum_{k=1}^{2} \int_\Omega |\psi^k(0)|^2\, dx + \sum_{k,l=1}^{2} \iint_{\mathcal{O}_d^{kl}\times(0,T)} \rho^2(t)|\gamma^{l,kl}|^2\, dxdt \leq C \sum_{k=1}^{2} \iint_{\mathcal{O}\times(0,T)} |\psi^k|^2\, dxdt. \tag{27}$$

(ii) *For $i, j = 1,2$, let (ψ^j, γ^{ji}) the solution of*

$$\begin{cases} -\psi^i_t - \Delta\psi^i = \sum_{p=1}^{2} A_{pi}\psi^p + \gamma^{ii}\mathbb{1}_{O_d^i} & \text{in } Q, \\ \gamma^{ii}_t - \Delta\gamma^{ii} = \sum_{p=1}^{2} A_{ip}\gamma^{pi} - \frac{1}{\mu_i}\psi^i \mathbb{1}_{\mathcal{O}_i} & \text{in } Q, \\ \gamma^{ji}_t - \Delta\gamma^{ji} = \sum_{p=1}^{2} A_{jp}\gamma^{pi}, \quad j \neq i, & \text{in } Q, \\ \psi^i = \gamma^{ii} = \gamma^{ji} = 0 & \text{in } \Sigma, \\ \psi^i(T) = \psi^T_i, \quad \gamma^{ii}(0) = \gamma^{ji}(0) = 0 & \text{in } \Omega. \end{cases} \tag{28}$$

Under the same assumptions as in Theorem 2.2, there exist $C > 0$ and a weight function $\hat{\rho}(t)$ where $\lim_{t\to T} \hat{\rho}(t) = 0$, such that the following observability *inequality holds:*

$$\sum_{k=1}^{2}\int_{\Omega}|\psi^k(0)|^2\,dx+\iint_{\mathcal{O}_d^k\times(0,T)}\rho^2(t)|\gamma^{kk}|^2\,dxdt \leq C\iint_{\mathcal{O}\times(0,T)}|\psi^i|^2\,dxdt. \tag{29}$$

As we already mentioned, if we prove the observability inequality (27), then we are solving Theorem 2.1, while if we prove (29), then we are solving Theorem 2.2.

It is important to mention that it is also not clear that systems (26) and (28) possess solutions for any value of $\{\mu_{kl}\}$ or $\{\mu_k\}$. By similar reasons as for the optimality system (8)–(9) (see Remark 2.4), we can prove the well-posedness for these systems by taking $\mu=\min\{\mu_{k,l}\}$ and $\widetilde{\mu}=\min\{\mu_k\}$ sufficiently large independently of the data ψ^T. In this way, we can take μ and $\widetilde{\mu}$ large enough so that either the optimality systems or the respective adjoint systems are well-posed.

We remark that the search for controllability/observability results for systems of parabolic equations is an extensive research area in control theory, and many positive/negative conclusions are known. We can refer to [1] for a survey and [2] where, for some specific coupling properties, the authors have proved the existence of a minimal time of controllability. More recently, Duprez, M. and Lissy, P. in [8] and [9] have found some sufficient conditions for the controllability of systems with fewer controls, by applying algebraic methods.

Now, we have the following Lemma, which is already used in [4] for the case of one single equation. Here, we have to adapt it to a more general situation.

Lemma 3.2. *Let Λ be a finite set and $\{q_m\}_{m\in\Lambda}$ a family of disjoint open subsets of $\mathcal{O}$ such that there exists $\widetilde{\mathcal{O}}\subset\subset\mathcal{O}$ where $q_m\subset\widetilde{\mathcal{O}}$ for every $m\in\Lambda$. Then, there exists a family of functions $\{\eta_m\}_{m\in\Lambda}$ in $C^2(\overline{\Omega})$, such that*

$$\begin{cases}\eta_m>0 & \text{in }\Omega, \qquad \eta_m=0 \quad \text{on }\partial\Omega,\\ \|\nabla\eta_m\|>C & \text{in }\overline{\Omega\setminus q_m}, \qquad \eta_n=\eta_m \quad \text{in }\overline{\Omega\setminus\widetilde{\mathcal{O}}} \quad \text{for } n\neq m.\end{cases} \tag{30}$$

Proof. This result is already proved in [4] for the case $\Lambda=\{1,2\}$. To generalize it to more sets, we assume that $\Lambda=\{1,\dots,N_0\}$ for some N_0. The general case can be obtained by a simple identification argument.

It is well known that (see [10]) there exists a function η^1 satisfying

$$\begin{cases}\eta_1>0 & \text{in }\Omega, \qquad \eta_1=0 \quad \text{on }\partial\Omega,\\ \|\nabla\eta_1\|>C & \text{in }\overline{\Omega\setminus q_1}.\end{cases}$$

Using [4, Lemma 5], we get that, for each $m \neq 1$, we can take $\widetilde{\mathcal{O}}^m \subset \widetilde{\mathcal{O}}$, such that $q^1 \cup q^m \subset \widetilde{\mathcal{O}}^m$, and a weight function n_m such that

$$\begin{cases} \eta_m > 0 & \text{in } \Omega, \qquad \eta_m = 0 \quad \text{on } \partial\Omega, \\ \|\nabla \eta_m\| > C & \text{in } \overline{\Omega \setminus q^m}, \qquad \eta_m = \eta_1 \quad \text{in } \overline{\Omega \setminus \widetilde{\mathcal{O}}^m}. \end{cases}$$

It is clear that the family $\{\eta_m\}_{m=1}^{N_0}$ satisfies (30) and the lemma is proved. ■

Now, for any finite set Λ, let $\{\eta_m\}_{m\in\Lambda}$ be a family given by Lemma 3.2. We define the following weight functions, very common when dealing with Carleman estimates:

$$\begin{aligned} \sigma_m(x,t) &:= \frac{e^{4\lambda\|\eta_m\|_\infty} - e^{\lambda(2\|\eta_m\|_\infty + \eta_m(x))}}{t(T-t)}, \\ \xi_m(x,t) &:= \frac{e^{\lambda(2\|\eta_m\|_\infty + \eta_m(x))}}{t(T-t)}, \qquad m \in \Lambda. \end{aligned} \tag{31}$$

For $n \in \mathbb{N}$, we introduce the notations

$$I_n^m(\psi) := s^{n-4}\lambda^{n-3} \iint_Q e^{-2s\sigma_m}(\xi_m)^{n-4}(|\psi_t|^2 + |\Delta\psi|^2)\, dxdt + L_n^m(\psi),$$

where

$$\begin{aligned} L_n^m(\psi) := {} & s^{n-2}\lambda^{n-1} \iint_Q e^{-2s\sigma_m}(\xi_m)^{n-2}|\nabla\psi|^2\, dxdt \\ & + s^n\lambda^{n+1} \iint_Q e^{-2s\sigma_m}(\xi_m)^n|\psi|^2\, dxdt. \end{aligned}$$

For $u^T \in L^2(\Omega)$ and $f, f_1, \ldots, f_r \in L^2(Q)$, let u be the solution of the following equation:

$$\begin{cases} -u_t - \Delta u = f + \sum_{k=1}^{r} \partial_k f_k & \text{on } \sum, \\ u = 0 & \text{on } \sum, \\ u(\cdot, T) = u^T & \text{in } \Omega. \end{cases} \tag{32}$$

Then, the following Carleman estimate holds.

Proposition 3.3. *Let Λ be a finite set, $n \in \mathbb{N}$, $\{q_m\}_{m\in\Lambda}$ a family of open subsets of Ω, and let $\{\eta_m\}_{m\in\Lambda}$ be the functions given by Lemma 3.2. For each $m \in \Lambda$, there exists a constant $C(\Omega, q_m) > 0$ so that, for every $s \geq s^m = C(\Omega, q_m)(T + T^2)$ and every $\lambda \geq C$, the following estimates hold for every solution u of (32) with $u^T \in L^2(\Omega)$, in each of the following cases:*

(i.) *if $f \in L^2(\Omega)$ and $f_k = 0$ for every $k \in \{1, \dots, n\}$, then*

$$I_n^m(u) \le C\left(s^n\lambda^{n+1}\iint_{q_m\times(0,T)} e^{-2s\sigma_m}(\xi_m)^n|u|^2\,dxdt + s^{n-3}\lambda^{n-3}\iint_Q e^{-2s\sigma_m}(\xi_m)^{n-3}|f|^2\,dxdt\right);$$

(ii.) *if $f \in L^2(\Omega)$ and $f_k \in L^2(\Omega)$ for every $k \in \{1, \dots, r\}$, then*

$$L_n^m(u) \le C\left(s^n\lambda^{n+1}\iint_{q_m\times(0,T)} e^{-2s\sigma_m}(\xi_m)^n|u|^2\,dxdt + s^{n-3}\lambda^{n-3}\iint_Q e^{-2s\sigma_m}(\xi_m)^{n-3}|f|^2\,dxdt + s^{n-1}\lambda^{n-1}\sum_{k=1}^{r}\iint_Q e^{-2s\sigma_m}(\xi_m)^{n-1}|f_k|^2\,dxdt\right).$$

The estimates given in Propositions 3.3 are classical in control theory and are well known nowadays, see references [10] and [13] for a proof.

In the next section, we prove new Carleman estimates to the solutions of (26) and (28). These estimates are the principal tools we are going to use to prove observability inequalities (27) and (29), respectively.

4. New Carleman estimates

We have the following result.

Proposition 4.1. (i.) *Assume that, for each $l = 1, 2$, the sets $\mathcal{O}_d^{kl}$ satisfy one of the conditions (10) or (11) and also assume (12) for every $k, l, p \in \{1, 2\}$, with $p \neq l$. Moreover, in the case we have (13) for some $l, k, \tilde{k} \in \{1, 2\}$, we suppose that condition (14) holds. Then, there exists $C > 0$ such that, for $\lambda \ge C$ and $s \ge \max_{k,l,p,j}\{C(T + T^2), CT^2\|A_{pj}\|^{\frac{2}{3}}\}$ sufficiently large, the following estimate holds:*

$$\sum_{j,k,l=1}^{2} I_n^{kl}(\gamma^{j,kl}) + s^{n+3}\lambda^{n+4}\sum_{j=1}^{2}\iint_Q e^{-2s\sigma_{jj}}(\xi_{jj})^{n+3}|\psi^j|^2\,dxdt \le Cs^{n+8}\lambda^{n+9}\sum_{p,j=1}^{2}\iint_{\mathcal{O}\times(0,T)} e^{-2s\sigma_{pj}}\xi_{pj}^{n+8}|\psi^p|^2\,dxdt, \tag{33}$$

for every solution $\{\psi^j, \gamma^{j,lk}\}_{j,k,l}$ of (26).

(ii.) *Assume that conditions* (18) *and* (19) *hold for some* $(i, j) \in \{(1,2),(2,1)\}$. *Then, there exists* $C > 0$ *such that, for* $\lambda \geq C$ *and* $s \geq \max_{i,j}\{C(T+T^2), CT^2\|A_{ij}\|_\infty^2\}$ *sufficiently large, it holds*

$$s^n\lambda^{n+1}\iint_Q e^{-2s\sigma_i}(\xi_i)^n|\psi^i|^2\,dxdt + I_{n-3}^j(\psi^j) + \sum_{k=1}^2\left(I_{n-3}^i(\gamma^{ki}) + I_{n-6}^j(\gamma^{kj})\right) \leq Cs^{n+9}\lambda^{n+10}\iint_{\mathcal{O}\times(0,T)}(e^{-2s\sigma_i}(\xi_i)^{n+9} + e^{-2s\sigma_j}(\xi_j)^{n+3})|\psi^i|^2\,dxdt, \tag{34}$$

for every $\{\psi^j, \gamma^{ji}\}_{j,i}$ *solution of* (28).

Proof. For the proof of (i.), we can deal with three possible cases:

(A) Condition (10) holds for every l in $\{1,2\}$.

(B) Condition (10) holds for only one value of l in $\{1,2\}$.

(C) Condition (11) holds for every $l = 1, 2$.

Condition (C) is the most difficult to handle and we are going to concentrate on it, in Remark 4.2 we make some comments concerning the other possibilities. We remind that, if (11) is true for some $l \in \{1,2\}$, then condition (13) holds for some $(k,\tilde{k}) \in \{(1,2),(2,1)\}$, this will be important in forthcoming computations. In this way, the proof of case (C) is divided into two others, case (C1) where for any $l \in \{1,2\}$, condition (13) holds for any $k \neq \tilde{k}$, and case (C2) where for only one $l \in \{1,2\}$ condition (13) holds for any $k \neq \tilde{k}$.

Proof of item (i.), *case* (C1)*:* Consider a family of open sets satisfying (20) and (21). In this proof we can assume (22) for any $l \in \{1,2\}$ and any $k \neq \tilde{k}$.

From now on, we are going to make use of the weight functions (30) and (31) for $\Lambda = \{1,2\}\times\{1,2\}$, $m = (k,l)$ and $\{q_m\} = \{\omega^{kl}\}$.

For $(k,l) \in \Lambda$ fixed, we apply item (i.) of Proposition 3.3 for $u = \gamma^{j,kl}$ (see (26)), $q_m = \omega^{kl}$, and we sum the resulting estimates to obtain that

$$\sum_{j=1}^2 I_n^{kl}(\gamma^{j,kl}) \leq C\sum_{j=1}^2\Bigg(s^n\lambda^{n+1}\iint_{\omega^{kl}\times(0,T)} e^{-2s\sigma_{kl}}(\xi_{kl})^n|\gamma^{j,kl}|^2\,dxdt + s^{n-3}\lambda^{n-3}\iint_Q e^{-2s\sigma_{kl}}(\xi_{kl})^{n-3}\sum_{p=1}^2\|A_{jp}\|_\infty^2|\gamma^{p,kl}|^2\,dxdt\Bigg) + Cs^{n-3}\lambda^{n-3}\iint_{\mathcal{O}_{kl}\times(0,T)} e^{-2s\sigma_{kl}}(\xi_{kl})^{n-3}|\psi^k|^2\,dxdt, \tag{35}$$

for every $\lambda \geq C$ and $s \geq s^{kl} = C(\Omega, \omega^{kl})(T + T^2)$. Given a small $\varepsilon > 0$, we use the fact that $T^6 \xi_{kl} \geq C$, and we get that

$$\begin{aligned}
& s^{n-3}\lambda^{n-3} \iint_Q e^{-2s\sigma_{kl}} (\xi_{kl})^{n-3} \sum_{p=1}^{2} \|A_{jp}\|_\infty^2 |\gamma^{p,kl}|^2 \, dxdt \\
& \quad \leq C T^6 \|A_{jp}\|_\infty^2 s^{n-3}\lambda^{n-3} \iint_Q e^{-2s\sigma_{kl}} (\xi_{kl})^n \sum_{p=1}^{2} |\gamma^{p,kl}|^2 \, dxdt \\
& \quad \leq \varepsilon I_n^{kl}(\gamma^{j,kl}), \quad k,l = 1,2,
\end{aligned} \tag{36}$$

for $\lambda \geq C$ and $s \geq \max\{s^{kl}, C T^2 \|A_{jp}\|^{\frac{2}{3}}\}$. Combining (35) and (36), we obtain that

$$\begin{aligned}
\sum_{j=1}^{2} I_n^{kl}(\gamma^{j,kl}) \leq C s^n \lambda^{n+1} \sum_{p=1}^{2} \iint_{\omega^{kl}\times(0,T)} e^{-2s\sigma_{kl}} (\xi_{kl})^n |\gamma^{p,kl}|^2 \, dxdt \\
+ s^{n-3}\lambda^{n-3} \iint_{\mathcal{O}_{kl}\times(0,T)} e^{-2s\sigma_{kl}} (\xi_{kl})^{n-3} |\psi^k|^2 \, dxdt,
\end{aligned} \tag{37}$$

for all $k,l = 1,2$.

Since (21) holds for $p \neq l$, we can take positive functions $\widehat{\theta}^{kl} \in C_0^2(\widehat{\omega}^{kl})$ such that $\widehat{\theta}^{kl} = 1$ in ω^{kl}, and we can show that

$$\begin{aligned}
& s^n \lambda^{n+1} \iint_{\omega^{kl}\times(0,T)} e^{-2s\sigma_{kl}} (\xi_{kl})^n |\gamma^{p,kl}|^2 \, dxdt \\
& \quad \leq C s^n \lambda^{n+1} \iint_{\widehat{\omega}^{kl}\times(0,T)} \widehat{\theta}^{kl} e^{-2s\sigma_{kl}} (\xi_{kl})^n A_{lp} |\gamma^{p,kl}|^2 \, dxdt \\
& \quad = s^n \lambda^{n+1} \iint_{\widehat{\omega}^{kl}\times(0,T)} \widehat{\theta}^{kl} e^{-2s\sigma_{kl}} (\xi_{kl})^n \\
& \qquad \cdot \Big(\gamma_t^{l,kl} - \Delta\gamma^{l,kl} - A_{ll}\gamma^{l,kl} + \frac{1}{\mu_{kl}} \psi^k \mathbb{1}_{\mathcal{O}_{kl}}\Big) \gamma^{p,kl} \, dxdt \\
& \quad \leq \varepsilon I_n^{kl}(\gamma^{p,kl}) + C s^{n+4} \lambda^{n+5} \iint_{\widehat{\omega}^{kl}\times(0,T)} e^{-2s\sigma_{kl}} \xi_{kl}^{n+4} |\gamma^{l,kl}|^2 \, dxdt \\
& \qquad + C s^n \lambda^{n+1} \iint_{(\mathcal{O}_{kl}\cap\widehat{\omega}^{kl})\times(0,T)} e^{-2s\sigma_{kl}} \xi_{kl}^n |\psi^k|^2 \, dxdt,
\end{aligned} \tag{38}$$

for all $p,k,l = 1,2$ with $p \neq l$ and ε sufficiently small.

Then, we combine (37) and (38), and we use the fact that $\mathcal{O}_{kl} \cap \widehat{\omega}^{kl} = \emptyset$, to get that

$$\sum_{j=1}^{2} I_n^{kl}(\gamma^{j,kl}) \leq C s^{n+2}\lambda^{n+3} \iint_{\widehat{\omega}^{kl}\times(0,T)} e^{-2s\sigma_{kl}}(\xi_{kl})^n |\gamma^{l,kl}|^2\,dxdt$$
$$+ C s^{n-3}\lambda^{n-3} \iint_{\mathcal{O}_{kl}\times(0,T)} e^{-2s\sigma_{kl}}(\xi_{kl})^{n-3}|\psi^k|^2\,dxdt, \qquad (39)$$

for $k,l = 1,2$, for $\lambda \geq C$ and $s \geq \max\{s^{kl}, CT^2\|A_{jp}\|^{\frac{2}{3}}\}$.

Let $\widetilde{\mathcal{O}}$ be given as in Lemma 3.2 and let $\theta \in C^2(\overline{\Omega})$ be such that

$$\begin{cases} \theta(x) = 0 & \text{for } x \in \widetilde{\mathcal{O}}, \\ \theta(x) = 1 & \text{for } x \in \Omega \setminus \mathcal{O}. \end{cases} \qquad (40)$$

For $i = 1,2$, we have that the functions $\theta\psi^j$ satisfy the equation

$$\begin{cases} -(\theta\psi^j)_t - \Delta(\theta\psi^j) = \theta\left(\displaystyle\sum_{p=1}^{2} A_{pj}\psi^p + \sum_{p=1}^{2}\gamma^{j,pj}\mathbb{1}_{O_d^{pj}}\right) & \\ \qquad\qquad - 2\nabla\cdot(\nabla\theta\cdot\psi^j) + 2\Delta\theta\psi^j & \text{in } Q, \\ \theta\psi^j = 0 & \text{on } \Sigma, \\ (\theta\psi^j)(\cdot,T) = \theta\psi_T^j & \text{in } Q. \end{cases}$$

We apply item (ii.) of Proposition 4.1, replacing n by $n+3$, taking $m = (j,j)$, $u = \theta\psi^j$ and $\{q_m\} = \{\omega^{jj}\}$, and we have, for every $j = 1,2$, that

$$L_{n+3}^{jj}(\theta\psi^j) \leq C\Bigg(s^{n+3}\lambda^{n+4}\iint_{\omega^{jj}\times(0,T)} e^{-2s\sigma_{jj}}(\xi_{jj})^{n+3}|\theta\psi^j|^2\,dxdt$$
$$+ s^n\lambda^n\sum_{p=1}^{2}\iint_Q e^{-2s\sigma_{jj}}(\xi_{jj})^n|\theta|^2|\gamma^{j,pj}\mathbb{1}_{O_d^{pj}}|^2\,dxdt$$
$$+ s^n\lambda^n\sum_{p=1}^{2}\|A_{pj}\|_\infty^2\iint_Q e^{-2s\sigma_{jj}}(\xi_{jj})^{n+2}|\theta|^2|\psi^p|^2\,dxdt$$
$$+ s^{n+2}\lambda^{n+2}\int_0^T\!\!\int_{\mathcal{O}} e^{-2s\sigma_{jj}}(\xi_{jj})^{n+2}|\psi^j|^2\,dxdt\Bigg).$$

Using the definition of θ (see (40)), taking $\lambda \geq C$ and $s \geq \max_{p,j}\{s^{jj}, CT^2, CT^2\|A_{pj}\|^{\frac{2}{3}}\}$, we obtain

$$s^{n+3}\lambda^{n+4}\sum_{j=1}^{2}\iint_Q e^{-2s\sigma_{jj}}(\xi_{jj})^{n+3}|\psi^j|^2\,dxdt$$

$$\leq C \sum_{j=1}^{2} \Big(s^{n+3}\lambda^{n+4} \iint_{\mathcal{O}\times(0,T)} e^{-2s\sigma_{jj}} (\xi_{jj})^{n+3} |\psi^j|^2 \,dx\,dt$$
$$+ s^n\lambda^n \sum_{p=1}^{2} \iint_Q e^{-2s\sigma_{jj}} (\xi_{jj})^n |\theta|^2 |\gamma^{j,pj} \mathbb{1}_{O_d^{pj}}|^2 \,dx\,dt \Big).$$

By summing this last estimate with (39) and using the fact that all the weights coincide in the support of θ (see (30)), we absorb the terms of $\gamma^{j,pj}$ obtaining

$$\sum_{j,k,l=1}^{2} I_n^{kl}(\gamma^{j,kl}) + s^{n+3}\lambda^{n+4} \sum_{j=1}^{2} \iint_Q e^{-2s\sigma_{jj}} (\xi_{jj})^{n+3} |\psi^j|^2 \,dx\,dt$$
$$\leq C \sum_{j=1}^{2} \Big(s^{n+3}\lambda^{n+4} \iint_{\mathcal{O}\times(0,T)} e^{-2s\sigma_{jj}} (\xi_{jj})^{n+3} |\psi^j|^2 \,dx\,dt$$
$$+ \sum_{p=1}^{2} s^{n+2}\lambda^{n+3} \iint_{\hat{\omega}^{pj}\times(0,T)} e^{-2s\sigma_{pj}} (\xi_{pj})^{n+2} |\gamma^{j,pj}|^2 \,dx\,dt \Big), \quad (41)$$

for $\lambda \geq C$ and $s \geq \max_{k,l,p,j}\{s^{kl}, CT^2, CT^2\|A_{pj}\|^{\frac{2}{3}}\}$ sufficiently large.

The next computations are dedicated to estimating the local terms of $\gamma^{j,pj}$ in the right-hand side of (41). Using the fact that, for each $l \in \{1,2\}$, the sets $\tilde{\omega}^{kl}$ can be taken satisfying (22) for every $k \neq \tilde{k}$, we can see that

$$-\psi_t^j - \Delta\psi^j = \sum_{p=1}^{2} A_{pj}\psi^p + \gamma^{j,kj} \mathbb{1}_{O_d^{kj}} \quad \text{in } \tilde{\omega}^{kj}\times(0,T),$$

for any $k \in \{1,2\}$.

Let $\tilde{\theta}^{kj} \in C_0^2(\tilde{\omega}^{kj})$ such that $\tilde{\theta}^{kj} = 1$ in $\hat{\omega}^{kj}$. Then, for any $k \in \{1,2\}$ and ε small, we have that

$$s^{n+2}\lambda^{n+3} \iint_{\hat{\omega}^{kj}\times(0,T)} e^{-2s\sigma_{kj}} (\xi_{kj})^{n+2} |\gamma^{j,kj}|^2 \,dx\,dt$$
$$= s^{n+2}\lambda^{n+3} \iint_{\tilde{\omega}^{kj}\times(0,T)} \tilde{\theta}^{kj} (\xi_{kj})^{n+2} e^{-2s\sigma_{kj}} \gamma^{j,kj}$$
$$\cdot \Big(-\psi_t^j - \Delta\psi^j - \sum_{p=1}^{2} A_{pj}\psi^p \Big) \,dx\,dt$$
$$\leq \varepsilon I_n^{kj}(\gamma^{j,kj}) + Cs^{n+8}\lambda^{n+9} \sum_{p=1}^{2} \iint_{\tilde{\omega}^{pj}\times(0,T)} e^{-2s\sigma_{pj}} \xi_{pj}^{n+8} |\psi^p|^2 \,dx\,dt. \quad (42)$$

Finally, we combine (41) and (42) to obtain (33).

In what follows, we prove item (i.) for the case (C2), that is, for only one $l \in \{1,2\}$ that condition (13) will hold for any choice of $(k,\tilde{k})$ in $\{(1,2),(2,1)\}$.

Proof of (i.), *case* (C2)*:* For simplicity, we will assume that for $l = 1$ condition (13) holds only when $(k,\tilde{k}) = (1,2)$, and for $l = 2$, we assume that (13) holds for any $k \neq \tilde{k}$. Then, the open sets $\{\tilde{\omega}^{kl}\}$ can be taken in such a way that

$$\tilde{\omega}^{11} \subset \mathcal{O}_d^{21} \cap \mathcal{O} \quad \text{and} \quad \tilde{\omega}^{kl} \cap \mathcal{O}_d^{\tilde{k}l} = \emptyset, \tag{43}$$

for $l,k,\tilde{k} \in \{1,2\}$ with $k \neq \tilde{k}$ and $(k,l) \neq (1,1)$.

Defining $h^j = \gamma^{j,11} + \gamma^{j,21}$ for $j = 1,2$, the adjoint system (26) becomes

$$\begin{cases} -\psi_t^1 - \Delta\psi^1 = \displaystyle\sum_{p=1}^2 A_{p1}\psi^p + h^1 \mathbb{1}_{\mathcal{O}_d^{11}} - \gamma^{1,21}(\mathbb{1}_{\mathcal{O}_d^{11}} - \mathbb{1}_{\mathcal{O}_d^{21}}), & \text{in } Q, \\ -\psi_t^2 - \Delta\psi^2 = \displaystyle\sum_{p=1}^2 A_{p2}\psi^p + \sum_{p=1}^2 \gamma^{2,p2}\mathbb{1}_{\mathcal{O}_d^{p2}}, & \text{in } Q, \\ h_t^1 - \Delta h^1 = \displaystyle\sum_{p=1}^2 A_{1p}h^p - \sum_{p=1}^2 \frac{1}{\mu_{p1}}\psi^p \mathbb{1}_{\mathcal{O}_{p1}}, & \text{in } Q, \\ h_t^2 - \Delta h^2 = \displaystyle\sum_{k=1}^2 A_{2p}h^p, & \text{in } Q, \\ \gamma_t^{l,kl} - \Delta\gamma^{l,kl} = \displaystyle\sum_{p=1}^2 A_{lp}\gamma^{p,kl} - \frac{1}{\mu_{kl}}\psi^k \mathbb{1}_{\mathcal{O}_{kl}}, \quad (k,l) \neq (1,1) & \text{in } Q, \\ \gamma_t^{j,kl} - \Delta\gamma^{j,kl} = \displaystyle\sum_{p=1}^2 A_{jp}\gamma^{p,kl}, \quad (k,l) \neq (1,1) & \text{in } Q, \\ \psi^j = h^j = \gamma^{j,kl} = 0, \quad \text{for } j,k,l = 1,2 & \text{in } \Sigma, \\ \psi^j(T) = \psi^T, \quad h^j(0) = 0, \quad \gamma^{j,kl}(0) = 0, \quad \text{for } j,k,l = 1,2 & \text{in } \Omega. \end{cases} \tag{44}$$

Applying item (i.) of Proposition 3.3 to $\{(\gamma^{1,k2},\gamma^{2,k2})\}_{k=1}^2$, $(\gamma^{1,21},\gamma^{2,21})$ and (h^1,h^2), we obtain

$$\sum_{j=1}^2 \left(\sum_{\substack{k,l=1 \\ (k,l)\neq(1,1)}}^2 I_n^{kl}(\gamma^{j,kl}) + I_n^{11}(h^j) \right)$$
$$\leq C \sum_{j=1}^2 \left(s^n \lambda^{n+1} \sum_{\substack{k,l=1 \\ (k,l)\neq(1,1)}}^2 \iint_{\omega^{kl}\times(0,T)} e^{-2s\sigma_{kl}}(\xi_{kl})^n |\gamma^{j,kl}|^2 \, dxdt \right.$$

$$+ s^n\lambda^{n+1}\sum_{j=1}^{2}\iint_{\omega^{11}\times(0,T)} e^{-2s\sigma_{11}}(\xi_{11})^n|h^j|^2\,dxdt\Bigg)$$

$$+ Cs^{n-3}\lambda^{n-3}\sum_{p,k=1}^{2}\iint_{\mathcal{O}_{pk}\times(0,T)} e^{-2s\sigma_{pk}}(\xi_{pk})^{n-3}|\psi^p|^2\,dxdt, \qquad (45)$$

for $\lambda \geq C$ and $s \geq \max\{s^{kl}, CT^2\|A_{jp}\|^{\frac{2}{3}}\}_{k,l,j,p}$.

Using the equation of h^1 in (44), that is

$$A_{12}h^2 = h^1_t - \Delta h^1 - A_{11}h^1 + \sum_{p=1}^{2}\frac{1}{\mu_{p1}}\psi^p \mathbb{1}_{\mathcal{O}_{p1}},$$

and from assumption (21), and taking $\varepsilon > 0$ sufficiently small, we prove that

$$s^n\lambda^{n+1}\iint_{\omega^{11}\times(0,T)} e^{-2s\sigma_{11}}(\xi_{11})^n|h^2|^2\,dxdt$$
$$\leq \varepsilon I^{11}_n(h^2) + Cs^{n+4}\lambda^{n+5}\iint_{\widehat{\omega}^{11}\times(0,T)} e^{-2s\sigma_{11}}\xi_{11}^{n+4}|h^1|^2\,dxdt$$
$$+ Cs^n\lambda^{n+1}\sum_{p=1}^{2}\iint_{(\mathcal{O}_{p1}\cap\widehat{\omega}^{11})\times(0,T)} e^{-2s\sigma_{11}}\xi_{11}^n|\psi^p|^2\,dxdt. \qquad (46)$$

In a very similar way, we use (21) again, obtaining

$$s^n\lambda^{n+1}\iint_{\omega^{kl}\times(0,T)} e^{-2s\sigma_{kl}}(\xi_{kl})^n|\gamma^{j,kl}|^2\,dxdt$$
$$\leq \varepsilon I^{kl}_n(\gamma^{j,kl}) + Cs^{n+4}\lambda^{n+5}\iint_{\widehat{\omega}^{kl}\times(0,T)} e^{-2s\sigma_{kl}}\xi_{kl}^{n+4}|\gamma^{l,kl}|^2\,dxdt$$
$$+ Cs^n\lambda^{n+1}\iint_{(\mathcal{O}_{kl}\cap\widehat{\omega}^{kl})\times(0,T)} e^{-2s\sigma_{kl}}\xi_{kl}^n|\psi^k|^2\,dxdt, \qquad (47)$$

for $\lambda \geq C$ and $s \geq CT^2\|A_{jj}\|^{\frac{2}{3}}$, for $j,k,l = 1,2$ where $j \neq l$ and $(k,l) \neq (1,1)$.

Now, applying (ii.) of Proposition 3.3 for $\theta\psi^j$, where θ is given in (40), summing to (45) and using (46)–(47), we get

$$\sum_{j=1}^{2}\Bigg(\sum_{\substack{k,l=1\\(k,l)\neq(1,1)}}^{2} I^{kl}_n(\gamma^{j,kl}) + I^{11}_n(h^j)\Bigg)$$
$$+ s^{n+3}\lambda^{n+4}\sum_{j=1}^{2}\iint_Q e^{-2s\sigma_{jj}}(\xi_{jj})^{n+3}|\psi^j|^2\,dxdt$$

$$\leq C\Bigg(s^{n+3}\lambda^{n+4}\sum_{j=1}^{2}\iint_{\widetilde{\mathcal{O}}\times(0,T)} e^{-2s\sigma_{jj}}(\xi_{jj})^{n+3}|\psi^j|^2\,dxdt$$
$$+ s^{n+4}\lambda^{n+5}\iint_{\widehat{\omega}^{11}\times(0,T)} e^{-2s\sigma_{11}}\xi_{11}^{n+4}|h^1|^2\,dxdt$$
$$+ s^{n+2}\lambda^{n+3}\sum_{\substack{k,l=1\\(k,l)\neq(1,1)}}^{2}\iint_{\widehat{\omega}^{kl}\times(0,T)} e^{-2s\sigma_{kl}}(\xi_{kl})^{n+2}|\gamma^{l,kl}|^2\,dxdt\Bigg), \quad (48)$$

for $\lambda \geq C$ and $s \geq \max\{s^{kl}, CT^2\|A_{jp}\|^{\frac{2}{3}}\}_{k,l,j,p}$ sufficiently large. To absorb the local terms of $\gamma^{l,kl}$ and h^1, we use (43) and the first equation (44), to have

$$-\psi_t^1 - \Delta\psi^1 = \sum_{p=1}^{2} A_{p1}\psi^p + h^1 \quad \text{in } \widehat{\omega}^{11}\times(0,T). \quad (49)$$

Also, using again (43) and the first two equations of (44), we get

$$-\psi_t^l - \Delta\psi^l = \sum_{p=1}^{2} A_{pl}\psi^p + \gamma^{l,kl} \quad \text{in } \widehat{\omega}^{kl}\times(0,T), \quad (50)$$

for every $k, l = 1, 2$, $(k,l) \neq (1,1)$. Finally, from (48), (49) and (50), we can proceed in a similar way as in (42), and we obtain that

$$\sum_{j=1}^{2}\Bigg(\sum_{\substack{k,l=1\\(k,l)\neq(1,1)}}^{2} I_n^{kl}(\gamma^{j,kl}) + I_n^{11}(h^j)\Bigg)$$
$$+ s^{n+3}\lambda^{n+4}\sum_{j=1}^{2}\iint_{Q} e^{-2s\sigma_{jj}}(\xi_{jj})^{n+3}|\psi^j|^2\,dxdt$$
$$\leq Cs^{n+3}\lambda^{n+4}\sum_{j=1}^{2}\iint_{\widetilde{\mathcal{O}}\times(0,T)} e^{-2s\sigma_{jj}}(\xi_{jj})^{n+3}|\psi^j|^2\,dxdt. \quad (51)$$

Since we have h^j and $\gamma^{j,2j}$ in the left-hand side of (48), we can write $\gamma^{j,1j} = h^j - \gamma^{j,2j}$ to add the global terms of $\gamma^{j,1j}$ in the left-hand side of (52), and hence estimate (33) follows.

Remark 4.2. We have proved item (i.) of Proposition 4.1 by assuming that condition (C) holds, which corresponds to the case where (11) is valid for every $l = 1, 2$, and for two specific cases. The case where, for each $l \in \{1, 2\}$, property (13) can be verified for only one choice of $(k, \tilde{k}) \in \{(1, 2), (2, 1)\}$ is not considered, and it can be treated by adapting the proof of case (C2). Indeed, if we assume, for instance, that for $l = 1$

condition (13) holds only for $(k,\tilde{k})=(1,2)$ and for $l=2$ it holds only for $(k,\tilde{k})=(2,1)$, then the open sets $\{\widetilde{\omega}^{kl}\}$ can be taken in such a way that

$$\widetilde{\omega}^{ii}\subset\mathcal{O}_d^{ji}\cap\mathcal{O}\quad\text{and}\quad\widetilde{\omega}^{ji}\cap\mathcal{O}_d^{ii}=\emptyset\quad\text{for every } j\neq i.$$

In this case, we can see that equation (50) does not hold for $(k,l)=(2,2)$, and then we cannot absorb the local term of $\gamma^{2,22}$ in the right-hand side of (48). To overcome that, we define the function $p^j=\gamma^{j,12}+\gamma^{j,22}$, we use the equation satisfied by ψ^2 in $\widetilde{\omega}^{22}$, which is similar to (49), and we bound the local terms of p^j in a very similar way as we did for h^j.

Now, if (10) is valid for some $l\in\{1,2\}$, the analysis is much more simpler. Indeed, to fix the ideas, let us assume that

$$\mathcal{O}_d^{11}=\mathcal{O}_d^{21}.$$

In this case, we can define the same functions $h^j=\gamma^{j,11}+\gamma^{j,22}$ and we see that the equation satisfied by $\{\psi^j,h^j,\gamma^{j,kl}\}_{j,k,l=1}^2$ is similar to (44), with the only difference that the equation of ψ^1 turns into

$$-\psi_t^1-\Delta\psi^1=\sum_{p=1}^{2}A_{p1}\psi^p+h^1\mathbb{1}_{\mathcal{O}_d^{11}},\quad\text{in } Q.$$

This equation allows us to estimate a local term of h^1 in $\widetilde{\omega}^{11}$ in terms of ψ^1.

Now, we proceed to the proof of (ii.).

Proof of (ii.): In this case, we are going to take open sets satisfying (23), and also (24) and (25) for some (i,j). We are going to use item (i.) of Proposition 3.3 with $\{q_m\}=\{\omega^1,\omega^2\}$ and the functions $\{\eta_m\}$ given in Lemma 3.2.

Let $\{(\psi^i,\gamma^{ii},\gamma^{ji})\}_{i,j=1}^2$ a solution of (28). To fix the ideas, we are assuming that (24) and (25) are valid for $(i,j)=(1,2)$.

Using item (i.) of Proposition 3.3 for $(\gamma^{11},\gamma^{21})$ and replacing n by $n-3$, we get that

$$\begin{aligned}&I_{n-3}^1(\gamma^{11})+I_{n-3}^1(\gamma^{21})\\&\quad\le C\Bigg(s^{n-3}\lambda^{n-2}\iint_{\omega^1\times(0,T)}e^{-2s\sigma_1}(\xi_1)^{n-3}\big(|\gamma^{11}|^2+|\gamma^{21}|^2\big)\,dxdt\\&\qquad+s^{n-6}\lambda^{n-6}\iint_{\mathcal{O}_1\times(0,T)}e^{-2s\sigma_1}(\xi_1)^{n-6}|\psi^1|^2\,dxdt\Bigg),\end{aligned}\tag{52}$$

for $\lambda\ge C$ and $s\ge C(T+T^2(1+\max\{\|A_{ip}\|^{\frac{2}{3}}\}))$. Let $\widehat{\theta}^1\in C_0^2(\widehat{\omega}^1)$ be a positive function such that $\widehat{\theta}^1=1$ in ω^1. Using that $A_{12}\ge C_0>0$ in $\omega^1\times(0,T)$ (see (24)), we obtain that

$$
\begin{aligned}
s^{n-3}\lambda^{n-2} & \iint_{\omega^1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n-3}|\gamma^{21}|^2\,dxdt \\
& \le C s^{n-3}\lambda^{n-2} \iint_{\widehat{\omega}^1\times(0,T)} \widehat{\theta}^1 e^{-2s\sigma_1}(\xi_1)^{n-3}\gamma^{21} \\
& \quad \cdot\left(-\gamma^{11}_t - \Delta\gamma^{11} - A_{11}\gamma^{11} + \frac{1}{\mu_1}\psi^1 1\!\!1_{\mathcal{O}_1}\right)dxdt \\
& \le \varepsilon I^1_{n-3}(\gamma^{21}) + C s^{n+1}\lambda^{n+2} \iint_{\widehat{\omega}^1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n+1}|\gamma^{11}|^2\,dxdt \\
& \quad + s^{n-3}\lambda^{n-2} \iint_{(\widehat{\omega}^1\cap\mathcal{O}_1)\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n-3}|\psi^1|^2\,dxdt, \qquad (53)
\end{aligned}
$$

for $\varepsilon > 0$ sufficiently small. Combining (52), (53), and using that $\mathcal{O}\cap\mathcal{O}_1 = \emptyset$, we get that

$$
\begin{aligned}
& I^1_{n-3}(\gamma^{11}) + I^1_{n-3}(\gamma^{21}) \\
& \quad \le C\Bigg(s^{n+1}\lambda^{n+2} \iint_{\widehat{\omega}^1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n+1}|\gamma^{11}|^2\,dxdt \\
& \qquad + s^{n-6}\lambda^{n-6} \iint_{\mathcal{O}_1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n-6}|\psi^1|^2\,dxdt\Bigg), \qquad (54)
\end{aligned}
$$

for $\lambda \ge C$ and $s \ge C(T + T^2(1 + \max\{\|A_{ip}\|^{\frac{2}{3}}\}))$.

In a completely analogous way, using that $A_{21} \ge C_0 > 0$ in $\omega^2 \times (0,T)$ (see (25)), we can prove that

$$
\begin{aligned}
& I^2_{n-6}(\gamma^{22}) + I^2_{n-6}(\gamma^{12}) \\
& \quad \le C\Bigg(s^{n-2}\lambda^{n-1} \iint_{\widehat{\omega}^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-2}|\gamma^{22}|^2\,dxdt \\
& \qquad + s^{n-9}\lambda^{n-9} \iint_{\mathcal{O}_2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-9}|\psi^2|^2\,dxdt\Bigg), \qquad (55)
\end{aligned}
$$

for $\lambda \ge C$ and $s \ge C(T + T^2(1 + \max\{\|A_{ip}\|^{\frac{2}{3}}\}))$. Summing (54) and (55) we obtain

$$
\begin{aligned}
\sum_{i,j=1}^{2} I^i_{n-3i}(\gamma^{ji}) \le C\Bigg(& s^{n+1}\lambda^{n+2} \iint_{\widehat{\omega}^1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n+1}|\gamma^{11}|^2\,dxdt \\
& + s^{n-2}\lambda^{n-1} \iint_{\widehat{\omega}^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-2}|\gamma^{22}|^2\,dxdt
\end{aligned}
$$

$$
+ s^{n-6}\lambda^{n-6} \iint_{\mathcal{O}_1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n-6}|\psi^1|^2\,dxdt
+ s^{n-9}\lambda^{n-9} \iint_{\mathcal{O}_2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-9}|\psi^2|^2\,dxdt\Bigg). \quad (56)
$$

Let $\widetilde{\mathcal{O}}$ be an open set given in Lemma 3.2, and $\theta \in C^2(\overline{\Omega})$ be a function given by (40). From the first equation of system (28), we find that

$$
\begin{cases}
-(\theta\psi^1)_t - \Delta(\theta\psi^1) = \theta\Big(\displaystyle\sum_{p=1}^{2} A_{p1}\psi^p + \gamma^{11}\mathbb{1}_{\mathcal{O}_d^1}\Big) & \\
\qquad\qquad + \psi^1\Delta\theta - 2\nabla(\psi^1\nabla\theta) & \text{in } Q,\\
\theta\psi^1 = 0 & \text{on } \Sigma,\\
(\theta\psi^1)(\cdot,T) = \theta\psi^1_T & \text{in } Q.
\end{cases}
$$

By applying item (ii.) of Proposition 3.3, we obtain that

$$
\begin{aligned}
&s^n\lambda^{n+1}\iint_Q e^{-2s\sigma_1}(\xi_1)^n|\psi^1|^2\,dxdt\\
&\quad\le C\Bigg(s^n\lambda^{n+1}\iint_{\mathcal{O}\times(0,T)} e^{-2s\sigma_1}(\xi_1)^n|\psi^1|^2\,dxdt\\
&\qquad + s^{n-3}\lambda^{n-3}\sum_{p=1}^{2}\iint_Q e^{-2s\sigma_1}(\xi_1)^{n-3}|\theta|^2|A_{p1}\psi^p|^2\,dxdt\\
&\qquad + s^{n-3}\lambda^{n-3}\iint_{\mathcal{O}_d^1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n-3}|\theta|^2|\gamma^{11}|^2\,dxdt\Bigg)
\end{aligned}\quad (57)
$$

for $\lambda \ge C$ and $s \ge s^1$. Now, using item (i.) of Proposition 3.3 for ψ^2, we get

$$
\begin{aligned}
I^2_{n-3}(\psi^2) \le C\Bigg(&s^{n-3}\lambda^{n-2}\iint_{\omega^2\times(0,T)} (\xi_2)^{n-3}e^{-2s\sigma_2}|\psi^2|^2\,dxdt\\
&+ s^{n-6}\lambda^{n-6}\sum_{p=1}^{2}\iint_Q e^{-2s\sigma_2}(\xi_2)^{n-6}|A_{p2}\psi^p|^2\,dxdt\\
&+ s^{n-6}\lambda^{n-6}\iint_{\mathcal{O}_d^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-6}|\gamma^{22}|^2\,dxdt\Bigg),
\end{aligned}\quad (58)
$$

for $\lambda \ge C$ and $s \ge s^2$.

Summing (57) and (58), we can absorb the global terms of ψ^1 and ψ^2 by taking s large enough, obtaining

$$\begin{aligned} s^n\lambda^{n+1} \iint_Q e^{-2s\sigma_1}(\xi_1)^n|\psi^1|^2\,dxdt + I^2_{n-3}(\psi^2) \\ \leq C\Big(s^n\lambda^{n+1} \iint_{\mathcal{O}\times(0,T)} e^{-2s\sigma_1}(\xi_1)^n|\psi^1|^2\,dxdt \\ + s^{n-3}\lambda^{n-2} \iint_{\omega^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-3}|\psi^2|^2\,dxdt \\ + s^{n-3}\lambda^{n-3} \iint_{\mathcal{O}^1_d\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n-3}|\gamma^{11}|^2\,dxdt \\ + s^{n-6}\lambda^{n-6} \iint_{\mathcal{O}^2_d\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-6}|\gamma^{22}|^2\,dxdt \Big) \end{aligned} \tag{59}$$

for $\lambda \geq C$ and $s \geq \max_{i,j}\{s^j, CT^2\|A_{ij}\|^2_\infty\}$.

Summing (56) and (59), and absorbing the global terms of ψ^i and γ^{ii}, we obtain

$$\begin{aligned} s^n\lambda^{n+1} \iint_Q e^{-2s\sigma_1}(\xi_1)^n|\psi^1|^2\,dxdt + I^2_{n-3}(\psi^2) + \sum_{i,j=1}^{2} I^i_{n-3i}(\gamma^{ji}) \\ \leq C\Big(s^n\lambda^{n+1} \iint_{\mathcal{O}\times(0,T)} e^{-2s\sigma_1}(\xi_1)^n|\psi^1|^2\,dxdt \\ + s^{n-3}\lambda^{n-2} \iint_{\omega^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-3}|\psi^2|^2\,dxdt \\ + s^{n+1}\lambda^{n+2} \iint_{\widehat{\omega}^1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n+1}|\gamma^{11}|^2\,dxdt \\ + s^{n-2}\lambda^{n-1} \iint_{\widehat{\omega}^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-2}|\gamma^{22}|^2\,dxdt \Big), \end{aligned} \tag{60}$$

$\lambda \geq C$ and $s \geq \max_{i,j}\{s^j, CT^2(1+\|A_{ij}\|^2_\infty)\}$ large enough.

In order to absorb the local terms of γ^{ii} in the right-hand side of (60), we are going to use the equation of ψ^i in (28), and the fact that the sets $\widehat{\omega}^i$ satisfy (24). Indeed, let $\tilde{\theta}^i \in C^2_0(\widetilde{\omega}^i)$ be positive functions such that $\tilde{\theta}^i = 1$ in $\widehat{\omega}^i$. Then, we have that

$$\begin{aligned} s^{n+1}\lambda^{n+2} \iint_{\widehat{\omega}^1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n+1}\tilde{\theta}^1|\gamma^{11}|^2\,dxdt \\ = s^{n+1}\lambda^{n+2} \sum_{p=1}^{2} \iint_{\widetilde{\omega}^1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n+1}\tilde{\theta}^1\gamma^{11}(-\psi^1_t - \Delta\psi^1 - A_{11}\psi^1)\,dxdt \end{aligned}$$

$$\leq \varepsilon I^1_{n-3}(\gamma^{11}) + s^{n+9}\lambda^{n+10}\iint_{\widetilde{\omega}^1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n+9}|\psi^1|^2\,dxdt, \tag{61}$$

and

$$\begin{aligned} s^{n-6}\lambda^{n-5}&\iint_{\widehat{\omega}^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n+1}\tilde{\theta}^2|\gamma^{22}|^2\,dxdt \\ &\leq s^{n-6}\lambda^{n-5}\sum_{p=1}^{2}\iint_{\widehat{\omega}^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-6}\tilde{\theta}^2\gamma^{22} \\ &\qquad\qquad \cdot(-\psi^2_t - \Delta\psi^2 - A_{12}\psi_1 - A_{22}\psi^2)\,dxdt \\ &\leq \varepsilon I^2_{n-6}(\gamma^{22}) + s^{n-6}\lambda^{n-5}\iint_{\widetilde{\omega}^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-6}|\psi^1|^2\,dxdt \\ &\quad + s^{n-2}\lambda^{n-1}\iint_{\widetilde{\omega}^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-2}|\psi^2|^2\,dxdt, \end{aligned} \tag{62}$$

for $\varepsilon > 0$ sufficiently small. Combining (60), (61) and (62), we get that

$$\begin{aligned} s^n\lambda^{n+1}&\iint_Q e^{-2s\sigma_1}(\xi_1)^n|\psi^1|^2\,dxdt + I^2_{n-3}(\psi^2) + \sum_{i,j=1}^{2} I^i_{n-3i}(\gamma^{ji}) \\ &\leq C\Bigg(s^{n+9}\lambda^{n+10}\iint_{\mathcal{O}\times(0,T)}\big(e^{-2s\sigma_1}(\xi_1)^{n+9} + e^{-2s\sigma_2}(\xi_2)^{n-6}\big)|\psi^1|^2\,dxdt \\ &\quad + s^{n-2}\lambda^{n-1}\iint_{\widetilde{\omega}^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-2}|\psi^2|^2\,dxdt\Bigg). \end{aligned} \tag{63}$$

Since we are assuming (25), we can take a positive function $\theta^2_* \in C^2_0(\omega^2_*)$ such that $\theta^2_* = 1$ in $\widetilde{\omega}^2$, and then

$$\begin{aligned} s^{n-2}\lambda^{n-1}&\iint_{\widetilde{\omega}^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-2}\theta_*|\psi^2|^2\,dxdt \\ &= s^{n-2}\lambda^{n-1}\iint_{\omega^2_*\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-2}\theta^2_*\psi^2(-\psi^1_t - \Delta\psi^1 - A_{11}\psi^1)\,dxdt \\ &\leq \varepsilon I^2_{n-3}(\psi^2) + s^{n+3}\lambda^{n+4}\iint_{\omega^2_*\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n+3}|\psi^1|^2\,dxdt. \end{aligned} \tag{64}$$

Combining (63) and (64) we obtain (34) for $\lambda \geq C$ and $s \geq \max_{i,j}\{s^j, CT^2\|A_{ij}\|^2_\infty\}$ large enough. We remind that for the proof of (ii.), we have considered the case $(i, j) = (1, 2)$. The case where $(i, j) = (2, 1)$ is completely analogous. ∎

Next section is dedicated to the proof of Theorem 3.1.

5. Observability inequalities

In this section, we are going to use Proposition 4.1 to prove Theorem 3.1. To do that, we are going to combine the Carleman estimates given in Proposition 4.1 with suitable energy estimates to the solutions of (26). Here, we will concentrate on the proof of (27), since the proof of (29) follows in a completely analogous way.

Let $\{\psi^j, \gamma^{j,kl}\}$ be a solution of (26). By energy estimates, we have that

$$\sum_{p=1}^{2} \|\gamma^{p,kl}(\cdot,t)\|_2^2 \leq \frac{C}{\mu_{kl}} \int_0^t \|\psi^k(\cdot,s)\|_2^2\, ds, \quad k,l = 1,2,\ t \in [0,T], \tag{65}$$

and

$$\begin{aligned}\sum_{p=1}^{2} \|\psi^p(\cdot,t)\|_2^2 \leq C &\sum_{p=1}^{2} \|\psi^p(\cdot,t')\|_2^2 \\ &+ (1 + \max_{i,j}\{\|A_{ij}\|_\infty^2\}) \int_t^{t'} \sum_{p=1}^{2} \|\psi^p(\cdot,s)\|^2\, ds \\ &+ C \int_t^{t'} \sum_{j,p=1}^{2} \|\gamma^{j,pj}(\cdot,s)\|^2\, ds.\end{aligned} \tag{66}$$

Combining (65) and (66), we have that

$$\begin{aligned}\sum_{p=1}^{2} \|\psi^p(\cdot,t)\|_2^2 \leq C &\sum_{p=1}^{2} \|\psi^p(\cdot,t')\|_2^2 \\ &+ (1 + \max_{i,j}\{\|A_{ij}\|_\infty^2\}) \int_t^{t'} \sum_{p=1}^{2} \|\psi^p(\cdot,s)\|^2\, ds \\ &+ C \sum_{p,j=1}^{2} \frac{T}{\mu_{pj}} \int_0^{t'} \|\psi^j(\cdot,s)\|_2^2\, ds, \quad k,l = 1,2,\end{aligned}$$

for every $0 \leq t < t' \leq T$. Now, we apply Gronwall's lemma and we get that

$$\begin{aligned}\sum_{p=1}^{2} \|\psi^p(\cdot,t)\|_2^2 \leq C e^{(1+\max_{ij}\{\|A_{ij}\|_\infty^2\})(t'-t)} &\sum_{p=1}^{2} \Bigg(\|\psi^p(\cdot,t')\|_2^2 \\ &+ \sum_{j=1}^{2} \frac{T}{\mu_{pj}} \int_0^{t'} \|\psi^j(\cdot,s)\|_2^2\, ds \Bigg).\end{aligned}$$

Integrating over $[0, t']$ and taking, if necessary, μ_{pj} even larger, we obtain that

$$\sum_{p=1}^{2} \int_0^{t'} \|\psi^p(\cdot, s)\|_2^2 \, ds \le C e^{(1+\max_{ij}\{\|A_{ij}\|_\infty^2\})(t'-t)} \sum_{p=1}^{2} \|\psi^p(\cdot, t')\|_2^2. \tag{67}$$

Combining (65) and (67), we get that

$$\sum_{p=1}^{2} \left(\|\psi^p(\cdot, t)\|_2^2 + \sum_{k,l=1}^{2} \|\gamma^{p,kl}(\cdot, t)\|_2^2 \right) \le C e^{(1+\max_{ij}\{\|A_{ij}\|_\infty^2\})(t'-t)} \sum_{p=1}^{2} \|\psi^p(\cdot, t')\|_2^2, \quad 0 \le t < t' \le T. \tag{68}$$

We remark that energy estimates such as (68) are necessary for the well-posedness of system (26). In this way, when we have assumed at the beginning of this paper that μ (and $\widetilde{\mu}$) is large (see Remark 2.4), we are assuming implicitly that (68) is valid. So, in this sense, we do not have to assume here that μ or $\widetilde{\mu}$ are larger than before.

Now, let $\upsilon : [0, T] \to \mathbb{R}$ be such that $\upsilon = 1$ in $[0, T/4]$ and $\upsilon = 0$ in $[3T/4, T]$. If we apply a similar argument to proving (68) to the functions $\{\upsilon\gamma^j, \upsilon\gamma^{j,kl}\}_{j,k,l}$, and taking $t' = T$, we find that

$$\begin{aligned} \sum_{p=1}^{2} \left(\|\upsilon(t)\psi^p(\cdot, t)\|_2^2 + \sum_{k,l=1}^{2} \|\upsilon(t)\gamma^{p,kl}(\cdot, t)\|_2^2 \right) &\le C e^{(1+\max_{ij}\{\|A_{ij}\|_\infty^2\})(T-t)} \int_0^T \sum_{p=1}^{2} \Bigg(\|\upsilon_t(t)\psi^p(\cdot, s)\|_2^2 \\ &\quad + \sum_{k,l=1}^{2} \|\upsilon_t(s)\gamma^{p,kl}(\cdot, s)\|_2^2 \Bigg) ds, \end{aligned}$$

for $t \in [0, T]$. In particular, we have that

$$\begin{aligned} \sum_{p=1}^{2} \left(\|\psi^p\|_{L^\infty(0,T/2;L^2(\Omega))}^2 + \sum_{k,l=1}^{2} \|\gamma^{p,kl}\|_{L^\infty(0,T/2;L^2(\Omega))}^2 \right) &\le C e^{(1+\max_{ij}\{\|A_{ij}\|_\infty^2\})(T-t)} \sum_{p=1}^{2} \Bigg(\|\psi^p\|_{L^2(T/2,3T/4;L^2(\Omega))}^2 \\ &\quad + \sum_{k,l=1}^{2} \|\gamma^{p,kl}\|_{L^2(T/2,3T/4;L^2(\Omega))}^2 \Bigg). \end{aligned} \tag{69}$$

Now, we define the new weight functions

$$l(t) = \begin{cases} T^2/4 & \text{in } [0, T/2], \\ t(T-t) & \text{in } [T/2, T], \end{cases}$$

and for a family of sets $\{\eta_m\}_{m\in\Lambda}$ given by Lemma 3.2, we define

$$\tilde{\sigma}_m(x,t) := \frac{e^{4\lambda\|\eta_m\|_\infty} - e^{\lambda(2\|\eta_m\|_\infty + \eta_m(x))}}{l(t)},$$
$$\tilde{\xi}_m(x,t) := \frac{e^{\lambda(2\|\eta_m\|_\infty + \eta_m(x))}}{l(t)}, \quad m \in \Lambda.$$

Then, we have that

$$\min\{e^{-2s\sigma_{kl}}\xi_{kl}^q, e^{-2s\tilde{\sigma}_{kl}}\tilde{\xi}_{kl}^q\} \geq e^{-C_{\max}s/T^2}\frac{1}{T^{2q}} \quad \text{in } \Omega \times \left(\frac{T}{4}, \frac{3T}{4}\right),\ k,l = 1,2,$$

and

$$\max\{e^{-2s\sigma_{kl}}\xi_{kl}^q, e^{-2s\tilde{\sigma}_{kl}}\tilde{\xi}_{kl}^q\} \leq e^{-C_{\min}s/T^2}\frac{1}{T^{2q}} \quad \text{in } Q,\ k,l = 1,2,$$

where

$$C_{\max} = e^{4\lambda\|\eta_m\|_\infty} - e^{2\lambda\|\eta_m\|_\infty} \quad \text{and} \quad C_{\min} = e^{4\lambda\|\eta_m\|_\infty} - e^{3\lambda\|\eta_m\|_\infty}.$$

In this way, using (69), we have that

$$\begin{aligned}
&\sum_{p=1}^{2}\left(\int_\Omega |\psi^p(0)|^2\,dx + \sum_{k,l=1}^{2}\int_\Omega\int_0^{T/2} e^{-2s\tilde{\sigma}_{kl}}\tilde{\xi}_{kl}^n|\gamma^{p,kl}|^2\,dx\,dt\right) \\
&\leq \frac{C}{T^{2n}}e^{-C_{\min}s/T^2}e^{(1+\max_{ij}\{\|A_{ij}\|_\infty^2\})T}\sum_{p=1}^{2}\Bigg(\|\psi^p\|^2_{L^2(T/2,3T/4;L^2(\Omega))} \\
&\quad + \sum_{k,l=1}^{2}\|\gamma^{p,kl}\|^2_{L^2(T/2,3T/4;L^2(\Omega))}\Bigg) \\
&\leq Ce^{(C_{\max}-C_{\min})s/T^2}e^{(1+\max_{ij}\{\|A_{ij}\|_\infty^2\})T}\Bigg(\sum_{j,k,l=1}^{2} I_n^{kl}(\gamma^{j,kl}) \\
&\quad + \sum_{j=1}^{2}\iint_Q e^{-2s\sigma_{jj}}(\xi_{jj})^{n+3}|\psi^j|^2\,dx\,dt\Bigg).
\end{aligned} \tag{70}$$

Combining (70) and (33) we get that

$$\sum_{p=1}^{2}\left(\int_\Omega |\psi^p(0)|^2\,dx + \sum_{k,l=1}^{2}\int_\Omega\int_0^{T/2} e^{-2s\tilde{\sigma}_{kl}}\tilde{\xi}_{kl}^n|\gamma^{p,kl}|^2\,dxdt\right)$$
$$\leq Ce^{C(1+\frac{1}{T}+T+\max\|A_{ij}\|^{\frac{2}{3}}+T\max_{ij}\|A_{ij}\|_\infty)}\sum_{p=1}^{2}\iint_{\mathcal{O}\times(0,T)}|\psi^p|^2\,dxdt.$$

To complete the proof, we just use the fact that the weights coincide in $(T/2,T)$, and we obtain

$$\sum_{k,l=1}^{2}\int_\Omega\int_{T/2}^{T} e^{-2s\tilde{\sigma}_{kl}}\xi_{kl}^n|\gamma^{p,kl}|^2\,dxdt$$
$$\leq I_n^{kl}(\gamma^{j,kl}) \leq C\sum_{p=1}^{2}\iint_{\mathcal{O}\times(0,T)}|\psi^p|^2\,dxdt.$$

Hence, we have proved (27) with $C_* = Ce^{C(1+\frac{1}{T}+T+\max\|A_{ij}\|^{\frac{2}{3}}+T\max_{ij}\|A_{ij}\|_\infty)}$ and $\rho(t) = \min_{x\in\Omega} e^{-2s\tilde{\sigma}_{kl}}\xi_{kl}^n$.

The proof of (29) is entirely analogous.

6. Comments and open questions

6.1. On the conditions on $\mathcal{O}_d^{kl}$ and $\mathcal{O}_d^k$

In this paper, we have assumed some similar conditions to the ones in [4, 5], these conditions are given essentially in (10)–(14) and (18)–(19). It is an interesting open question how to prove Theorem 2.1 when, for some $l \in \{1,2\}$, the sets $\{\mathcal{O}_d^{k,l}\}_{k=1}^2$ coincide only inside $\mathcal{O}$. We remark that, even for the case of one single equation, a similar open problem arises (see [4]). Another open question is the one of proving Theorem 2.2 with weaker conditions than (18) and (19). Again, we do not know how to deal with the case where $\{\mathcal{O}_d^k\}_{k=1}^2$ coincide only inside $\mathcal{O}$ and, additionally, how to proceed if $\mathcal{O}_d^i \cap \mathcal{O} \subset \operatorname{supp} A_{ij}$ for every $i \neq j$.

6.2. On the functionals (4) and (15)

In the results of [11], the authors have solved a similar result to Theorem 2.2 for the specific case where

$$J^i(\{v^j\}) = \alpha_i \iint_{\mathcal{O}_d^i\times(0,T)} |y-y_d^1|^2 + |y-y_d^2|^2\,dxdt + \mu_i\iint_{\mathcal{O}_i\times(0,T)}|v^i|\,dxdt, \tag{71}$$

and assuming

$$\mathcal{O}_d^1 = \mathcal{O}_d^2.$$

The reason why we have defined functionals (15) instead of (71) is that, by doing so, we simplify some computations appearing in the proofs. In any case, by applying similar arguments to the ones contained in here, and assuming similar condition as (18) and (19), one can solve the problem of [11] even when $\mathcal{O}_d^1 \neq \mathcal{O}_d^2$.

6.3. On the quantity of equations considered

A natural open question which arises is the one of proving similar results when we have m equations for $m > 2$. The difficulty in this case is the proof of estimates (38), (61), and (62). For each (k,l) fixed, the variables $\{\gamma^{j,kl}\}_{j=1}^m$ are solutions of a system of m equations, and for each $j \neq l$ we have to estimate the local terms of $\gamma^{j,kl}$ by local terms of $\gamma^{l,kl}$. This problem can be compared to the one of controlling a general system of m equations with one single control, which is a completely open problem. Even if the coupling coefficients are in cascade, we do not know how we can do it.

6.4. On the boundary control problem

It is an interesting problem to consider the case where some of the controls are positioned into the boundary of the domain $\partial\Omega$. Three cases are of particular interest:

(i) Distributed leader and boundary followers. In this case, the followers are positioned in sub-regions $\mathcal{S}_{ij}$ of the boundary $\partial\Omega$ and the control system is the following:

$$\begin{cases} y_t^1 - \Delta y^1 = \displaystyle\sum_{p=1}^{2} A_{1p} y^p + f^1 \mathbb{1}_{\mathcal{O}} & \text{in } Q, \\ y_t^2 - \Delta y^2 = \displaystyle\sum_{p=1}^{2} A_{2p} y^p + f^2 \mathbb{1}_{\mathcal{O}} & \text{in } Q, \\ y^1 = v^{11}\rho_{11} + v^{12}\rho_{12} & \text{in } \Sigma, \\ y^2 = v^{21}\rho_{21} + v^{22}\rho_{22} & \text{in } \Sigma, \\ y^1(0) = y_0^1, \quad y^2(0) = y_0^2 & \text{in } \Omega, \end{cases}$$

where ρ_{ij} are $C^2(\partial\Omega)$ such that

$$0 < \rho_{i,j} \leq 1 \quad \text{on } \mathcal{S}_{ij} \quad \text{and} \quad \rho_{i,j} = 0 \quad \text{on } \partial\Omega \setminus \mathcal{S}_{ij}, \quad \text{for } i,j = 1,2.$$

In this case, the functionals are defined by

$$J^{kl}(\{v^{ij}\}_{i,j=1}^{2}) = \frac{1}{2}\iint_{\mathcal{O}_d^{kl}\times(0,T)} |y^l - y_d^{kl}|^2\,dx\,dt + \frac{\mu_{kl}}{2}\iint_{\mathcal{S}_{kl}\times(0,T)} |v^{kl}|^2\,dx\,dt, \quad k,l = 1,2. \tag{72}$$

If we search for a Nash equilibrium to the costs (72), then similar conditions to (6) must hold. In this case, for $j = 1,2$, the optimality system becomes

$$\begin{cases} y_t^j - \Delta y^j = \sum_{p=1}^{2} A_{jp} y^p + f^j \mathbb{1}_{\mathcal{O}}, & \text{in } Q, \\ y^j = \sum_{p=1}^{2} \frac{1}{\mu_{jp}} \frac{\partial}{\partial \nu} \varphi^{p,jp} \rho_{jp}, & \text{in } \Sigma, \\ y^j(0) = y_0^j, & \text{in } \Omega, \end{cases}$$

where $\{\varphi^{j,kl}\}$ satisfies the same equation (8). Note that in this case, the controllability problem becomes controlling a system of 10×10 equations with two control forces (f^1, f^2) where some of the couplings are localized into the boundary. In this case, the strategy can be the proof of an observability inequality with the same aspect as in (27), but now the adjoint system is

$$\begin{cases} -\psi_t^j - \Delta\psi^j = \sum_{p=1}^{2} A_{pj}\psi^p + \sum_{p=1}^{2} \gamma^{j,pj} \mathbb{1}_{O_d^{pj}} & \text{in } Q, \\ \gamma_t^{j,kl} - \Delta\gamma^{j,kl} = \sum_{p=1}^{2} A_{jp}\gamma^{p,kl}, \quad j \neq l, & \text{in } Q, \\ \gamma_t^{l,kl} - \Delta\gamma^{l,kl} = \sum_{p=1}^{2} A_{lp}\gamma^{p,kl} & \text{in } Q, \\ \psi^j = \gamma^{j,kl} = 0, \quad j \neq l, & \text{in } \Sigma, \\ \gamma^{l,kl} = \frac{1}{\mu_{kl}} \frac{\partial}{\partial\nu} \psi^k \rho_{kl}, & \text{in } \Sigma, \\ \psi^j(T) = \psi_j^T, \quad \gamma^{j,kl}(0) = 0, & \text{in } \Omega. \end{cases}$$

We can deal with this situation in a very similar way. The main difference is that in place of using the usual Carleman estimate of [10], as in (35) for $\{\gamma^{j,kl}\}$, we apply a refined version of it, proved in [12], for the cases where we have non-homogeneous boundary conditions. The residual terms appearing from that can be absorbed by standard energy estimates. We can cite [3] where the authors deal with this situation in the context of one single heat equation. The adaptation to the case considered in this paper is straightforward.

(ii) Boundary leader and distributed followers. In this case, there is a sub-region $\mathcal{S}$ where the leader (f_1, f_2) actuate. The followers are localized in sub-regions in the interior of Ω. Then, the control system is

$$\begin{cases} y_t^1 - \Delta y^1 = \sum_{p=1}^{2} A_{1p} y^p + v^{11} \mathbb{1}_{\mathcal{O}_{11}} + v^{12} \mathbb{1}_{\mathcal{O}_{12}} & \text{in } Q, \\ y_t^2 - \Delta y^2 = \sum_{p=1}^{2} A_{2p} y^p + v^{21} \mathbb{1}_{\mathcal{O}_{21}} + v^{22} \mathbb{1}_{\mathcal{O}_{22}} & \text{in } Q, \\ y^1 = f_1 \mathbb{1}_{\mathcal{S}}, \quad y^2 = f_2 \mathbb{1}_{\mathcal{S}} & \text{in } \Sigma, \\ y^1(0) = y_0^1, \quad y^2(0) = y_0^2 & \text{in } \Omega. \end{cases}$$

Again, the strategy consists in combining the ideas of this article with the ones in [3]. The functions to be minimized are defined by the same formula (4). The optimality system is

$$\begin{cases} y_t^j - \Delta y^j = \sum_{p=1}^{2} A_{jp} y^p - \sum_{p=1}^{2} \frac{1}{\mu_{jp}} \varphi^{p,jp} \mathbb{1}_{\mathcal{O}_{jp}}, & \text{in } Q, \\ y^j = f^j \mathbb{1}_{\mathcal{S}}, & \text{in } \Sigma, \\ y^j(0) = y_0^j, & \text{in } \Omega, \end{cases}$$

for $j = 1, 2$, where $\{\varphi^{j,kl}\}$ satisfies system (8). Therefore, we can see that the corresponding adjoint system coincides with system (26). The main difference in this case is that, in place of the distributed observability estimate (27), we will need a boundary version of it of the form

$$\sum_{k=1}^{2} \int_{\Omega} |\psi^k(0)|^2 \, dx + \sum_{k,l=1}^{2} \iint_{\mathcal{O}_d^{kl} \times (0,T)} \rho^2(t) |\gamma^{l,kl}|^2 \, dx dt \\ \leq C \sum_{k=1}^{2} \iint_{\mathcal{S} \times (0,T)} |\psi^k|^2 \, dx dt. \tag{73}$$

The strategy to prove (73) follows similarly to the proof for one heat equation (see [3]). Here we will present only the steps to follow, the detailed proof can be made by following the computation of [3, Appendix B]:

(1°) Use boundary Carleman estimates for $\{\psi^k\}$ and $\{\gamma^{j,kl}\}$ in the region $\mathcal{S}$.

(2º) For $k,l \in \{1,2\}$, we use a weighted energy estimate for the system $\{\gamma^{j,kl}\}_{j=1}^{2}$ obtaining an estimate for

$$\|\rho^*\nabla\gamma^{j,kl}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\rho^*\Delta\gamma^{j,kl}\|^2_{L^\infty(0,T;L^2(\Omega))}$$

in terms of $\frac{1}{\mu_{kl}^2}\|\hat{\rho}\psi\|^2_{L^2(\mathcal{O}\times(0,T))}$, where ρ^* and $\hat{\rho}$ goes exponentially to zero as $t \to T$.

(3º) The conclusions follow by combining a trace theorem for $\{\gamma^{j,kl}\}$, the energy estimate obtained in the (2º) step, and taking μ_{kl} sufficiently large.

(iii) Boundary leader and boundary followers. In this case, the leader and followers are assumed in the boundary,

$$\begin{cases} y_t^1 - \Delta y^1 = \displaystyle\sum_{p=1}^{2} A_{1p}y^p & \text{in } Q, \\ y_t^2 - \Delta y^2 = \displaystyle\sum_{p=1}^{2} A_{2p}y^p & \text{in } Q, \\ y^1 = f_1\rho 1\!\!1_{\mathcal{S}} + v^{11}\rho_{\mathcal{S}_{11}} + v^{12}\rho_{\mathcal{S}_{12}} & \text{in } \Sigma, \\ y^2 = f_2 1\!\!1_{\mathcal{S}} + v^{21}\rho_{\mathcal{S}_{21}} + v^{22}\rho_{\mathcal{S}_{22}} & \text{in } \Sigma, \\ y^1(0) = y_0^1, \quad y^2(0) = y_0^2 & \text{in } \Omega. \end{cases}$$

In this case, the optimality system becomes

$$\begin{cases} y_t^j - \Delta y^j = \displaystyle\sum_{p=1}^{2} A_{jp}y^p & \text{in } Q, \\ y^j = f^j 1\!\!1_{\mathcal{S}} + \displaystyle\sum_{p=1}^{2} \frac{1}{\mu_{jp}}\frac{\partial}{\partial\nu}\varphi^{p,jp}\rho_{jp} & \text{in } \Sigma, \\ y^j(0) = y_0^j & \text{in } \Omega, \end{cases}$$

where $\{\varphi^{j,kl}\}$ satisfies system (8). This case is more complicated than the others since we have a boundary controllability problem with boundary couplings. Up to now, we have no ideas on how we can deal with this case, but it will be considered in future works.

6.5. Semilinear systems

Another interesting question is the possible extension of the main results of this paper to semilinear parabolic systems. Consider a system of coupled heat equations of the form

$$\begin{cases} y^1_t - \Delta y^1 = \sum_{p=1}^{2} A_{1p} y^p + \mathcal{F}_1(y^1, y^2) + f^1 \mathbb{1}_{\mathcal{O}} + v^{11}\mathbb{1}_{\mathcal{O}_{11}} + v^{12}\mathbb{1}_{\mathcal{O}_{i2}} & \text{in } Q, \\ y^2_t - \Delta y^2 = \sum_{p=1}^{2} A_{2p} y^p + \mathcal{F}_2(y^1, y^2) + f^2 \mathbb{1}_{\mathcal{O}} + v^{21}\mathbb{1}_{\mathcal{O}_{21}} + v^{22}\mathbb{1}_{\mathcal{O}_{22}} & \text{in } Q, \\ y^1 = y^2 = 0 & \text{on } \Sigma, \\ y^1(0) = y^1_0, \quad y^2(0) = y^2_0 & \text{in } \Omega, \end{cases} \tag{74}$$

where (f^1, f^2) is the leader, $\{v^{ij}\}_{i,j=1}^2$ the followers and, for $i = 1, 2$, the functions $\mathcal{F}_i : \mathbb{R}^2 \to \mathbb{R}$ are locally Lipschitz-continuous. The main issue when considering nonlinear systems is that functions $\{v^{ij}\}_{i,j=1}^2$ satisfying (6) are not necessarily minimum of the corresponding functionals $\{J^{kl}\}_{k,l=1}^2$. In this way, in order to ensure that the critical points are indeed a Nash equilibrium, we can analyze the positiveness of the second derivatives of $\{J^{kl}\}_{k,l=1}^2$. Let us give a sketch on how the results of this paper can be extended to a semilinear framework.

Following the ideas of [5], we see that functions $\{v^{ij}\}$ satisfy (6) if and only if, for each $k, l = 1, 2$,

$$v^{kl} = -\frac{1}{\mu_{kl}} \varphi^{l,kl} \tag{75}$$

where $\{\varphi^{i,kl}\}_{i=1}^2$ are solutions of the system

$$\begin{cases} -\varphi^{l,kl}_t - \Delta \varphi^{l,kl} = \sum_{p=1}^{2} A_{pl} \varphi^{p,kl} + \sum_{i=1}^{2} \frac{\partial \mathcal{F}_i}{\partial x_l}(y^1, y^2)\varphi^{i,kl} \\ \qquad\qquad + (y^l - y^{kl}_d)\mathbb{1}_{\mathcal{O}^{kl}_d}, & \text{in } Q, \\ -\varphi^{j,kl}_t - \Delta \varphi^{j,kl} = \sum_{p=1}^{2} A_{pj} \varphi^{p,kl} + \sum_{i=1}^{2} \frac{\partial \mathcal{F}_i}{\partial x_j}(y^1, y^2)\varphi^{i,kl}, \quad j \neq l, & \text{in } Q, \\ \varphi^{j,kl} = 0, & \text{on } \Sigma, \\ \varphi^{j,kl}(\cdot, T) = 0, & \text{in } \Omega, \end{cases} \tag{76}$$

for $j, k, l = 1, 2$.

At this point, there are two main problems:

(I.) prove the null controllability of (74)–(76) by the action of the leaders (f_1, f_2);

(II.) prove that the followers given by (75) are a Nash equilibrium, which is not immediate since we are in a nonlinear framework.

The proof of (I.) can be obtained by following very similar ideas as the ones contained in [5, Section 3.2]. After alinearization around a given trajectory, we prove a linear controllability result very similar to the one stated in Theorem 2.1 of the present paper. After that, we can combine the Aubin Lions Compactness Theorem and Schauder's Fixed Point Theorem to get the null controllability of (74). In this step it is convenient to assume that $\mathcal{F}_i \in W^{1,\infty}(\mathbb{R}^2)$, for $i = 1, 2$.

The proof of (II.) requires some additional assumptions and to see that clearly we will give a sketch of it. To simplify the writing, we will fix the functional J^{11} and will show that the followers characterized by (75) satisfy

$$J^{11}(\{v^{ij}\}_{i,j=1}^2) = \min_{\hat{v}^{11} \in L^2(\mathcal{O}_{11}\times(0,T))} J^{11}(\hat{v}^{11}, v^{12}, v^{21}, v^{22}). \tag{77}$$

The computations for the other functional follows similarly.

Proceeding in a very similar way as in [5, Section 3.3], if $(v^{11}, v^{12}, v^{21}, v^{22})$ are functions satisfying (6), one can obtain that

$$\begin{aligned}&\langle D_1^2 J^{11}(v^{11}, v^{12}, v^{21}, v^{22}), (\hat{v}^{11})^2\rangle \\ &\quad = \iint_{\mathcal{O}_d^{11}\times(0,T)} \theta^1 \hat{v}^{11}\, dxdt + \mu_{11} \iint_{\mathcal{O}_{11}\times(0,T)} |\hat{v}^{11}|^2\, dxdt,\end{aligned}$$

where

$$\begin{cases}
-\theta_t^1 - \Delta\theta^1 = \displaystyle\sum_{p=1}^2 A_{p1}\theta^p + \sum_{i=1}^2 \left(\nabla \frac{\partial \mathcal{F}_i}{\partial x_1}(y^1, y^2)\cdot(w^1, w^2)\phi^i \right. & \\
\qquad\qquad \left. + \displaystyle\frac{\partial \mathcal{F}_i}{\partial x_1}(y^1, y^2)\theta^i\right) + w^1 \mathbb{1}_{\mathcal{O}_d^{11}} & \text{in } Q,\\
-\theta_t^2 - \Delta\theta^2 = \displaystyle\sum_{p=1}^2 A_{p2}\theta^p + \sum_{i=1}^2 \left(\nabla \frac{\partial \mathcal{F}_i}{\partial x_2}(y^1, y^2)\cdot(w^1, w^2)\phi^i \right. & \\
\qquad\qquad \left. + \displaystyle\frac{\partial \mathcal{F}_i}{\partial x_2}(y^1, y^2)\theta^i\right) & \text{in } Q,\\
-\phi_t^1 - \Delta\phi^1 = \displaystyle\sum_{p=1}^2 A_{p1}\phi^p + \sum_{i=1}^2 \frac{\partial \mathcal{F}_i}{\partial x_1}(y^1, y^2)\phi^i + (y^1 - y_d^{11})\mathbb{1}_{\mathcal{O}_d^{11}} & \text{in } Q,\\
-\phi_t^2 - \Delta\phi^2 = \displaystyle\sum_{p=1}^2 A_{p2}\phi^p + \sum_{i=1}^2 \frac{\partial \mathcal{F}_i}{\partial x_2}(y^1, y^2)\phi^i & \text{in } Q,\\
w_t^1 - \Delta w^1 = \displaystyle\sum_{p=1}^2 A_{1p}w^p + \nabla\mathcal{F}_1(y^1, y^2)\cdot(w^1, w^2) + \hat{v}^{11}\mathbb{1}_{\mathcal{O}_{11}} & \text{in } Q,\\
w_t^2 - \Delta w^2 = \displaystyle\sum_{p=1}^2 A_{2p}w^p + \nabla\mathcal{F}_2(y^1, y^2)\cdot(w^1, w^2) & \text{in } Q,
\end{cases} \tag{78}$$

supplemented with null Dirichlet boundary conditions and initial data. In this way, to prove that $\{v^{ij}\}$ satisfies (77), it is sufficient to show that $D_1^2 J^{11}(v^{11}, v^{12}, v^{21}, v^{22})$ is a positive definite bilinear form.

Using the fifth equation of (78) and by integration by parts we can obtain that

$$\iint_{\mathcal{O}_d^{11}\times(0,T)} \theta^1 \hat{v}^{11}\, dxdt = \sum_{i=1}^{2} \iint_Q [w^1\, w^2][\mathcal{H}\mathcal{F}_i(y^1, y^2)][w^1\, w^2]^T \phi^i\, dxdt + \iint_{\mathcal{O}_d^{11}\times(0,T)} |w^1|^2\, dxdt,$$

where $[\mathcal{H}\mathcal{F}_i(y^1, y^2)]$ is the Hessian matrix. Therefore

$$\begin{aligned}\langle D_1^2 J^{11}&(v^{11}, v^{12}, v^{21}, v^{22}), (\hat{v}^{11})^2\rangle \\ &= \mu_{11} \iint_{\mathcal{O}_{11}\times(0,T)} |\hat{v}^{11}|^2\, dxdt + \iint_{\mathcal{O}_d^{11}\times(0,T)} |w^1|^2\, dxdt \\ &\quad + \sum_{i=1}^{2} \iint_Q [w^1\, w^2][\mathcal{H}\mathcal{F}_i(y^1, y^2)][w^1\, w^2]^T \phi^i\, dxdt. \end{aligned} \tag{79}$$

Then, for proving that the second derivative of J^{11} is positive definite, we just have to bound the trilinear form

$$\iint_Q |w^i w^j \phi^i|\, dxdt \le C \|\hat{v}^{11}\|^2_{L^2(\mathcal{O}_{kl}\times(0,T))}, \quad i, j = 1, 2.$$

To prove that we can follow very similar ideas as the ones of [5, Section 3.3], we obtain the following result.

Theorem 6.1. *For $i = 1, 2$ and $k, l = 1, \dots, 4$, assume that $\mathcal{F}_i \in W^{2,\infty}(\mathbb{R}^2)$ and $y_d^{kl} \in L^\infty(\mathcal{O}_d^{kl} \times (0, T))$. If $y_0^i \in H_0^1(\Omega)$ (resp. $y_0^i \in L^2(\Omega)$) and $N \le 14$ (resp. $N \le 12$), we can take μ_{kl} sufficiently large so that $(v^{11}, v^{12}, v^{21}, v^{22})$ satisfying (6), also satisfies (5).*

6.6. Stackelberg–Nash controllability for Stokes and Navier–Stokes systems

Another interesting question is the study of a multi-objective control problem for equations coming from fluid mechanics. Indeed, consider the Stokes system

$$\begin{cases} y_t - \Delta y + (w\cdot\nabla)y + \nabla p = f 1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} + v^3 1_{\mathcal{O}_3} & \text{in } Q, \\ \nabla\cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on} \Sigma, \\ y(x,0) = y^0(x) & \text{in } \Omega, \end{cases} \tag{80}$$

where Ω, T, $\mathcal{O}$ and the $\mathcal{O}_i$ are as above. For $N = 2$ or $N = 3$ the data y^0 belongs to the Hilbert space

$$H := \{z \in L^2(\Omega)^N;\ \nabla \cdot z = 0 \text{ in } \Omega,\ z \cdot n = 0 \text{ on } \Gamma\},$$

the field w belongs to $L^\infty(0, T; H)$ and the controls $f = \{f^k\}_{k=1^2}$ and $v^k = \{v^{kl}\}_{i,k=1}^3$ satisfy

$$f \in L^2(\mathcal{O} \times (0, T))^N, \quad v^i \in L^2(\mathcal{O}_i \times (0, T))^N.$$

In this way, we can consider a very similar problem to the one considered here with functionals J^{kl} (resp. J^k) defined similarly to (4) (resp. (15)).

The situation is obviously much more difficult to analyze and, up to now, we are not aware on how we can adapt the ideas contained in here to this case. Positive results could lead to a local Stackelberg–Nash associated to a null controllability result to the Navier–Stokes equation ($w = y$). The existence of Nash equilibria or quasi-equilibria for each f and, of course, the existence of a leader control responsible for a null controllability property to (80) are questions being investigated in an ongoing work.

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