Portugal. Math. (N.S.)
Vol. 65, Fasc. 1, 2008, 49–65

Portugaliae Mathematica

A free boundary problem for a torrential flow

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(Communicated by Hugo Beirão da Veiga)

Abstract. A theoretical method based on the hodograph transformation is presented to solve the problem of an irrotational and steady flow of an inviscid and incompressible fluid, over a two-dimensional obstacle lying on the bottom of a channel. The suggested method for the solution of the fully non-linear problem is presented for a super-critical flow (Froude number $\mathrm{Fr} > 1$). The results obtained are based on those established in [6] and [7].

Mathematics Subject Classification (2000). 35R35.

Keywords. Torrential flow, hodograph, free boundary, Bernoulli.

Introduction

The study of the propagation of surface waves for an out-flow of fluid in a channel over an obstacle has been performed by different authors with different methods. The literature on this topic is abundant. We mention in particular Forbes [3], King and Bloor [5], Dias and Vanden-Broeck [2], as well the bibliography contained therein.

In this paper we consider the flow of a fluid over an obstacle lying on the bottom of a channel (see Fig. 1). Our aim is to study the perturbation effect of the obstacle on the free surface when this obstacle is smooth enough; it is described by a C^1 function on $\mathbb{R}$ and matches smoothly with the bottom.

We begin to formulate the problem in the (x, y) plane where the domain S occupied by the fluid is unknown (precisely the free surface of the domain is unknown).

We assume that the depth at infinity is the same in both directions. The movement is supposed to be bi-dimensional, uniform, irrotational and the fluid is supposed to be incompressible and inviscid, the depth and velocity at infinity being H and c, respectively. Here c is the uniform velocity of the flow downstream of the obstacle. A theoretical method is presented for the solution of the fully non-linear problem for a super-critical flow (torrential flow) which corresponds to Froude number $\mathrm{Fr} = \frac{c}{\sqrt{g_0 H}} > 1$, where g_0 is the acceleration due to gravity. The fluvial case ($\mathrm{Fr} < 1$) has been studied in [1].

Our results are obtained by considering the fluid equations and boundary conditions in complex potential coordinates $\phi + i\psi$, where ϕ is the potential fluid and ψ is the stream function. Here we apply the hodograph transformation which has been used in [6] to solve the non-linear wave resistance problem. This transformation changes the unknown domain S to a fixed domain A_H. Then we solve two equations by defining two operators on the upper and the lower boundaries of A_H.

The plan of the paper is as follows. In Section 2 we give the governing equations of the model, and by the hodograph transformation we reformulate these equations and define some spaces and operators. In Section 3 we give the linearized problem and we solve it. Section 4 contains the existence and uniqueness proof of the solution of the non-linear problem by an implicit function theorem argument.

1. Outline of the problem

We consider an irrotational flow of an ideal and incompressible fluid over a small obstacle lying on the bottom of a channel.

We denote by ω the complex velocity function defined by $\omega(z) = u(x, y) - iv(x, y)$, $z = x + iy$ is a complex variable and (u, v) is the velocity vector of the fluid. The bottom is described by a C^1 function b on $\mathbb{R}$.

We set

$$b(x) = \begin{cases} \varepsilon f(x) & \text{if } |x| < x^*, \\ 0 & \text{if } |x| \geq x^*, \end{cases}$$

where x^* is a positive real number, f is a function of class C^1 and ε is a small positive real number. Moreover, the function f satisfies the following relation:

$$\int_{-x^*}^{x^*} \frac{f'(x)^2}{(x - x^*)(x + x^*)} dx < \infty. \tag{1.1}$$

Here f' is the derivative of f. We will see the utility of this hypothesis in the third section. Our problem is to find a function $h \in C^1(\mathbb{R})$ describing the free surface and the complex velocity ω holomorphic in the domain

$$S = \{(x, y) \in \mathbb{R}^2 \mid b(x) < y < H(x)\}.$$

The functions ω and h must satisfy the following boundary conditions:

$$\frac{1}{2}\left|\omega\big(x,h(x)\big)\right|^2 + g_0 h(x) = \text{constant} \qquad \text{for all } x \in \mathbb{R}, \tag{1.2}$$

$$v\big(x,h(x)\big) = h'(x)u\big(x,h(x)\big) \qquad \text{for all } x \in \mathbb{R}, \tag{1.3}$$

$$v\big(x,b(x)\big) = b'(x)u\big(x,b(x)\big) \qquad \text{for all } x \in \mathbb{R}, \tag{1.4}$$

$$\lim_{|z|\to\infty} \omega(z) = c, \tag{1.5}$$

$$\lim_{|x|\to\infty} h(x) = H. \tag{1.6}$$

Equation (1.2) is the Bernoulli condition on the free surface (g_0 is the gravity constant) while equations (1.3), (1.4) indicate that the free surface and the bottom are streamlines. Equations (1.5) and (1.6) indicate that upstream and downstream of the obstacle the flow is horizontal with uniform velocity $(c, 0)$, and the constant H is the depth of the unperturbed fluid (see Fig. 1).

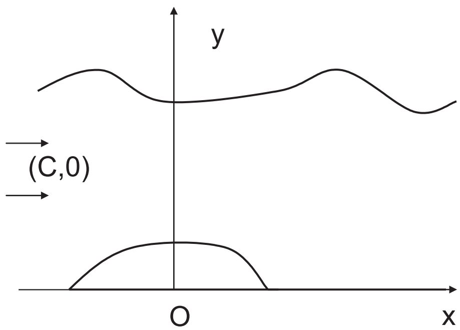

Figure 1.

2. The hodograph transformation

In this section we reformulate the problem by using as new independent variables the velocity potential $\varphi = \varphi(x, y)$ and the stream function $\psi = \psi(x, y)$. Let W be the complex potential

$$W = \varphi + i\psi, \qquad W'(z) = \omega(z).$$

Then

$$u = \frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y} \qquad v = \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}. \tag{2.1}$$

The incompressibility of the fluid and the irrotationality of the flow permit us to write

$$\Delta\varphi = \Delta\psi = 0 \tag{2.2}$$

in S, where Δ is the Laplace operator.

The lines $\varphi =$ constant are the equipotential lines, while $\psi =$ constant are the streamlines. The region S being simply connected, the potential W is determined by the complex velocity up to an additive constant. Let us now fix the complex constant: the real part is fixed by requiring that $\varphi(0, y) = 0$ due to the symmetry of the horizontal component u of the velocity. The imaginary part is fixed by requiring that the streamline $y = h(x)$ is represented by the equation $\psi = cH$. Then $\psi\big(x, h(x)\big) - \psi\big(x, b(x)\big) = cH$ implies that $\psi\big(x, b(x)\big) = 0$.

We recall now that we are looking for a solution which is a small perturbation of the free parallel flow $(y = H)$. Hence it is reasonable to assume that $u(x, y) > 0$ in S and therefore $\omega(z) \neq 0$ in S. Moreover, by the equation (2.1), the maps $x \mapsto \varphi(x, y)$ and $y \mapsto \psi(x, y)$ are strictly increasing. It follows that there is a conformal map, called the hodograph,

$$z \mapsto W(z), \tag{2.3}$$

which maps the domain S of the physical plane onto a strip A_H in the hodograph plane (φ, ψ) given by

$$A_H = \{(\varphi, \psi) \in \mathbb{R}^2 \,|\, 0 < \psi < cH\}. \tag{2.4}$$

We note that W is one-to-one so that the inverse map

$$W \mapsto z(W)$$

is well defined on A_H and satisfies

$$\frac{dz}{dW} = \frac{1}{\omega(z)} = \Omega(\omega). \tag{2.5}$$

By writing $\Omega = U - iV$, the flow is better described in the hodograph plane and the above relation takes the form

$$U = \frac{\partial x}{\partial \varphi} = \frac{\partial y}{\partial \psi}, \qquad V = \frac{\partial x}{\partial \psi} = -\frac{\partial y}{\partial \varphi}. \tag{2.6}$$

By noting that

$$U = \frac{u}{u^2 + v^2}, \qquad V = -\frac{v}{u^2 + v^2} \tag{2.7}$$

we easily verify

$$\Omega \to \frac{1}{c} \qquad \text{for } |\varphi| \to \infty. \tag{2.8}$$

Then we can write explicitly

$$x(\varphi,\psi) = \int_0^{\varphi} U(s,\psi)\,ds, \qquad y(\varphi,\psi) = \frac{1}{c}\psi + \int_{\varphi}^{\infty} V(s,\psi)\,ds. \tag{2.9}$$

Now we formulate the problem with respect to the (φ,ψ) plane; for this, we describe the bottom and the free surface by defining the following sets:

$$B = \{(\varphi,\psi) \mid \psi = 0, \varphi \in \mathbb{R}\}, \tag{2.10}$$

$$F = \{(\varphi,\psi) \mid \psi = cH, \varphi \in \mathbb{R}\}. \tag{2.11}$$

The kinematic free surface condition is already taken into account by requiring that the free surface is part of the streamline $\psi = cH$, while the Bernoulli condition (1.2) takes the form

$$\frac{1}{2}|\Omega|^{-4}\frac{\partial|\Omega|^2}{\partial\varphi} + g_0 V = 0 \tag{2.12}$$

on F.

Let us now write the condition on B. The equation (1.4) becomes

$$-\frac{V(\varphi,\psi)}{U(\varphi,\psi)}\bigg|_{\psi=0} = b'\big(x(\varphi,\psi)\big)|_{\psi=0} \tag{2.13}$$

or

$$-\frac{V(\varphi,0)}{U(\varphi,0)} = \begin{cases} \varepsilon f'\big(x(\varphi,0)\big) & \text{if } |\varphi| < \varphi^*, \\ 0 & \text{if } |\varphi| \geq \varphi^*, \end{cases} \tag{2.14}$$

where φ^* corresponds to the hodograph transformation of x^*. Some precisions about φ^* will be given at the end in Remark 11, before the conclusion.

We also define the beam:

$$I = \{(\varphi,\psi) \mid \psi = 0, |\varphi| < \varphi^*\}. \tag{2.15}$$

Now we are able to formulate the problem for the hododraph plane:

$$\frac{1}{2}|\Omega|^{-4}\frac{\partial|\Omega|^2}{\partial\varphi} + g_0 V = 0 \qquad \text{on } F, \tag{2.16}$$

$$V(\varphi,0) + b'\big(x(\varphi,0)\big)U(\varphi,0) = 0 \qquad \text{for all } \varphi \in \mathbb{R}, \tag{2.17}$$

$$\Omega(\varphi,\psi) \to \frac{1}{c}, \qquad |\varphi| \to \infty, \tag{2.18}$$

where $\Omega = U - iV$ is holomorphic in A_H.

By setting

$$\rho = \frac{\varphi}{\varphi^*}, \qquad \sigma = \frac{\psi}{\varphi^*}, \qquad \zeta = \rho + i\sigma, \tag{2.19}$$

the strip A_H becomes

$$A^* = \left\{ (\rho, \sigma) \in \mathbb{R}^2 \,|\, 0 < \sigma < \frac{cH}{\varphi^*} \right\}. \tag{2.20}$$

The sets B and F become:

$$B^* = \{ (\rho, \sigma) \,|\, \sigma = 0, \rho \in \mathbb{R} \}, \tag{2.21}$$

$$F^* = \left\{ (\rho, \sigma) \,|\, \sigma = \frac{cH}{\varphi^*}, \rho \in \mathbb{R} \right\}. \tag{2.22}$$

In particular the beam (2.15) maps onto the interval $(-1, 1)$ of the ρ-axis.

We now observe that, for $\varepsilon = 0$, the problem (2.16)–(2.18) admits the constant solution $\Omega = 1/c$. Then we define the new unknown $\chi = \xi - i\eta$ by subtracting this solution from Ω and dividing by ε; namely we set

$$U(\varphi, \psi) = \frac{1}{c}\big(1 + \varepsilon\xi(\rho, \sigma)\big), \qquad V(\varphi, \psi) = \frac{\varepsilon}{c}\eta(\rho, \sigma). \tag{2.23}$$

We want to write the non-linear boundary conditions (2.16)–(2.17) as formal operator equations in the new variables. We first note that the relations (2.9) take the form

$$x(\varphi, \psi) = \frac{\varphi^*}{c}\int_0^\rho \big(1 + \varepsilon\xi(s, \sigma)\big)\, ds, \qquad y(\varphi, \psi) = \frac{\varphi^*}{c}\left\{\sigma + \varepsilon\int_\rho^\infty \eta(s, \sigma)\, ds\right\} \tag{2.24}$$

and we can define on $(-1, 1)$ the function

$$G(\rho) = f'\big(x(\varphi, 0)\big) = f'\left(\frac{\varphi^*}{c}\int_0^\rho \big(1 + \varepsilon\xi(s, 0)\big)\, ds\right). \tag{2.25}$$

We now set

$$B^1(\chi, \varepsilon) = \{\eta + G(\cdot)(1 + \varepsilon\xi)\}\big|_{\sigma=0,\, |\rho|<1}, \tag{2.26}$$

$$B^2(\chi, \varepsilon) = \left\{ \frac{|1 + \varepsilon\chi|^{-4}}{2\varepsilon} \frac{\partial}{\partial\rho}|1 + \varepsilon\chi|^2 + \frac{g_0\varphi^*}{c^3}\eta \right\}\Bigg|_{\sigma = cH/\varphi^*,\, \rho\in\mathbb{R}}, \tag{2.27}$$

$$B(\chi, \varepsilon) = \big(B^1(\chi, \varepsilon), B^2(\chi, \varepsilon)\big). \tag{2.28}$$

Then it is easily verified that the equation

$$B(\chi, \varepsilon) = 0 \tag{2.29}$$

is equivalent to the conditions (2.16)–(2.17). Moreover, the function χ must be holomorphic in A^*, vanishing for $|\rho| \to \infty$ and satisfying the linear condition

$$\eta(\rho, 0) = 0.$$

3. The linearized problem

We have already remarked that when $\varepsilon = 0$ the problem (2.16)–(2.18) admits the trivial solution $\Omega = 1/c$. Now we assume that Ω can be expanded in powers of ε and according to (2.23) we set

$$\chi(\rho, \sigma) = \tilde{\chi}(\rho, \sigma) + \mathcal{O}(\varepsilon). \tag{3.1}$$

By inserting this relation into (2.26), (2.27) and by taking the limit $\varepsilon \to 0$, we get a problem satisfied by the holomorphic function $\tilde{\chi} = \tilde{\xi} - i\tilde{\eta}$ in the fixed domain A^*:

$$\frac{\partial \tilde{\xi}}{\partial \rho} + \varphi^* \frac{g_0}{c^3} \tilde{\eta} = 0 \qquad \text{for } \sigma = \frac{cH}{\varphi^*},\ \rho \in \mathbb{R}, \tag{3.2}$$

$$\tilde{\eta}(\rho, 0) = -f'\left(\varphi^* \frac{\rho}{c}\right) \qquad \text{for } |\rho| < 1, \tag{3.3}$$

$$\tilde{\eta}(\rho, 0) = 0 \qquad \text{for } |\rho| \geq 1, \tag{3.4}$$

$$\lim_{|\rho| \to \infty} \tilde{\chi}(\rho, \sigma) = 0. \tag{3.5}$$

By substituting $-\frac{\partial \tilde{\eta}}{\partial \sigma}$ for $\frac{\partial \tilde{\xi}}{\partial \rho}$ we obtain a boundary value problem for the harmonic function $\tilde{\eta}$ (the harmonic conjugate $\tilde{\xi}$ is then determined by the requirement of vanishing at infinity). Then we have the linear problem:

Find $\tilde{\eta}$ harmonic in A^* such that

$$-\frac{\partial \tilde{\eta}}{\partial \sigma} + v^* \tilde{\eta} = 0 \qquad \text{for } \sigma = \frac{cH}{\varphi^*},\ \rho \in \mathbb{R}, \tag{3.6}$$

$$\tilde{\eta}(\rho, 0) = -f'\left(\varphi^* \frac{\rho}{c}\right) \qquad \text{for } |\rho| < 1, \tag{3.7}$$

$$\tilde{\eta}(\rho, 0) = 0 \qquad \text{for } |\rho| \geq 1, \tag{3.8}$$

$$\lim_{|\rho| \to \infty} \tilde{\eta}(\rho, \sigma) = 0, \tag{3.9}$$

where v^* is the constant $\varphi^* \frac{g_0}{c^3}$.

We consider the linear problem

$$\Delta\eta = 0 \qquad \text{in } A^*, \tag{3.10}$$

$$\frac{\partial\eta}{\partial\sigma} - v^*\eta = 0 \qquad \text{for } \sigma = \frac{cH}{\varphi^*},\ \rho \in \mathbb{R}, \tag{3.11}$$

$$\eta = g \qquad \text{for } \sigma = 0,\ \rho \in \mathbb{R}, \tag{3.12}$$

where $g(\rho) = \begin{cases} -f'\left(\frac{\varphi^*\rho}{c}\right) & \text{for } |\rho| < 1, \\ 0 & \text{for } |\rho| \geq 1. \end{cases}$

Remark 1. Using [4], we prove that g is in $H^{1/2}(\mathbb{R})$.

3.1. A variational solution for the linear problem. For $g \in H^{1/2}(\mathbb{R})$ there exists a function $v_0 \in H^1(A^*)$ such that $v_0 = g$ for $\sigma = 0$.

A variational form of the problem can now be given in the Sobolev space $H^1(A^*)$ endowed with the equivalent norm

$$\|v\|^2 = \int_{A^*} |\nabla v|^2 + \int_{F^*} |v|^2. \tag{3.13}$$

Let us put $v = v_0 + v_1$ and consider the subspace $H^1_* \subset H^1(A^*)$ of the functions vanishing on B^*. Then the weak form of (3.10)–(3.12) is:

find $v_1 \in H^1_*$ such that

$$\int_{A^*} \nabla v_1 \nabla w - v^* \int_{F^*} v_1 w = -\int_{A^*} \nabla v_0 \nabla w + v^* \int_{F^*} v_0 w \qquad \text{for all } w \in H^1_*. \tag{3.14}$$

We can now state:

Theorem 2. *For any given* $g \in H^{1/2}(\mathbb{R})$ *and* $c^2 > g_0 H$ *(*g_0 *is the gravity acceleration), there is a unique weak solution in* $H^1(A^*)$ *of the problem* (3.10)–(3.12). *Furthermore, we have the bound*

$$\|v\|_{H^1(A^*)} \leq c\|gt\|_{H^{1/2}(B^*)} \tag{3.15}$$

for some positive constant c.

Proof. We must show that the bilinear form

$$a(v_1, w) = \int_{A^*} \nabla v_1 \nabla w - v^* \int_{F^*} v_1 w \tag{3.16}$$

is continuous and coercive in the space H^1_* and that the linear form

$$l(w) = -\int_{A^*} \nabla v_0 \nabla w + v^* \int_{F^*} v_0 w \tag{3.17}$$

is continuous in the space H^1_*.

First, we show the following result which will be useful for the coercivity of $a(\cdot, \cdot)$.

For any $v_1 \in H^1_*$ we write

$$v_1\left(x, \frac{cH}{\varphi^*}\right) = \int_0^{cH/\varphi^*} \frac{\partial v_1}{\partial y} dy. \tag{3.18}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}} |v_1|^2\left(x, \frac{cH}{\varphi^*}\right) dx &= \int_{\mathbb{R}} \left| \int_0^{cH/\varphi^*} \frac{\partial v_1}{\partial y} dy\right|^2 dx \\
&\leq \int_{\mathbb{R}} \Big(\int_0^{cH/\varphi^*} \left|\frac{\partial v_1}{\partial y}\right|^2 dy \int_0^{cH/\varphi^*} dy\Big) dx \\
&\leq \frac{cH}{\varphi^*} \int_{\mathbb{R}} \Big(\int_0^{cH/\varphi^*} \left|\frac{\partial v_1}{\partial y}\right|^2 dy\Big) dx \\
&\leq \frac{cH}{\varphi^*} \int_{A^*} |\nabla v_1|^2.
\end{aligned} \tag{3.19}$$

So

$$\begin{aligned}
a(v_1, v_1) &= \int_{A^*} |\nabla v_1|^2 - v^* \int_{F^*} |v_1|^2 \\
&\geq \int_{A^*} |\nabla v_1|^2 - \frac{g_0 \varphi^*}{c^3} \frac{cH}{\varphi^*} \int_{A^*} |\nabla v_1|^2 \\
&\geq \left(1 - \frac{g_0 H}{c^2}\right) \int_{A^*} |\nabla v_1|^2
\end{aligned}$$

where $\left(1 - \frac{g_0 H}{c^2}\right)$ is strictly positive because $v^* < \frac{\varphi^*}{cH}$. Then the bilinear form $a(\cdot, \cdot)$ is coercive on H^1_*.

Now we prove the continuity of $a(\cdot, \cdot)$ and $l(\cdot)$. For w in H^1_* we have:

$$\begin{aligned}
|a(v_1, w)| &= \left| \int_{A^*} \nabla v_1 \nabla w - v^* \int_{F^*} v_1 w \right| \\
&\leq \|\nabla v_1\|_{L^2(A^*)} \|\nabla w\|_{L^2(A^*)} + v^* \|v_1\|_{L^2(F^*)} \|w\|_{L^2(F^*)} \\
&\leq \|v_1\|_{H^1_*} \|w\|_{H^1_*} + \alpha v^* \|v_1\|_{H^1_*} \|w\|_{H^1_*} \\
&\leq (1 + \alpha v^*) \|v_1\|_{H^1_*} \|w\|_{H^1_*}.
\end{aligned}$$

where the constant α comes from the continuity of the trace operator from $H^1(A^*)$ to $H^{1/2}(F^*)$.

The same arguments give the continuity of the linear form $l(\cdot)$ and by putting $w = v_1$ in (3.14) we easily obtain (3.15). □

We now investigate the regularity and the decay at infinity of the above solution.

Proposition 3. *Let v be the solution of* (3.10)–(3.12). *Then for every $R > 1$ we have*

$$\sup_{|\rho| \geq R, 0 \leq \sigma \leq cH/\varphi^*} e^{\lambda_1 |\rho|} |v(\rho, \sigma)| < \infty, \tag{3.20}$$

where λ_1 is the first positive solution of

$$\tan\left(\lambda \frac{cH}{\varphi^*}\right) = \frac{\lambda}{v^*}. \tag{3.21}$$

Proof. Let us consider the restriction of v to the domain $(R, +\infty) \times \left(0, \frac{cH}{\varphi^*}\right)$. Clearly v is harmonic and square integrable in this domain and satisfies the conditions (3.11)–(3.12) on the upper and lower bound respectively. Then by separation of variables we obtain

$$v = \sum_{n \geq 1} c_n e^{-\lambda_n \rho} \sin(\lambda_n \sigma)$$

where $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ are the positive solutions of equation (3.21) and the coefficients c_n are uniquely determined by the values of the function $v(R, \cdot)$. Thus $v \sim Ce^{-\lambda_1 \rho}$ for large positive values of ρ, uniformly with respect to σ. Clearly, the same conclusion holds for large negative values of ρ. Hence the bound (3.20) follows. □

Now we give a result which will be useful for the next section. We consider the following boundary value problem:

$$\Delta \eta = 0 \quad \text{in } A^*, \tag{3.22}$$

$$\frac{\partial \eta}{\partial \sigma} - v^* \eta = l \quad \text{for } \rho \in \mathbb{R},\ \sigma = \frac{cH}{\varphi^*}, \tag{3.23}$$

$$\eta(\rho, 0) = k \quad \text{for } \rho \in \mathbb{R}. \tag{3.24}$$

Proposition 4. *Assume that $k \in H^{3/2}(\mathbb{R})$ and $l \in H^{1/2}(\mathbb{R})$. Then there exists a unique solution $\eta \in H^2(A^*)$ of the problem* (3.22)–(3.24).

Proof. By the same arguments as in Theorem 2 there exists a solution $\eta \in H^1(A^*)$ of (3.22)–(3.24).

Moreover, the condition

$$\frac{\partial \eta}{\partial \sigma} = -\sigma^* \eta + l$$

with l and η in $H^{1/2}(\mathbb{R})$ and $k \in H^{3/2}(\mathbb{R})$ implies that $\eta \in H^2(A^*)$. □

Corollary 5. *Assume that for some* $\rho_0 > 1$ *there exists* $\mu > \lambda^* > 0$ *with* $\lambda^* = \frac{\varphi^* \lambda_1}{c}$ *and* λ_1 *is the first positive solution of the equation* (3.21) *such that* $\sup_{|\rho| \geq \rho_0} e^{\mu|\rho|} |l(\rho)| < \infty$. *Then there is a unique holomorphic function*

$$\chi = \xi - i\eta$$

which belongs to $H^2(A^*)$ *and satisfies the boundary conditions* (3.23)–(3.24). *The following bounds hold:*

$$\sup_{A^*} e^{\lambda^*|\rho|} |\chi(\rho, \sigma)| < \infty, \tag{3.25}$$

$$\sup_{|\rho| \geq \rho_0} e^{\lambda^*|\rho|} \left| \frac{\partial \xi}{\partial \rho}(\rho, 0) \right| < \infty. \tag{3.26}$$

Proof. We note that the solution of the problem (3.22)–(3.24) can be written in the form $\tilde{\eta} = \eta_0 + \eta_1$, where $\eta_1(\rho, 0) = 0$, η_0, η_1 are harmonic in A^* satisfying the boundary conditions:

$$\frac{\partial \eta_0}{\partial \sigma} - v^* \eta_0 = 0 \qquad \text{for } \sigma = \frac{cH}{\varphi^*},\ \rho \in \mathbb{R}, \tag{3.27}$$

$$\eta_0 = k - \eta_1 \qquad \text{for } \sigma = 0,\ \rho \in \mathbb{R}, \tag{3.28}$$

$$\frac{\partial \eta_1}{\partial \sigma} - v^* \eta_1 = l \qquad \text{for } \sigma = \frac{cH}{\varphi^*},\ \rho \in \mathbb{R}. \tag{3.29}$$

We observe that if η_1 is known, the problem for η_0 is similar to problem (3.10)–(3.12). By Proposition 3 the bounds (3.25), (3.26) hold for the harmonic function $\chi_0 = \xi_0 - i\eta_0$ (where ξ_0 is the harmonic conjugate of η_0 vanishing at infinity). Thus we are reduced to prove the bounds for the function η_1 satisfying (3.29) (and for the harmonic conjugate ξ_1). Let us define $H^* = \frac{cH}{\varphi^*}$. By elementary calculations, η_1 has the representation

$$\eta_1(\rho, \sigma) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ip\rho} \widehat{K}_\sigma(p) \widehat{l}(p)\, dp, \tag{3.30}$$

where

$$\widehat{K}_\sigma(p) = \frac{\sinh(p\sigma)}{p\cosh(pH^*) - \nu^*\sinh(pH^*)}.$$

and $\widehat{l}(p)$ is the Fourier transform of l. We point out that the function $\widehat{K}_\sigma$ is not singular since the equation $\nu^*\tanh(pH^*) = p$ has only the real solution $p = 0$ for $\nu^*H^* < 1$. We further note that the integral (3.30) is convergent also for $\sigma = \frac{cH}{\varphi^*}$. In fact, $\widehat{l}$ belongs to $\mathbb{L}^2(\mathbb{R})$ and we easily verify that $\widehat{K}_\sigma$ belongs to $\mathbb{L}^2(\mathbb{R})$.

By the convolution theorem we have

$$\eta_1(\rho,\sigma) = \int_{\mathbb{R}} K_\sigma(\rho-\rho')l(\rho')\,d\rho' \tag{3.31}$$

where

$$K_\sigma(\rho) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ip\rho}\frac{\sinh(p\sigma)}{p\cosh(pH^*) - \nu^*\sinh(pH^*)}\,dp. \tag{3.32}$$

Note that the function $p \mapsto \frac{\sinh(p\sigma)}{p\cosh(pH^*)-\nu^*\sinh(pH^*)}$ has countable poles which are all pure imaginary.

For $|\rho| > 0$, we can evaluate (3.32) by the residual formula and find that

$$K_\sigma(\rho) = \sum_{n=1}^{\infty} c_n(\sigma)e^{-\lambda_n^*|\rho|} \tag{3.33}$$

where

$$c_n(\sigma) = \frac{\sin(\lambda_n^*\sigma)}{(1-\nu^*H^*)\cos(\lambda_n^*H^*) - \lambda_n^*H^*\sin(\lambda_n^*H^*)}$$

and λ_n^* are the positive solutions of the equation

$$\tan(\lambda H^*) = \frac{\lambda}{\nu^*}.$$

For large values of n we have $\lambda_n^*H^* \approx (n-1/2)\pi$ and so

$$c_n(0) \sim -\frac{1}{n\pi}$$

and from (3.33) we get the estimate

$$|K_\sigma(\rho)| \le ce^{-\lambda_1^*|\rho|} \tag{3.34}$$

for $|\rho| > 0$ with c independent of σ.

We can now prove the bounds (3.25), (3.26) for the holomorphic function $\chi_1 = \xi_1 - i\eta_1$. Put $I_0 = (-\rho_0, \rho_0)$, $I_{\rho,\delta} = (\rho - \delta, \rho + \delta)$. Then, by the representation (3.31), the estimate (3.34) and the decaying property of l, we obtain for $|\rho| > \rho_0 + \delta$,

$$\begin{aligned} |\eta_1(\rho,\sigma)| &\le \int_{I_{\rho,\delta}} |K_\sigma(\rho-\rho')|\,|l(\rho')|\,d\rho' + \int_{I_0} |K_\sigma(\rho-\rho')|\,|l(\rho')|\,d\rho' \\ &\quad + \int_{\mathbb{R}/\{I_{\rho,\delta}\cup I_0\}} |K_\sigma(\rho-\rho')|\,|l(\rho')|\,d\rho' \\ &\le C\Big\{(\|K_\sigma\|_{L^2} + \|l\|_{L^p})e^{-\lambda_1^*|\rho|} + \int_{\mathbb{R}/\{I_{\rho,\delta}\cup I_0\}} e^{-\lambda_1^*(|\rho-\rho'|+|\rho'|)}\,d\rho'\Big\} \\ &\le Ce^{-\lambda_1^*|\rho|} \end{aligned} \tag{3.35}$$

with C independent of σ.

By (3.31), (3.32) we see that the same bound holds for every derivative of η_1 if $\sigma \ge \sigma_0 > 0$. The limit at infinity of (3.35) extends this bound to $\sigma = 0$ for $|\rho| \ge \rho_0$.

By condition (3.29) and the assumption on l we obtain the same estimate for $\frac{\partial \eta_1}{\partial \sigma}$; the same bound holds for the function $\frac{\partial \xi_1}{\partial \rho}$ (we have used the Cauchy–Riemann relations). So (3.26) holds. Furthermore we have

$$|\xi_1(\rho,\sigma)| = \left|\int_{-\infty}^{\rho} \frac{\partial \xi_1}{\partial t}(t,\sigma)\,dt\right| = \left|\int_{-\infty}^{\rho} \frac{\partial \eta_1}{\partial \sigma}(t,\sigma)\,dt\right| \le Ce^{-\lambda_1^*|\rho|} \tag{3.36}$$

for $\rho < -\rho_0$ and

$$|\xi_1(\rho,\sigma)| = \left|\int_{\rho}^{+\infty} \frac{\partial \xi_1}{\partial t}(t,\sigma)\,dt\right| = \left|\int_{\rho}^{+\infty} \frac{\partial \eta_1}{\partial \sigma}(t,\sigma)\,dt\right| \le Ce^{-\lambda_1^*|\rho|} \tag{3.37}$$

for $\rho > \rho_0$. It follows that ξ_1 satisfies the bound (3.35). Thus (3.25) is proved. □

Remark 6. By recalling the relation $\xi_\rho = -\eta_\sigma$, which holds in A^*, we can rephrase the boundary condition (3.23) in the form

$$\xi_\rho + v^*\eta = -l. \tag{3.38}$$

4. The solution of the non-linear problem

To solve the non-linear problem, we define some Banach spaces in order to apply the implicit function theorem to the equation $B(\chi, \varepsilon) = 0$ where B is defined by (2.28). Here we use the results given in Corollary 5.

Take $\rho_0 > 1$ and denote by $Q_0 \subset \overline{A^*}$ the closed region $\left[0, \frac{cH^*}{\varphi^*}\right] \times \mathbb{R} \backslash (-\rho_0, \rho_0)$. Let us define the following set:

$$X = \{\chi = \xi - i\eta \text{ holomorphic in } A^*, \, \chi \in H^2(A^*), \, \eta(\cdot, 0) = 0,$$
$$|\rho| > 1, \, \sup_{A^*} e^{\lambda^*|\rho|}|\chi(\rho, \sigma)| < \infty\}$$

where λ^* is the first positive root of

$$\frac{\lambda}{v} = \tan\left(\lambda \frac{cH}{\varphi^*}\right).$$

Then X is a Banach space of continuous functions vanishing at infinity equipped with the norm

$$\|\chi\| = \|\chi\|_{H^2(A^*)} + \sup_{A^*} e^{\lambda^*|\rho|}|\chi(\rho, \sigma)|.$$

We now define Y_0 and Y by

$$\begin{aligned} Y_0 &= \{l \in H^{1/2}(\mathbb{R}) | \sup_{|\rho| \succeq \rho_0} e^{\lambda^*|\rho|}|l(\rho)| < \infty\} \\ Y &= H^{3/2}(-1, 1) \times Y_0 \end{aligned} \tag{4.1}$$

In the following theorem for the proof that the operator B is continuously differentiable we need a definition.

Definition 7. We call the Nemitski operator associated to a function f the application defined by $u \mapsto f \circ u$.

Theorem 8. *Let f be a C^2 function defined in an interval $J \supset \left[-\frac{\varphi^*}{c}, \frac{\varphi^*}{c}\right]$ and suppose that the Nemitski operator associated to f'' is continuous from $H^{3/2}(-1, 1)$ to $H^{3/2}(-1, 1)$. Then there exists $\varepsilon_0 > 0$ and a bounded open set $U \subset X$ containing the solution $\bar{\chi}$ of the problem*

$$\frac{\partial \xi}{\partial \sigma} + \sigma^* \eta = 0 \qquad \textit{for } \sigma = \frac{cH}{\varphi^*}\rho \in \mathbb{R}, \tag{4.2}$$

$$\eta(\rho, 0) = -f'\left(\varphi^* \frac{\rho}{c}\right) \qquad \textit{for } |\rho| < 1, \tag{4.3}$$

$$\eta(\rho, 0) = 0 \qquad \textit{for } |\rho| \geq 1, \tag{4.4}$$

$$\lim_{|\rho| \to \infty} \tilde{\chi}(\rho, \sigma) = 0 \tag{4.5}$$

such that the operator

$$B : U \times [0, \varepsilon_0] \to Y$$

defined by (2.28) *is continuously differentiable.*

Proof. By recalling the expression (2.26) of B^1, we may choose ε_0 and U such that if $(\chi, \varepsilon) \in U \times [0, \varepsilon_0)$, then the relation $\frac{\varphi^*}{c}\int_0^\rho [1 + \varepsilon\xi(t, 0)]\, dt \in J$ holds for every $\rho \in [-1, 1]$. Then, by our assumptions on f'' and the continuity of the product between functions of $H^2(-1, 1)$, the derivative (G-differential) of B^1 at $\chi^* = \xi^* - i\eta^*$ in the direction $\chi = \xi - i\eta$ exists and is equal to

$$d_G B^1(\chi^*, \varepsilon)\chi = \eta(\rho, 0) - \varepsilon f'\Big(\frac{\varphi^*}{c}\int_0^\rho [1 + \varepsilon\xi^*(t, 0)]\, dt\Big)\xi(\rho, 0) - \varepsilon\frac{\varphi^*}{c} f''\Big(\frac{\varphi^*}{c}\int_0^\rho [1 + \varepsilon\xi^*(t, 0)]\, dt\Big)[1 + \varepsilon\xi^*(\rho, 0)]\int_0^\rho \xi \tag{4.6}$$

with $\rho \in (-1, 1)$. Furthermore, the right-hand side term of (4.6) defines a bounded linear operator

$$d_G B^1(\chi^*, \varepsilon) : X \to H^2(-1, 1) \tag{4.7}$$

and one can easily check that the map

$$(\chi^*, \varepsilon) \mapsto d_G B^1(\chi^*, \varepsilon) \tag{4.8}$$

is continuous. Then B^1 is Fréchet differentiable with continuous derivative in $U \times [0, \varepsilon_0)$. The differentiability of B^1 with respect to ϵ is readily verified.

Let us now consider the operator B^2 given by (2.27) and take ε_0 small enough such that $|1 + \varepsilon\chi| > 0$ for every $\chi \in U$. Then by a straightforward calculation we can write

$$B^2(\chi, \varepsilon) = \left\{ |1 + \varepsilon\chi|^{-4} \left| \frac{\partial\xi}{\partial\rho} + \varepsilon\Big(\frac{\partial\xi}{\partial\rho}\xi + \frac{\partial\eta}{\partial\rho}\eta\Big) \right| + v^*\eta \right\}\Bigg|_{\rho \in \mathbb{R},\, \sigma = cH/\varphi^*}. \tag{4.9}$$

By the above expression, by the continuity of the application $f, g \mapsto f \cdot g$ from $H^{3/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R})$ into $H^{1/2}(\mathbb{R})$, and by Corollary 4, we find that B^2 is a well-defined continuous operator from X into Y_0. Moreover the G-derivative at χ^* is given by

$$d_G B^2(\chi^*, \varepsilon)\chi = \frac{\partial\xi}{\partial\rho} + v^*\eta + \mathcal{O}(\varepsilon), \tag{4.10}$$

where $\mathcal{O}(\varepsilon)$ represents a function depending on χ, χ^* and their derivatives whose norm for $\varepsilon \to 0$ (and χ, χ^* in a bounded set of X) is $\mathcal{O}(\varepsilon)$. Hence, we obtain as before that $d_G B^2(\chi^*, \varepsilon) : X \twoheadrightarrow Y_0$ is a bounded linear operator and that the map $(\chi^*, \varepsilon) \mapsto d_G B^2(\chi^*, \varepsilon)$ is continuous in $U \times [0, \varepsilon_0)$. Finally, again from (4.9) we easily infer the differentiability of B^2 with respect to ε. □

By denoting with $B' = B'(\chi^*, \varepsilon)$ the Fréchet differential of B with respect to χ, we get from (4.6) and (4.10)

$$B'(\chi^*, 0)\chi = (\eta_{|\rho|<1, \sigma=0}, \{\xi_\rho + v^*\eta\}_{\rho \in \mathbb{R}, \sigma=cH/\varphi^*}). \tag{4.11}$$

By Proposition 4, Corollary 5 and Remark 6 we obtain the following result.

Corollary 9. *For every $\chi \in U$ the operator $B'(\chi, 0)$ is invertible.*

Now, by applying the implicit function theorem, we arrive at the main result.

Theorem 10. *Under the assumptions of Theorem 8 on f there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in [0, \varepsilon_0)$, the equation $B(\chi, \varepsilon) = 0$ has a unique solution $\chi^\varepsilon \in U$. Moreover, the map $\varepsilon \mapsto \chi^\varepsilon$ is continuously differentiable.*

Remark 11. Here we verify that φ^* is uniquely determined from x^*. Indeed, from (2.24) it follows that

$$x^* = \frac{\varphi^*}{c} \int_0^1 \big(1 + \varepsilon\xi(s, 0)\big)\, ds. \tag{4.12}$$

Put

$$\Im(\varepsilon, \varphi^*) = x^* - \frac{\varphi^*}{c} \int_0^1 \big(1 + \varepsilon\xi(s, 0)\big)\, ds. \tag{4.13}$$

Then

$$\Im(0, cx^*) = 0 \tag{4.14}$$

and

$$\frac{\partial \Im}{\partial \varphi^*}(\varepsilon, \varphi^*) = -\frac{1}{c} \int_0^1 \big(1 + \varepsilon\xi(s, 0)\big)\, ds - \frac{\varphi^*}{c} \int_0^1 \varepsilon \frac{\partial \xi}{\partial \varphi^*}(s, 0)\, ds. \tag{4.15}$$

Hence

$$\frac{\partial \Im}{\partial \varphi^*}(0, cx^*) = -\frac{1}{c} \neq 0. \tag{4.16}$$

From the implicit function theorem, we deduce that there exists a neighborhood V of zero such that for all ε in V, there exists a unique solution φ^* satisfying $\mathfrak{I}(\varepsilon, \varphi^*) = 0$.

Summary. In this paper we presented a result of existence and uniqueness for the non-linear free surface problem concerning a torrential flow over an obstacle. For this we have used the hodograph transformation to solve, by the implicit function theorem, an equation defined in the upper and lower bound in the hodograph plane. We point out that the free surface profile $h(x)$ has disappeared among the unknowns and it will be recovered by the inverse image of the level line $\psi = cH$.

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Received April 3, 2006

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