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An existence result for a new variant of the nonconvex sweeping process

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Abstract. In this paper we prove the existence of solutions of the following new variant of the nonconvex sweeping process with perturbation $-Au(t) \in N_{C(t)}\big(\dot{u}(t)\big) + \ddot{u}(t) + F\big(t, \dot{u}(t)\big)$ a.e. on $[0, T]$ $(T > 0)$.

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1. Introduction

In [17] Moreau introduced and studied the following differential inclusion

$$-\dot{u}(t) \in N_{C(t)}\big(u(t)\big) \text{ a.e. on } I, \qquad u(0) = u_0 \in C(0), \tag{1.1}$$

where $I := [0, T]$ $(T > 0)$, $C : I \to \mathbb{H}$ is a set-valued mapping defined from I to a Hilbert space $\mathbb{H}$ with closed convex values, and $N_{C(t)}\big(u(t)\big)$ denotes the outward normal cone, in the sense of convex analysis, to the set $C(t)$ at $u(t)$. The differential inclusion (1.1) is known as the sweeping process problem. This problem is equivalent to the following evolution variational inequality: find $u(t) \in C(t)$ a.e. on I such that

$$\langle \dot{u}(t), v - u \rangle \geq 0 \tag{1.2}$$

for all $v \in C(t)$. Consequently the sweeping process includes as a special case (by taking $C(t) = f(t) + K$) the following evolution variational inequality. Find $u(t) \in K$ a.e. on I such that

$$\langle \dot{u}(t), v - u \rangle \geq \langle f(t), v - u \rangle \tag{1.3}$$

for all $v \in K$, with K a closed subset of $\mathbb{H}$, $u : I \to \mathbb{H}$, $f \in L^2(I, \mathbb{H})$.

Several extensions of the sweeping process in diverse ways were obtained; see for example [11], [12], [13], [14], [15], [16], [18], [20], [21]. In [20], the authors considered the following variant of the sweeping process: find $u : I \to \mathbb{H}$ such that $\dot{u}(t) \in C(t)$ and

$$-u(t) \in N_{C(t)}\big(\dot{u}(t)\big). \tag{1.4}$$

They proved the existence and uniqueness of the solution of (1.4). In [2], the first author proved the existence and uniqueness of solutions for (1.4) in the nonconvex case.

In this paper we are interested in a new variant of sweeping process with a perturbation,

$$-Au(t) \in N^c_{C(t)}\big(\dot{u}(t)\big) + \ddot{u}(t) + F\big(t, \dot{u}(t)\big) \text{ a.e. in } I \tag{1.5}$$

with $u(0) = u_0 \in C(0)$, $\dot{u}(0) = v_0 \in \mathbb{H}$. Here $N^c_{C(t)}(\cdot)$ denotes the Clarke normal cone to $C(t)$ and F is a multifunction. Problem (1.5) includes as a special case the following evolution quasi-variational inequality:

$$\begin{aligned}&\text{Find } u : I \to \mathbb{H},\ u(0) = u_0 \in K(0),\ \dot{u}(0) = u_1, \text{ such that } \dot{u}(t) \in K(t)\\ &\quad\text{a.e. on } I, \text{ and}\\ &\langle l(t), w - \dot{u}(t)\rangle \le \langle \ddot{u}(t), w - \dot{u}(t)\rangle + a\big(u(t), w - \dot{u}(t)\big) + j(w) - j\big(\dot{u}(t)\big) \quad (1.6)\\ &\text{for all } w \in K(t).\end{aligned}$$

Here $a(\cdot\,,\cdot)$ is a real bilinear, symmetric, bounded, and elliptic form on $\mathbb{H} \times \mathbb{H}$, $l \in H^{1,2}\big((0,T);\mathbb{H}\big)$, and $j(\cdot)$ denotes a non-negative, convex, positively homogenous and Lipschitz continuous functional from $\mathbb{H}$ to $\mathbb{R}$. $K(t) \subset \mathbb{H}$ is a set of constraints. The variational inequality of type (1.6) is the dynamic analogue of the Signorini problem (see [10]). Let A be a linear and bounded operator on $\mathbb{H}$ associated with $a(\cdot\,,\cdot)$, that is, $a(u,v) = \langle Au, v\rangle$ for all $u, v \in \mathbb{H}$ and put $F(t,\cdot) := \partial j(\cdot) - l(t)$. Also assume that K has convex values. Then the variational inequality of type (1.6) can be rewritten in the form of (1.5). The main goal of this paper is to prove an existence result for sweeping processes described by (1.5) when the set-valued mapping K is not necessarily convex. Here we use ideas and techniques from nonsmooth analysis. The result is proved by showing that a new projection algorithm converges to a solution of (1.5).

2. Notation and preliminaries

Throughout this paper $\mathbb{H}$ denotes a real separable Hilbert space. Let S be a closed subset of $\mathbb{H}$. We denote by $d(\cdot, S)$ or $\big(d_S(\cdot)\big)$ the usual distance function associated with S, i.e., $d(x, S) := \inf_{u\in S}\|x - u\|$. First we need to recall some notation and definitions that will be used throughout the paper. Let $f : \mathbb{H} \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous (l.s.c) function and let x be any point where f is finite. We recall that the *proximal subdifferential* $\partial^P f(x)$ is the set of all $\xi \in \mathbb{H}$ for which there exist $\delta, \sigma > 0$ such that

$$\langle \xi, x' - x \rangle \leq f(x') - f(x) + \sigma \|x' - x\|^2$$

for all $x' \in x + \delta\mathbb{B}$. Here $\mathbb{B}$ is the closed unit ball centered at the origin of $\mathbb{H}$. By convention we set $\partial^P f(x) = \emptyset$ if $f(x)$ is not finite. Note that $\partial^P f(x)$ is always convex but may not be closed (see for instance [8]). Let S be a nonempty closed subset of $\mathbb{H}$ and x be a point in S. We recall (see [8]) that the proximal normal cone of S at x is defined by $N_S^P(x) := \partial^P \psi_S(x)$, where ψ_S denotes the indicator function of S, i.e., $\psi_S(x') = 0$ if $x' \in S$ and $+\infty$ otherwise. Note that the proximal normal cone may be given by

$$N_S^P(x) = \{\xi \in \mathbb{H} \mid \text{there exists } \alpha > 0 \text{ such that } x \in \mathrm{Proj}_S(x + \alpha\xi)\}$$

where

$$\mathrm{Proj}_S(u) := \{y \in S \mid d(u, S) := \|u - y\|\}.$$

Recall that for a given $r \in]0, +\infty]$ a subset S is uniformly r-prox-regular (see [19]) or equivalently r-proximally smooth (see [8]) if and only if every nonzero proximal normal to S can be realized by an r-ball, that is, for all $\bar{x} \in S$ and all $0 \neq \xi \in N_S^P(\bar{x})$ one has

$$\left\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \right\rangle \leq \frac{1}{2r} \|x - \bar{x}\|^2,$$

for all $x \in S$. We make the convention $\frac{1}{r} = 0$ for $r = +\infty$. Note that for $r = +\infty$ the uniform r-prox-regularity of S is equivalent to the convexity of S. The following proposition summarizes some important consequences of the uniform prox-regularity needed in the sequel. For the proof of these results we refer the reader to [19].

Proposition 2.1. *Let S be a nonempty closed subset in $\mathbb{H}$ and let $r \in]0, +\infty]$. If the subset S is uniformly r-prox-regular then the following hold:*

(i) *for all $x \in \mathbb{H}$ with $d(x, S) < r$, the projection* $\mathrm{Proj}_S(x)$ *is a singleton;*

(ii) *the proximal subdifferential of $d(\cdot, S)$ coincides with all the subdifferentials contained in the Clarke subdifferential at all points $x \in \mathbb{H}$ satisfying $d(x, S) < r$. So in this a case the subdifferential $\partial d(x, S) := \partial^P d(x, S) = \partial^C d(x, S)$ is a closed convex set in $\mathbb{H}$.*

As a consequence of (ii) we get that for uniformly r-prox-regular sets, the proximal normal cone to S coincides with all the normal cones contained in the Clarke normal cone at all points $x \in S$, i.e., $N_S^P(x) = N_S^C(x)$. In such case we put $N_S(x) := N_S^P(x) = N_S^C(x)$. Here $\partial^C d(x, S)$ and $N_S^C(x)$ denote the Clarke subdifferential of $d(\cdot, S)$ and the Clarke normal cone to S respectively (see [8]). We give an important result, due to Bounkhel and Thibault [5], of the closedness of the proximal subdifferential of the distance function to images of set-valued mappings with prox-regular values (for some applications of this result to economics we refer to [3]). Before stating the next proposition we recall the definition of Lipschitz continuity for set-valued mappings. We say that a set-valued mapping $C : \mathbb{R} \to \mathbb{H}$ is Lipschitz continuous with ratio $\lambda > 0$ provided that

$$d\big(x, C(t)\big) - d\big(x, C(s)\big) \le \lambda|t - s| \quad \text{for all } t, s \in \mathbb{R} \text{ and all } x \in \mathbb{H}. \tag{2.1}$$

Proposition 2.2. *Let $r \in]0, +\infty]$. Assume that $C : I \to \mathbb{H}$ is Lipschitz continuous set-valued mapping with uniformly r-prox-regular values for some open interval I of $\mathbb{R}$. For a given $0 < \delta < r$, the following closedness property of the proximal subdifferential of the distance function holds:*

for any $\bar{t} \in I$, $\bar{x} \in C(\bar{t}) + (r - \delta)\mathbb{B}$, $x_n \to \bar{x}$, $t_n \to \bar{t}$ with $t_n \in I$ (x_n is not necessarily in $C(t_n)$) and $\xi_n \in \partial^P d\big(x_n, C(t_n)\big)$ with $\xi_n \overset{w}{\to} \bar{\xi}$, one has $\bar{\xi} \in \partial^P d\big(\bar{x}, C(\bar{t})\big)$.

Here $\overset{w}{\to}$ means the weak convergence in $\mathbb{H}$.

Let now B be a bounded set of a normed space E. Then the Kuratowski measure of noncompacteness of B, $\alpha(B)$, is defined by

$$\alpha(B) = \inf\Big\{d > 0 \,|\, B = \bigcup_{i=1}^{m} B_i \text{ for some } m \text{ and } B_i \text{ with } \operatorname{diam}(B_i) \le d\Big\}.$$

Here $\operatorname{diam}(A)$ stands for the diameter of A given by $\operatorname{diam}(A) := \sup_{x,y \in A}\|x - y\|$. In the following lemma we recall (see for instance Proposition 9.1 in [9]) some useful properties for the measure of noncompactness α.

Lemma 1. *Let $\mathbb{H}$ be an infinite dimensional real Banach space and D_1, D_2 be two bounded subsets of $\mathbb{H}$.*

(i) $\alpha(D_1) = 0 \Leftrightarrow D_1$ *is relatively compact;*

(ii) $\alpha(\lambda D_1) = |\lambda|\alpha(D_1)$ *for all* $\lambda \in \mathbb{R}$;

(iii) $D_1 \subset D_2 \Rightarrow \alpha(D_1) \le \alpha(D_2)$;

(iv) $\alpha(D_1 + D_2) \le \alpha(D_1) + \alpha(D_2)$;

(v) *if* $x_0 \in \mathbb{H}$ *and* r *is a positive real number, then* $\alpha(x_0 + r\mathbb{B}) = 2r$.

3. Main result

The following existence theorem establishes our main result in this paper.

Theorem 3.1. *Let $r \in]0, +\infty]$, let $\mathbb{H}$ be a separable Hilbert space, and let $C : I \to \mathbb{H}$ $(T > 0)$ be a set-valued mapping with nonempty closed uniformly r-prox-regular values. Assume that C is Lipschitz continuous with ratio $\lambda > 0$ and there exists a convex strongly compact set K such that $C(t) \subset K$ for all $t \in I$. Let $F : I \times \mathbb{H} \to \mathbb{H}$ be a scalarly upper semicontinuous set-valued mapping with non-empty convex weakly compact values in $\mathbb{H}$. Assume also that F has linear growth, that is, there exists $L > 0$ such that $F(t, u) \subset L(1 + \|u\|)\mathbb{B}$ for all $(t, u) \in I \times \mathbb{H}$, and that $A : \mathbb{H} \to \mathbb{H}$ be a linear bounded operator. Then for any $u_0 \in C(0)$ and any $v_0 \in \mathbb{H}$, there exists at least one solution of* (1.5).

Proof. Let $\rho > 0$ such that $C(t) \subset K \subset \rho\mathbb{B}$ for all $t \in [0, T]$. Put $\beta = L(1 + \rho)$ and $\gamma = \rho(1 + T)$. Fix $n_0 \geq 1$ satisfying

$$(\beta + \rho\|A\| + \lambda)\frac{T}{2^{n_0}} \leq \frac{r}{2}. \tag{3.1}$$

For every $n \geq n_0$, we put

$$\mu_n := \frac{T}{2^n} \to 0 \qquad \text{as } n \to \infty \tag{3.2}$$

and we consider the following partition of I:

$$\begin{aligned} t_{n,i} &:= i\mu_n \qquad \text{for } 0 \leq i \leq 2^n, \\ I_{n,i+1} &:=]t_{n,i}, t_{n,i+1}] \qquad \text{for } 0 \leq i \leq 2^n - 1. \\ I_{n,0} &:= \{t_{n,0}\}. \end{aligned} \tag{3.3}$$

Algorithm 1. For every $n \geq n_0$, we choose by induction

- $z_{n,0} = u_0 \in C(0)$, $u_{n,0} = u_0$, and $g_{n,0} \in F(t_{n,0}, z_{n,0})$;
- $0 \leq i \leq 2^n - 1$: $z_{n,i+1} = \mathrm{Proj}_{C(t_{n,i+1})}(z_{n,i} - \mu_n A u_{n,i} - \mu_n g_{n,i})$;
- $u_{n,i+1} := u_{n,i} + \mu_n z_{n,i+1}$ and $g_{n,i+1} \in F(t_{n,i+1}, z_{n,i+1})$.

This algorithm is well defined. Indeed, for $i = 0$, we have by the Lipschitz property of C

$$\begin{aligned} &d\big(z_{n,0} - \mu_n A u_{n,0} - \mu_n g_{n,0}, C(t_{n,1})\big) \\ &\quad \leq \mu_n\|Au_{n,0}\| + \mu_n\|g_{n,0}\| + d\big(z_{n,0}, C(t_{n,1})\big) - d\big(z_0, C(t_{n,0})\big) \\ &\quad \leq \mu_n\|A\|\,\|u_0\| + \mu_n\|g_{n,0}\| + \lambda|t_{n,1} - t_{n,0}|. \end{aligned}$$

Since $\|u_0\| \le \rho$ and as F has linear growth we obtain

$$\|g_{n,0}\| \le L(1 + \|u_0\|) \le L(1+\rho) = \beta$$

and so

$$\begin{aligned} d\big(z_{n,0} - \mu_n A u_{n,0} - \mu_n g_{n,0}, C(t_{n,1})\big) &\le \mu_n \rho \|A\| + \beta\mu_n + \lambda\mu_n \\ &= \mu_n(\beta + \rho\|A\| + \lambda) \\ &\le (\rho\|A\| + \beta + \lambda)\frac{T}{2^{n_0}} \le \frac{r}{2} < r. \end{aligned}$$

The prox-regularity of the set $C(t_{n,1})$ and Proposition 2.1 (i) ensure the existence and the uniqueness of the projection $\mathrm{Proj}_{C(t_{n,1})}(z_{n,0} - \mu_n A u_{n,0} - \mu_n g_{n,0})$ and then we can take $z_{n,1} = \mathrm{Proj}_{C(t_{n,1})}(z_{n,0} - \mu_n A u_{n,0} - \mu_n g_{n,0})$, $u_{n,1} := u_{n,0} + \mu_n z_{n,1}$ and $g_{n,1} \in F(t_{n,1}, z_{n,1})$.

Assume now that $i \ge 1$. By Algorithm 1 we have $z_{n,i} \in C(t_{n,i}) \subset \rho\mathbb{B}$, that is, $\|z_{n,i}\| \le \rho$. Thus we get

$$\begin{aligned} \|u_{n,i}\| &\le \|u_{n,i-1}\| + \rho\mu_n \\ \|u_{n,i-1}\| &\le \|u_{n,i-2}\| + \rho\mu_n \\ &\vdots \\ \|u_{n,1}\| &\le \|u_0\| + \rho\mu_n. \end{aligned}$$

Adding these inequalities yields ($i \le 2^n$)

$$\|u_{n,i}\| \le \|u_0\| + i\rho\mu_n \le \rho + \rho T = \gamma.$$

Now by Algorithm 1 and by the fact that F has linear growth we obtain that

$$\|g_{n,i}\| \le L(1 + \|z_{n,i}\|) \le L(1+\rho) = \beta.$$

Therefore the Lipschitz property of C ensures that

$$\begin{aligned} & d\big(z_{n,i} - \mu_n A u_{n,i} - \mu_n g_{n,i}, C(t_{n,i+1})\big) \\ &\quad \le \mu_n\|A u_{n,i}\| + \mu_n\|g_{n,i}\| + d\big(z_{n,i}, C(t_{n,i+1})\big) - d\big(z_{n,i}, C(t_{n,i})\big) \\ &\quad \le \mu_n\rho\|A\| + \mu_n\beta + \lambda|t_{n,i+1} - t_{n,i}| \\ &\quad = (\rho\|A\| + \beta + \lambda)\mu_n \\ &\quad \le \frac{(\beta + \rho\|A\| + \lambda)T}{2^{n_0}} \le \frac{r}{2} < r, \end{aligned}$$

which implies by the prox-regularity of the set $C(t_{n,i+1})$ and Proposition 2.1 (i) the existence and the uniqueness of the projection $\mathrm{Proj}_{C(t_{n,i+1})}(z_{n,i} - \mu_n Au_{n,i} - \mu_n g_{n,i})$ and hence we can take $z_{n,i+1} = \mathrm{Proj}_{C(t_{n,i+1})}(z_{n,i} - \mu_n Au_{n,i} - \mu_n g_{n,i})$, $u_{n,i+1} := u_{n,i} + \mu_n z_{n,i+1}$ and $g_{n,i+1} \in F(t_{n,i+1}, z_{n,i+1})$.

Now we use the sequences $(u_{n,i})$, $(z_{n,i})$ and $(g_{n,i})$ to construct sequences of mappings u_n, v_n and g_n from I to $\mathbb{H}$ by defining their restrictions to each interval $I_{n,i}$ as follows:

For $t \in I_{n,0}$ set $g_n(t) = g_{n,0}$, $u_n(t) = u_0$ and $v_n(t) = v_0$; for $t \in I_{n,i+1}$ $(0 \le i \le 2^n - 1)$ set $g_n(t) = g_{n,i}$,

$$u_n(t) = u_{n,i} + z_{n,i+1}(t - t_{n,i}), \tag{3.4}$$

and

$$v_n(t) = z_{n,i} + (z_{n,i+1} - z_{n,i})\frac{(t - t_{n,i})}{\mu_n}. \tag{3.5}$$

It is clear by construction that both mappings u_n and v_n are differentiable a.e. on I with

$$\dot{u}_n(t) = z_{n,i+1} \quad \text{and} \quad \dot{v}_n(t) = \frac{z_{n,i+1} - z_{n,i}}{\mu_n} \text{ a.e. on } I. \tag{3.6}$$

By Algorithm 1 we have

$$z_{n,i+1} = \mathrm{Proj}_{C(t_{n,i+1})}(z_{n,i} - \mu_n Au_{n,i} - \mu_n g_{n,i})$$

and so by the definition of proximal normal cone we get

$$z_{n,i} - z_{n,i+1} - \mu_n Au_{n,i} - \mu_n g_{n,i} \in N^P_{C(t_{n,i+1})}(z_{n,i+1}).$$

By (3.6) we obtain

$$-\dot{v}_n(t) - Au_{n,i} - g_{n,i} \in N^P_{C(t_{n,i+1})}\big(\dot{u}_n(t)\big) \text{ a.e. on } I.$$

Now let us define the step functions from I to I by

$$\begin{aligned} \theta_n(t) &= t_{n,i}, \qquad t \in I_{n,i+1}, \\ \rho_n(t) &= t_{n,i+1}, \qquad t \in I_{n,i+1}. \end{aligned} \tag{3.7}$$

Then (3.4) and (3.7) yield that

$$-Au_n\big(\theta_n(t)\big) - \dot{v}_n(t) - g_n(t) \in N^P_{C(\rho_n(t))}\big(\dot{u}_n(t)\big) \text{ a.e. on } I.$$

On the other hand we have

$$\begin{aligned}\|z_{n,i+1} - z_{n,i}\| &\le \|z_{n,i+1} - z_{n,i} + \mu_n A u_{n,i} + \mu_n g_{n,i}\| + \mu_n\|Au_{n,i}\| + \mu_n\|g_{n,i}\| \\ &\le d_{C(t_{n,i+1})}(z_{n,i} - \mu_n A u_{n,i} - \mu_n g_{n,i}) + \mu_n\|Au_{n,i}\| + \mu_n\|g_{n,i}\| \\ &\le d_{C(t_{n,i+1})}(z_{n,i}) - d_{C(t_{n,i})}(z_{n,i}) + 2\mu_n\|A\|\,\|u_{n,i}\| + 2\mu_n\|g_{n,i}\| \\ &\le \lambda\mu_n + 2\mu_n\|A\|\,\|u_{n,i}\| + 2\mu_n\|g_{n,i}\|.\end{aligned}$$

So

$$\left\|\frac{z_{n,i} - z_{n,i+1}}{\mu_n}\right\| \le \lambda + 2\|A\|\,\|u_{n,i}\| + 2\|g_{n,i}\| \le \lambda + 2\rho\|A\| + 2\beta. \tag{3.8}$$

Then

$$\begin{aligned}\left\|\frac{z_{n,i} - z_{n,i+1}}{\mu_n} - Au_n\big(\theta_n(t)\big) - g_n(t)\right\| &\le \frac{1}{\mu_n}\|z_{n,i+1} - z_{n,i}\| + \|A\|\,\|u_{n,i}\| + \|g_{n,i}\| \\ &\le \lambda + 3\rho\|A\| + 3\beta\end{aligned}$$

and so by Proposition 4.1 in [4] we obtain that

$$-Au_n\big(\theta_n(t)\big) - \dot v_n(t) - g_n(t) \in (\lambda + 3\rho\|A\| + 3\beta)\partial^P d_{C(\rho_n(t))}\big(\dot u_n(t)\big) \text{ a.e. on } I. \tag{3.9}$$

As $\|g_n(t)\| \le \beta$ for all $t \in I$, we see that (g_n) is a bounded sequence in $L^\infty(I, \mathbb{H})$. Then by extracting a subsequence (because $L^\infty(I, \mathbb{H})$ is the dual space of the separable Banach space $L^1(I, \mathbb{H})$) we may suppose without loss of generality that (g_n) weakly-star converges in $L^\infty(I, \mathbb{H})$ to some mapping g.

Observe now that u_n and v_n are Lipschitz continuous on I with ratio β and $\lambda + 3\rho\|A\| + 3\beta$, respectively. So u_n is differentiable a.e. on I and

$$u_n(t) = u_0 + \int_0^t \dot u_n(s)\, ds.$$

Since $\dot u_n(t) \in C\big(\rho_n(t)\big) \in K$ it follows that

$$\{u_n(t) \mid n \ge n_0\} \subset u_0 + TK.$$

This implies the relative strong compactness of the set $\{u_n(t) \mid n \ge n_0\}$ in $\mathbb{H}$ for all $t \in [0, T]$. On the other hand, since $\dot u_n(t) \in C\big(\rho_n(t)\big)$, we get $\|\dot u_n(t)\| \le \rho$ and so Arzela–Ascoli's theorem (see for instance Theorem 0.4.4 in [1]) ensures the existence of a Lipschitz mapping $u : I \to \mathbb{H}$ with ratio ρ such that

- (u_n) converges uniformly to u on I, that is, $\lim_{n\to\infty} \max_{t\in[0,T]} \|u_n(t) - u(t)\| = 0$;
- $(\dot{u}_n)$ weakly converges to $\dot{u}$ in $L^1(I, \mathbb{H})$.

Return now to the compactness of the sequence v_n. By (3.6) and (3.8) we have

$$\|\dot{v}_n(t)\| \le \lambda + 2\rho\|A\| + 2\beta, \tag{3.10}$$

and it is clear that the sequence $\{v_n(t)\}$ is equi-Lipschitz with constant $\lambda + 2\rho\|A\| + 2\beta$. Now we show that the set $X(t) = \{v_n(t) \mid n \ge n_0\}$ is relatively compact in $\mathbb{H}$ for every $t \in [0, T]$. From the definition of (v_n) we have for all $t \in [0, T]$ and all $n \ge n_0$, $v_n(\theta_n(t)) \in C(\theta_n(t)) \subset K$. Then the set $\{v_n(\theta_n(t)) \mid n \ge n_0\}$ is relatively compact in $\mathbb{H}$ for all $t \in [0, T]$, and so by Lemma 1 we get

$$\alpha\big(\{v_n(\theta_n(t)) \mid n \ge n_0\}\big) = 0.$$

We have $X(t) = \{v_n(t) \mid n \ge n_0\} = \{v_n(t) - v_n(\theta_n(t)) + v_n(\theta_n(t)) \mid n \ge n_0\}$ for all $t \in [0, T]$. Then by Lemma 1 we obtain that

$$\begin{aligned}
\alpha(X(t)) &\le \alpha\big(\{v_n(t) - v_n(\theta_n(t)) \mid n \ge n_0\}\big) + \alpha\big(\{v_n(\theta_n(t)) \mid n \ge n_0\}\big) \\
&\le \alpha\left(\left\{\int_t^{\theta_n(t)} \dot{v}_n(s)\, ds \mid n \ge n_0\right\}\right) + 0 \\
&\le \alpha\left(B\left(0, \frac{T}{2^n}(\lambda + 2\rho\|A\| + 2\beta)\right)\right) \\
&= 2(\lambda + 2\rho\|A\| + 2\beta)\frac{T}{2^n} \to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

Hence by Lemma 1 the set $X(t)$ is relatively strongly compact in $\mathbb{H}$ for all $t \in I$. Then all the assumptions of the Arzela–Ascoli theorem are satisfied and hence there exists a Lipschitz mapping $v : I \to \mathbb{H}$ with ratio $\lambda + 2\rho\|A\| + 2\beta$ such that

- (v_n) converges uniformly to v on I, that is, $\lim_{n\to\infty} \max_{t\in[0,T]} \|v_n(t) - v(t)\| = 0$;
- $(\dot{v}_n)$ weakly converges to $\dot{v}$ in $L^1(I, \mathbb{H})$.

Consequently, we get

$$\begin{aligned}
\|\dot{u}_n(t) - v(t)\| &\le \|\dot{u}_n(t) - v_n(t)\| + \|v_n(t) - v(t)\| \\
&\le \|v_n(\rho_n(t)) - v_n(t)\| + \|v_n(t) - v(t)\| \\
&\le \mu_n + \|v_n(t) - v(t)\|
\end{aligned}$$

and so

$$\max_{t\in I}\|\dot{u}_n(t) - v(t)\| \le \mu_n + \max_{t\in I}\|v_n(t) - v(t)\| \to 0 \qquad \text{as } n \to \infty.$$

Thus, the sequence $\dot{u}_n$ converges to v uniformly on I and $\dot{u}(t) = v(t)$ and $\ddot{u}(t) = \dot{v}(t)$ a.e. on I. We proceed now to prove that

$$-Au(t) - g(t) - \ddot{u}(t) \in N^c_{C(t)}\big(\dot{u}(t)\big) \qquad \text{for almost all } t \in I.$$

Applying Castaing techniques (see for instance [6]), the uniform convergence of u_n to u, the weak convergence of $\dot{v}_n$ to $\dot{v}$ in $L^1(I, \mathbb{H})$, the weak star convergence of g_n to g in $L^\infty(I, \mathbb{H})$, and Mazur's lemma entail

$$-Au(t) - g(t) - \ddot{u}(t) \in \bigcap_n \overline{\text{co}}\big\{-Au_k\big(\theta_k(t)\big) - g_k(t) - \dot{v}_k(t) \,|\, k \ge n\big\}.$$

for almost all $t \in I$. Here $\overline{\text{co}}$ denotes the closed convex hull.

Fix any such $t \in I$ and consider any $\xi \in \mathbb{H}$. The last relation above yields

$$\langle \xi, -Au(t) - g(t) - \ddot{u}(t)\rangle \le \inf_n \sup_{k\ge n}\big\langle \xi, -Au_k\big(\theta_k(t)\big) - g_k(t) - \dot{v}_k(t)\big\rangle.$$

According to (3.8) we obtain that

$$\begin{aligned}\langle \xi, -Au(t) - g(t) - \ddot{u}(t)\rangle &\le \limsup_n \sigma\big((\lambda + 3\rho\|A\| + 3\beta)\partial^P d_{C(\rho_n(t))}\big(\dot{u}_n(t)\big), \xi\big)\\ &\le \sigma\big((\lambda + 3\rho\|A\| + 3\beta)\partial^P d_{C(t)}\big(\dot{u}(t)\big), \xi\big),\end{aligned}$$

where the last inequality follows from the upper semicontinuity property of the proximal subdifferential given in Proposition 2.2 and because of $\rho_n(t) \to t$ and $\dot{u}_n(t) \to v(t) = \dot{u}(t)$ strongly. Since the set $\partial^P d_{C(t)}\big(\dot{u}(t)\big)$ is closed convex (see Proposition 2.1) and $\dot{u}(t) \in C(t)$, we obtain that

$$-Au(t) - g(t) - \ddot{u}(t) \in (\lambda + 3\rho\|A\| + 3\beta)\partial^P d_{C(t)}\big(\dot{u}(t)\big) \subset N^P_{C(t)}\big(\dot{u}(t)\big)$$

and so

$$-Au(t) \in N^P_{C(t)}\big(\dot{u}(t)\big) + \ddot{u}(t) + g(t).$$

Now, as $g_n(t) \in F\big(\theta_n(t), v_n(t)\big)$ and by the upper semicontinuity of F and the convexity and the weak compactness of its values, we conclude (see Theorem V-14, [7]) that $g(t) \in F\big(t, \dot{u}(t)\big)$. Thus we get

$$-Au(t) \in N^P_{C(t)}\big(\dot{u}(t)\big) + \ddot{u}(t) + F\big(t, \dot{u}(t)\big) \qquad \text{for } a.e.\ t \in I.$$

This completes the proof of the theorem. □

As a direct consequence of our main theorem we obtain an existence result for the dynamic analogue of the Signorini problem given in (1.6).

Corollary 1. *Assume that $K : I \times \mathbb{H}$ is Lipschitz continuous with ratio $\lambda > 0$ and convex values such that $K(t) \subset \mathscr{K}$ for all $t \in I$ for some convex strongly compact set $\mathscr{K} \subset \mathbb{H}$. Assume that*

- *l is uniformly bounded, that is, there exists $L_2 > 0$ such that $\|l(t)\| \leq L_2$ for all $t \in I$;*
- *j is convex and Lipschitz continuous on $\rho\mathbb{B}$ with ratio $L_1 > 0$ (where ρ is a positive number satisfying $\mathscr{K} \subset \rho\mathbb{B}$).*

Then, for every $v_0 \in K(0)$ and any $v_1 \in \mathbb{H}$, there exists at least one solution of (1.6).

Proof. Since K has convex values, the variational inequality of type (1.6) can be rewritten in the form of (1.5) as follows:

$$-Au(t) \in N_{K(t)}\big(\dot{u}(t)\big) + \ddot{u}(t) + F\big(t, \dot{u}(t)\big) \text{ a.e. on } I,$$

with $u(0) = v_0 \in K(0)$, $\dot{u}(0) = v_1$ and $F(t, x) := \partial j(x) - l(t)$.

As j is Lipschitz continuous with ratio L_1 on $\rho\mathbb{B}$, it follows that $\partial j(x) \subset L_1\mathbb{B}$ for all $x \in K(t)$. Hence we obtain

$$\|F(t, x)\| \leq \|\partial j(x)\| + \|l(t)\| \leq L_1 + L_2 = L.$$

On the other hand it is well known that the subdifferential of Lipschitz convex functions is convex weakly compact and scalarly upper semicontinuous. Also we have by hypothesis that l is continuous. Then F is scalarly upper semicontinuous set-valued mapping with convex weakly compact values. Thus all the assumptions of Theorem 3.1 are satisfied and so the proof is complete. □

4. Solution sets

Throughout this section, let $r \in]0, +\infty]$, let $F : I \times \mathbb{H} \to \mathbb{H}$ be a set-valued mapping and let $C : I \to \mathbb{H}$ $(T > 0)$ be a Lipschitz set-valued mapping with ratio $\lambda > 0$ taking nonempty closed uniformly r-prox-regular values in $\mathbb{H}$. Let $v_0 \in \mathbb{H}$, $u_0 \in C(0)$. We denote by $S_F(u_0, v_0)$ the set of all couples (u, v) of Lipschitz mappings $u, v : I \to \mathbb{H}$ such that

$$\begin{aligned} u(0) &= u_0 \in C(0), \qquad v(0) = v_0, \\ u(t) &= u(0) + \int_0^t v(s)\,ds, \\ v(t) &\in C(t) \qquad \text{for all } t \in [0, T], \\ -Au(t) &\in N^c_{C(t)}\big(v(t)\big) + \dot v(t) + F\big(t, v(t)\big) \text{ a.e. in } I. \end{aligned} \tag{4.1}$$

In this section we are interested in the strong compactness of the graph of the set-valued mapping S_F defined from a strong compact set $\mathcal{K} \subset \mathbb{H} \times \mathbb{H}$ to $C(I, \mathbb{H} \times \mathbb{H})$.

Proposition 4.1. *Assume that the hypotheses of Theorem* 3.1 *are satisfied. Then the graph of the set-valued mapping S_F is strongly compact in $\mathcal{K} \times C(I, \mathbb{H} \times \mathbb{H})$.*

Proof. Let $((u_0^n, v_0^n))_n \in \mathcal{K}$ and $((u^n, v^n))_n \in C(I, \mathbb{H} \times \mathbb{H})$ with $(u^n, v^n) \in S_F((u_0^n, v_0^n))$. First, by the compactness of the set $\mathcal{K}$, we may assume without loss of generality that $((u_0^n, v_0^n))_n$ uniformly converges to some $(u_0, v_0) \in \mathcal{K}$. Now, according to the proof of Theorem 3.1, the sequence $((u^n, v^n))_n$ is equi-Lipschitz, and $\dot u^n(t) \le \rho$ and $\dot v^n(t) \le \lambda + 2\rho\|A\| + 2\beta$ a.e. on I. We also have the inclusion $\{(u^n(t), v^n(t)) \mid t \in [0, T]\} \subset (\rho + TK) \times K$ and so the set $\{(u^n(t), v^n(t)) \mid t \in [0, T]\}$ is relatively strongly compact in $\mathbb{H} \times \mathbb{H}$. Therefore, Arzelà–Ascoli's theorem gives the relative strong compactness of the sequence $((u^n, v^n))_n$ in $C(I, \mathbb{H} \times \mathbb{H})$ and so we may assume without loss of generality that $((u^n, v^n))_n$ uniformly converges to some $(u, v) \in C(I, \mathbb{H} \times \mathbb{H})$. By showing that $(u, v) \in S_F(u_0, v_0)$ the proof of the proposition will be complete. To do that, observe first that the closedness of $C(t)$ and the uniform convergence of both sequences $((u_0^n, v_0^n))_n$ and $((u^n, v^n))_n$ imply that $(u(0), v(0)) = (u_0, v_0)$ and that $v(t) \in C(t)$ for all $t \in [0, T]$. On the other hand one has

$$u(t) = \lim_n u^n(t) = u_0 + \lim_n \int_0^t v^n(s)\,ds = u_0 + \int_0^t v(s)\,ds.$$

for all $t \in [0, T]$. Now we have to show that

$$-Au(t) \in N^c_{C(t)}\big(v(t)\big) + \dot v(t) + F\big(t, v(t)\big) \text{ a.e. on } I.$$

Since $(u^n, v^n) \in S_F(u_0^n, v_0^n)$ we have for every n,

$$-Au^n(t) \in N^c_{C(t)}\big(v^n(t)\big) + \dot v^n(t) + F\big(t, v^n(t)\big) \text{ a.e. on } I. \tag{4.2}$$

Then there exists a measurable selection g^n, for every n, such that

$$g^n(t) \in F\big(t, v^n(t)\big) \quad \text{and} \quad -Au^n(t) - g^n(t) \in N^c_{C(t)}\big(v^n(t)\big) \tag{4.3}$$

for a.e. $t \in [0, T]$. By Theorem 3.1 and (3.8) one has

$$\|\dot{v}^n(t)\| \le \lambda + 2\|A\|(\rho + \rho T) \le \lambda + 2\|A\|(\rho + 2\rho T) \tag{4.4}$$

for n sufficiently large and

$$\|Au^n(t)\| \le \|A\|(\rho + \rho T) \le \|A\|(\rho + 2\rho T). \tag{4.5}$$

Since $v^n(t) \in C(t) \subset \rho\mathbb{B}$ and F has linear growth we obtain

$$\|g^n(t)\| \le L(1 + \|v^n(t)\|) \le L(1 + \rho).$$

Therefore, we suppose without loss of generality that $\dot{v}^n \to \dot{v}$ and $g^n \to g$ weakly star in $L^\infty(I, \mathbb{H})$. Since $F(t, \cdot)$ is scalarly upper semicontinuous with convex weakly compact values, then we get easily (see for instance Theorem V-14 in [7]) that $g(t) \in F\big(t, v(t)\big)$ a.e. $t \in [0, T]$. Now by (4.4), (4.5), (4.2) and Theorem 4.1 in [4], we have for $\delta := \lambda + 3\|A\|(\rho + 2\rho T) + \beta$,

$$-Au^n(t) - \dot{v}^n(t) - g^n(t) \in \delta\partial^P d_{C(t)}\big(v^n(t)\big) \text{ a.e. on } I.$$

Then by using Mazur's lemma and Propositions 3.1–4.1 in [5], it is easy to conclude that

$$-Au(t) - \dot{v}(t) - g(t) \in \delta\partial^P d_{C(t)}\big(v(t)\big) \subset N^P_{C(t)}\big(v(t)\big)$$

for a.e. $t \in I$. Thus we obtain that

$$-Au(t) \in N_{C(t)}\big(v(t)\big) + \dot{v}(t) + F\big(t, v(t)\big)$$

for a.e. $t \in [0, T]$, which completes the proof of the proposition. □

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