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Existence and iteration of positive solution for a multi-point boundary value problem with a p-Laplacian operator

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Abstract. In this paper we obtain the existence of a positive solution and establish a corresponding iterative scheme for the following boundary value problem:

$$\begin{cases} (\phi_p(u'))' + q(t)f(t,u) = 0, \quad 0 < t < 1, \\ u(0) = \sum_{i=1}^{q-1} \gamma_i u(\delta_i),\ u(1) = \sum_{i=1}^{m-1} \eta_i u(\xi_i). \end{cases}$$

The main tool is the monotone iterative technique. Here the coefficient $q(t)$ may be singular at $t = 0, 1$.

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1. Introduction

The existence of positive solutions to multi-point second order boundary value problems (BVPs) has been extensively studied by many authors; see, e.g., [5], [6], [8], [13], [14]. Also there has been considerable interest in p-Laplacian BVPs [1], [3], [7], [9], [10], [11], [15], [16]. As it is well known, when dealing with p-Laplacian BVPs, the main difficulty is that $\phi(x) = |x|^{p-2}x$ is nonlinear in x for $p \neq 2$. In fact, there are only a few papers in which positive solutions for the m-point ($m > 3$) second order BVP with p-Laplacian are obtained. In a recent paper [4], Bai and Fang studied the following BVP:

$$\begin{cases} (\phi_p(y'))' + a(t)f(t,u) = 0, \quad 0 < t < 1, \\ y(0) = 0,\ y(1) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i). \end{cases} \tag{1}$$

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Here multiple positive solutions for BVP (1) were obtained. The tool used in [4] is the fixed point index theory.

Here we study the following multi-point second order p-Laplacian BVP:

$$\begin{cases} (\phi_p(u'))' + q(t)f(t,u) = 0, \quad 0 < t < 1, \\ u(0) = \sum\limits_{i=1}^{q-1} \gamma_i u(\delta_i),\ u(1) = \sum\limits_{i=1}^{m-1} \eta_i u(\xi_i), \end{cases} \tag{2}$$

where

(i) $\phi_p(s) = |s|^{p-2}s,\ p > 1$;

(ii) $\delta_i \in (0,1),\ i = 1,2,\dots,q-1$, and $\xi_i \in (0,1),\ i = 1,2,\dots,m-1$;

(iii) $\gamma_i \in [0,1),\quad i = 1,2,\dots,q-1,\quad$ with $\sum_{i=1}^{q-1}\gamma_i \in [0,1)$ and $\eta_i \in [0,1),\quad i = 1,2,\dots,m-1$, with $\sum_{i=1}^{m-1}\eta_i \in [0,1)$.

The method of [4] is not applicable to our problem since we cannot change BVP (2) to an integral equation without a parameter which is critical in [4]. In this paper, by using the classical monotone iterative technique of Amann [2], we obtain not only the existence of positive solutions for BVP (2), but also give an iterative scheme for approximating the solutions. It is worth stating that the first term of our iterative scheme is a constant function. Therefore, the iterative scheme is significant and feasible.

In fact, this paper is a continuation to [12] in which the existence of positive solutions of the following multi-point p-Laplacian BVP

$$\begin{cases} (\phi_p(u'))'(t) + q(t)f(t,u(t)) = 0, \quad 0 < t < 1, \\ u'(0) = \sum\limits_{i=1}^{n} \alpha_i u'(\xi_i),\ u(1) = \sum\limits_{i=1}^{n} \beta_i u(\xi_i), \end{cases} \tag{3}$$

was obtained. We should point out that, due to the fact that the boundary conditions treated in this paper are different from those considered in [12], the process of changing the two BVPs into their corresponding equivalent integral equations are different. Thus the operator T defined according to the integral equation is different, and we must define a different cone such that $T: P \to P$. According to the properties of the function in P we can only get $Ta \le a$. Therefore only one iterative sequence is obtained in this paper, while we got both $Ta \le a$ and $b \le Tb$ and so obtained two iterative sequences in [12].

We know easily that when $p > 1$, $\phi_p(s)$ is strictly increasing on $(-\infty, +\infty)$. So ϕ_p^{-1} exists. Moreover, $\phi_p^{-1} = \phi_q$, where $\frac{1}{p} + \frac{1}{q} = 1$.

We list the following conditions for convenience.

(H_1) $f \in C([0,1] \times [0,+\infty), [0,+\infty))$;

(H_2) $q(t) \in L^1(0,1)$ is nonnegative and $q(t)$ is not identically zero on any subinterval of $(0,1)$.

By (H_2), it is clear that $q(t)$ satisfies

$$0 < \int_0^1 q(t)\,dt < +\infty.$$

2. Preliminaries

We consider the Banach space $E = C[0,1]$ equipped with norm $\|w\| = \max_{0 \le t \le 1}|w(t)|$. In this paper, by a positive solution w of BVP (2) we mean a function $w \in C^1[0,1]$ with $w(t) > 0$, $0 < t < 1$, and $\phi_p(w') \in AC[0,1]$, which fulfills the equation almost everywhere.

Definition 2.1. A functional $\tau \in E$ is said to be *concave* on $[0,1]$ provided that $\tau\big(tx + (1-t)y\big) \ge t\tau(x) + (1-t)\tau(y)$ for all $x, y \in [0,1]$ and $t \in [0,1]$.

Definition 2.2. Let $(E, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a *cone* provided that the following holds:

(i) if $y \in P$ and $\lambda \ge 0$ then $\lambda y \in P$,

(ii) if $y \in P$ and $-y \in P$ then $y = 0$.

We denote

$$\begin{aligned} C^+[0,1] &= \{w \in C[0,1] \mid w(t) \ge 0, t \in [0,1]\}, \\ P &= \{w \in C^+[0,1] \mid w(t) \text{ is concave } [0,1]\}. \end{aligned}$$

It is easy to see that P is a cone in E.

Lemma 2.1. *Suppose that $y \in C^1[0,1]$ with $\phi_p(y') \in AC[0,1]$ satisfies*

$$\begin{cases} -\big(\phi_p(y')\big)'(t) \ge 0, \quad 0 < t < 1, \\ y(0) = \sum\limits_{i=1}^{q-1} \gamma_i y(\delta_i),\ y(1) = \sum\limits_{i=1}^{m-1} \eta_i y(\xi_i). \end{cases}$$

Then $y(t)$ is concave on $[0,1]$ and $y(t) \ge 0$, $t \in [0,1]$, i.e., $y \in P$.

Proof. By $-\big(\phi_p(y')\big)'(t) \ge 0$ we get that $\phi_p(y')(t)$ is non-increasing, so $y'(t)$ is non-increasing, which means that $y(t)$ is concave on $[0,1]$. Without loss of generality, suppose that $y(0) = \min\{y(0), y(1)\}$. By the concavity of $y(t)$, we know that $y(\delta_i) \ge y(0)$, $i = 1, 2, \ldots, (q-1)$. So $y(0) = \sum_{i=1}^{q-1} \gamma_i y(\delta_i) \ge \sum_{i=1}^{q-1} \gamma_i y(0)$ and thus $y(0) \ge 0$. Therefore $y(t) \ge y(0) \ge 0$ since $y(t)$ is concave on $[0,1]$. □

Lemma 2.2. *If $u_i \in E$, $i = 1, 2$ satisfy*

$$\begin{cases} -\big(\phi_p\big(u_1'(t)\big)\big)' + \big(\phi_p\big(u_2'(t)\big)\big)' \geq 0, \quad 0 \leq t \leq 1, \\ u_1(0) = \sum_{i=1}^{q-1} \gamma_i u_1(\delta_i),\ u_1(1) = \sum_{i=1}^{m-1} \eta_i u_1(\xi_i), \\ u_2(0) = \sum_{i=1}^{q-1} \gamma_i u_2(\delta_i),\ u_2(1) = \sum_{i=1}^{m-1} \eta_i u_2(\xi_i). \end{cases}$$

Then $u_2(t) \leq u_1(t)$, $0 \leq t \leq 1$.

Proof. Suppose not, then there exists $t' \in [0,1]$ such that $(u_1 - u_2)(t') = \min_{t\in I}(u_1 - u_2)(t) = c < 0$.

Step 1. We conclude that there must exist $t_0 \in (0,1)$ with $(u_1 - u_2)(t_0) = \min_{t\in I}(u_1 - u_2)(t) < 0$. In fact,

(i) $t' = 0$. If, for each δ_i with $i = 1, 2, \ldots, (q-1)$, $(u_1 - u_2)(\delta_i) > (u_1 - u_2)(0)$, then, when $\sum_{i=1}^{q-1} \gamma_i > 0$, it follows that $(u_1 - u_2)(0) = \sum_{i=1}^{q-1} \gamma_i (u_1 - u_2)(\delta_i) \geq \sum_{i=1}^{q-1} \gamma_i (u_1 - u_2)(0) > (u_1 - u_2)(0)$, which is a contradiction; if $\sum_{i=1}^{q-1} \gamma_i = 0$, i.e., $\gamma_i = 0$ for $i = 1, 2, \ldots, (q-1)$, then we have $(u_1 - u_2)(0) = \sum_{i=1}^{q-1} \gamma_i (u_1 - u_2)(\delta_i) = 0$, which is also a contradiction.

Therefore, if $t' = 0$ there must exist δ_{i_0} such that $(u_1 - u_2)(\delta_{i_0}) = (u_1 - u_2)(0)$. Let $t_0 = \delta_{i_0} \in (0,1)$.

(ii) $t' = 1$. If, for each ξ_i with $i = 1, 2, \ldots, (m-1)$, $(u_1 - u_2)(\xi_i) > (u_1 - u_2)(1)$, then, when $\sum_{i=1}^{m-1} \eta_i > 0$, we have $(u_1 - u_2)(1) = \sum_{i=1}^{m-1} \eta_i (u_1 - u_2)(\xi_i) \geq \sum_{i=1}^{m-1} \eta_i (u_1 - u_2)(1) > (u_1 - u_2)(1)$, which is a contradiction; if $\sum_{i=1}^{m-1} \eta_i = 0$, i.e., $\eta_i = 0$ for $i = 1, 2, \ldots, (m-1)$, then $(u_1 - u_2)(1) = \sum_{i=1}^{m-1} \eta_i (u_1 - u_2)(\xi_i) = 0$, which is again a contradiction. Therefore, if $t' = 1$, there must exist ξ_{i_0} such that $(u_1 - u_2)(\xi_{i_0}) = (u_1 - u_2)(1)$. Let $t_0 = \xi_{i_0} \in (0,1)$.

(iii) $t' \in (0,1)$. Let $t_0 = t' \in (0,1)$.

Then we conclude that there must exist $t_0 \in (0,1)$ with $(u_1 - u_2)(t_0) = \min_{t\in I}(u_1 - u_2)(t) < 0$.

Step 2. From Step 1, we get

$$(u_1 - u_2)'(t_0) = 0. \tag{4}$$

Since $\big(\phi_p(u_1') - \phi_p(u_2')\big)'(t) \leq 0$ and $\phi_p\big(u_1'(t_0)\big) - \phi_p\big(u_2'(t_0)\big) = 0$, we have

$$\phi_p\big(u_1'(t)\big) - \phi_p\big(u_2'(t)\big) \leq 0, \quad t \in [t_0, 1], \qquad \text{and}$$

$$\phi_p\big(u_1'(t)\big) - \phi_p\big(u_2'(t)\big) \geq 0, \quad t \in [0, t_0].$$

So

$$u_1'(t) - u_2'(t) \leq 0, \quad t \in [t_0, 1], \qquad \text{and} \qquad u_1'(t) - u_2'(t) \geq 0, \quad t \in [0, t_0],$$

i.e.,

$$(u_1 - u_2)'(t) \le 0, \quad t \in [t_0, 1], \qquad \text{and} \qquad (u_1 - u_2)'(t) \ge 0, \quad t \in [0, t_0].$$

Thus, $(u_1 - u_2)(t_0) = \max_{t\in I}(u_1 - u_2)(t)$. Therefore, $(u_1 - u_2)(t) \equiv c < 0$.

So we have $c = (u_1 - u_2)(0) = \sum_{i=1}^{q-1} \gamma_i (u_1 - u_2)(\delta_i) = \sum_{i=1}^{q-1} \gamma_i c > c$, which is a contradiction. Thus the lemma is proved. □

For any $x \in C^+[0,1]$ suppose that u is a solution of the following BVP:

$$\begin{cases} \big(\phi_p(u')\big)'(t) + q(t) f\big(t, x(t)\big) = 0, \quad 0 < t < 1, \\ u(0) = \sum\limits_{i=1}^{q-1} \gamma_i u(\delta_i), \; u(1) = \sum\limits_{i=1}^{m-1} \eta_i u(\xi_i). \end{cases}$$

Then

$$u'(t) = \phi_p^{-1}\Big(A_x - \int_0^t q(s) f\big(s, x(s)\big)\, ds\Big),$$

$$u(t) = B_x + \int_0^t \phi_p^{-1}\Big(A_x - \int_0^s q(r) f\big(r, x(r)\big)\, dr\Big)\, ds,$$

where A_x, B_x satisfy the boundary conditions, i.e.,

$$B_x = \sum_{i=1}^{q-1} \gamma_i \Big[B_x + \int_0^{\delta_i} \phi_p^{-1}\Big(A_x - \int_0^s q(r) f\big(r, x(r)\big)\, dr\Big)\, ds\Big],$$

$$\begin{aligned} B_x &+ \int_0^1 \phi_p^{-1}\Big(A_x - \int_0^s q(r) f\big(r, x(r)\big)\, dr\Big)\, ds \\ &= \sum_{i=1}^{m-1} \eta_i \Big[B_x + \int_0^{\xi_i} \phi_p^{-1}\Big(A_x - \int_0^s q(r) f\big(r, x(r)\big)\, dr\Big)\, ds\Big] \end{aligned}$$

Thus

$$\begin{aligned} u(t) = {} & \frac{\sum\limits_{i=1}^{q-1} \gamma_i \int_0^{\delta_i} \phi_p^{-1}\big(A_x - \int_0^s q(r) f\big(r, x(r)\big)\, dr\big)\, ds}{1 - \sum\limits_{i=1}^{q-1} \gamma_i} \\ & + \int_0^t \phi_p^{-1}\Big(A_x - \int_0^s q(r) f(r, x)\, dr\Big)\, ds, \end{aligned}$$

where A_x satisfies

$$\frac{\sum\limits_{i=1}^{q-1}\gamma_i\int_0^{\delta_i}\phi_p^{-1}\left(A_x-\int_0^s q(\tau)f\left(\tau,x(\tau)\right)d\tau\right)ds}{1-\sum\limits_{i=1}^{q-1}\gamma_i}+\int_0^1\phi_p^{-1}\left(A_x-\in_0^s q(\tau)f\left(\tau,x(\tau)\right)d\tau\right)ds$$

$$=\sum_{i=1}^{m-1}\eta_i\frac{\sum\limits_{i=1}^{q-1}\gamma_i\int_0^{\delta_i}\phi_p^{-1}\left(A_x-\int_0^s q(\tau)f\left(\tau,x(\tau)\right)d\tau\right)ds}{1-\sum\limits_{i=1}^{q-1}\gamma_i}$$

$$+\int_0^{\xi_i}\phi_p^{-1}\Big(A_x-\int_0^s q(\tau)f\left(\tau,x(\tau)\right)d\tau\Big)ds. \tag{5}$$

Lemma 2.3. *For any $x\in C^+[0,1]$ there exists a unique $A_x\in(-\infty,+\infty)$ satisfying* (5).

Proof. For any $x\in C^+[0,1]$ define $H_x(c):R\to R$ as follows.

$$H_x(c)=\frac{\sum\limits_{i=1}^{q-1}\gamma_i\int_0^{\delta_i}\phi_p^{-1}\left(c-\int_0^s q(\tau)f\left(\tau,x(\tau)\right)d\tau\right)ds}{1-\sum\limits_{i=1}^{q-1}\gamma_i}$$

$$+\int_0^1\phi_p^{-1}\Big(c-\int_0^s q(\tau)f\left(\tau,x(\tau)\right)d\tau\Big)ds$$

$$-\sum_{i=1}^{m-1}\eta_i\frac{\sum\limits_{i=1}^{q-1}\gamma_i\int_0^{\delta_i}\phi_p^{-1}\left(c-\int_0^s q(\tau)f\left(\tau,x(\tau)\right)d\tau\right)ds}{1-\sum\limits_{i=1}^{q-1}\gamma_i}$$

$$+\int_0^{\xi_i}\phi_p^{-1}\Big(c-\int_0^s q(\tau)f\left(\tau,x(\tau)\right)d\tau\Big)ds.$$

Since $0\le\sum_{i=1}^{m-1}\eta_i\le1$, we rewrite $H_x(c)$ as follows.

$$H_x(c)=\frac{1-\sum\limits_{i=1}^{m-1}\eta_i}{1-\sum\limits_{i=1}^{q-1}\gamma_i}\sum_{i=1}^{q-1}\gamma_i\int_0^{\delta_i}\phi_p^{-1}\Big(c-\int_0^s q(\tau)f\left(\tau,x(\tau)\right)d\tau\Big)ds$$

$$+\left(1-\sum_{i=1}^{m-1}\eta_i\right)\int_0^1\phi_p^{-1}\Big(c-\int_0^s q(\tau)f\left(\tau,x(\tau)\right)d\tau\Big)ds$$

$$+\sum_{i=1}^{m-1}\eta_i\int_{\xi_i}^{1}\phi_p^{-1}\Big(c-\int_0^s q(\tau)f\big(\tau,x(\tau)\big)\,d\tau\Big)\,ds. \tag{6}$$

Obviously the function $H_x(c)$ is strictly increasing on $(-\infty,+\infty)$ and $H_x(0)\le 0\le H_x\big(\int_0^1 q(\tau)f\big(\tau,x(\tau)\big)\,d\tau\big)$. Therefore there exists a unique constant $c\in\big[0,\int_0^1 q(\tau)f\big(\tau,x(\tau)\big)\,d\tau\big]$ satisfying $H_x(c)=0$, which means that there exists a unique constant $A_x\in\big[0,\int_0^1 q(\tau)f\big(\tau,x(\tau)\big)\,d\tau\big]$ satisfying (5). □

Remark 2.1. From the proof of Lemma 2.3, we know that for any $x\in C^+[0,1]$, $A_x\in\big[0,\int_0^1 q(\tau)f\big(\tau,x(\tau)\big)\,d\tau\big]$.

For any $x\in C^+[0,1]$, let A_x be the unique constant satisfying (5) corresponding to x. Then we have the following lemma.

Lemma 2.4. *$A_x: C^+[0,1]\to R$ is continuous in x.*

Proof. The proof of this lemma is similar to the proof of Lemma 2.3 of [12]; we therefore omit it. □

For any $x\in C^+[0,1]$, let A_x be the unique constant from (5) corresponding to x. Let us define $T: C^+[0,1]\to C[0,1]$ as follows.

$$(Tx)(t)=\frac{\sum_{i=1}^{q-1}\gamma_i\int_0^{\delta_i}\phi_p^{-1}\big(A_x-\int_0^s q(r)f\big(r,x(r)\big)\,dr\big)\,ds}{1-\sum_{i=1}^{q-1}\gamma_i}$$
$$+\int_0^t\phi_p^{-1}\Big(A_x-\int_0^s q(r)f\big(r,x(r)\big)\,dr\Big)\,ds.$$

By Lemma 2.3, we know that Tx is well defined. It is clear that a fixed point x of T in P is equal to a positive solution of BVP (2). About T, we have the following result.

Lemma 2.5. *$T: P\to P$ is completely continuous, i.e., T is continuous and compact.*

Proof. For any $x\in P$, from the definition of Tx we know that $(Tx)\in C^1[0,1]$, $\phi_p\big((Tx)'\big)\in AC[0,1]$ and

$$\begin{cases}-\big(\phi_p\big((Tx)'\big)\big)'(t)=q(t)f\big(t,x(t)\big)\ge 0, & 0<t<1,\\ (Tx)(0)=\sum_{i=1}^{q-1}\gamma_i(Tx)(\delta_i),\ (Tx)(1)=\sum_{i=1}^{m-1}\eta_i(Tx)(\xi_i).\end{cases}$$

By Lemma 2.1, Tx is concave on $[0,1]$ and $(Tx)(t) \geq 0$, $t \in I$, i.e., $Tx \in P$. Thus $TP \subset P$. Similar to Lemma 2.4 of [12], we can prove that T is completely continuous. □

Lemma 2.6 ([1]). *For any* $0 < \delta < \frac{1}{2}$, $u \in P$ *has the following properties:*

(a) $u(t) \geq \|u\| t(1-t)$ *for all* $t \in [0,1]$.

(b) $u(t) \geq \delta^2 \|u\|$ *for all* $t \in [\delta, 1-\delta]$.

3. Existence and iteration of positive solution of BVP (2)

By Remark 2.1, we know that $A_x \in \left[0, \int_0^1 q(\tau) f\left(\tau, x(\tau)\right) d\tau\right]$. Suppose that $A_x = \int_0^{\sigma_x} q(\tau) f\left(\tau, x(\tau)\right) d\tau$, where $\sigma_x \in [0,1]$ is a parameter corresponding to x. Then

$$(Tx)(t) = \begin{cases} \dfrac{\sum_{i=1}^{q-1} \gamma_i \int_0^{\delta_i} \phi_p^{-1}\left(\int_s^{\sigma_x} q(r) f\left(r, x(r)\right) dr\right) ds}{1 - \sum_{i=1}^{q-1} \gamma_i} + \int_0^t \phi_p^{-1}\left(\int_s^{\sigma_x} q(r) f\left(r, x(r)\right) dr\right) ds, \\ \qquad\qquad 0 \leq t \leq \sigma_x, \\ \dfrac{\sum_{i=1}^{q-1} \gamma_i \int_0^{\delta_i} \phi_p^{-1}\left(\int_s^{\sigma_x} q(r) f\left(r, x(r)\right) dr\right) ds}{1 - \sum_{i=1}^{q-1} \gamma_i} + \int_0^1 \phi_p^{-1}\left(\int_s^{\sigma_x} q(r) f\left(r, x(r)\right) dr\right) ds \\ \quad + \int_t^1 \phi_p^{-1}\left(\int_{\sigma_x}^s q(r) f\left(r, x(r)\right) dr\right) ds, \quad \sigma_x \leq t \leq 1. \end{cases} \tag{7}$$

It is easily verified that $\|Tx\| = (Tx)(\sigma_x)$ for any $x \in P$.

For any $0 < \delta < \frac{1}{2}$, define

$$y(x) = \int_\delta^x \phi_p^{-1}\left(\int_s^x q(\tau)\, d\tau\right) ds + \int_x^{1-\delta} \phi_p^{-1}\left(\int_x^s q(\tau)\, d\tau\right) ds, \qquad x \in [\delta, 1-\delta].$$

By (H_2), we know that $y(t) > 0$ is continuous on $[\delta, 1-\delta]$. Denote

$$A = \frac{\left(1 - \sum_{i=1}^{q-1} \gamma_i\right)}{\left(1 - \sum_{i=1}^{q-1} \gamma_i + \sum_{i=1}^{q-1} \gamma_i \delta_i\right) \phi_p^{-1}\left(\int_0^1 q(s)\, ds\right)} > 0, \qquad B = \frac{2}{\min_{t \in [\delta, 1-\delta]} y(t)} > 0. \tag{8}$$

Theorem 3.1. *Assume that* (H_1) *and* (H_2) *hold. If there exists a constant* $\delta \in \left(0, \frac{1}{2}\right)$ *and two positive numbers* a, b *with* $b < a$ *such that*

(H_3) $f(t,x) : I \times [0,a] \to [0,+\infty)$ *is non-decreasing in x;*

(H_4) $\sup_{t\in[0,1]} f(t,a) \le (aA)^{p-1}$, $\inf_{t\in[\delta,1-\delta]} f(t,\delta^2 b) \ge (bB)^{p-1}$.

Then BVP (2) *has at least one solution* $w^* \in P$ *with*

$$b \le \|w^*\| \le a \quad \textit{and} \quad \lim_{n\to+\infty} w_n = \lim_{n\to+\infty} T^n w_0 = w^* \quad \textit{where } w_0(t) \equiv a,\ t\in[0,1].$$

Proof. We denote $P[b,a] = \{w \in P \mid b \le \|w\| \le a\}$. In what follows, we first prove that $TP[b,a] \subset P[b,a]$.

Let $w \in P[b,a]$. Then $0 \le w(t) \le \max_{t\in[0,1]} w(t) = \|w\| \le a$. By Lemma 2.6, $\min_{t\in[\delta,1-\delta]} w(t) \ge \delta^2\|w\| \ge \delta^2 b$. So, by the assumption (H_4),

$$0 \le f\big(t,w(t)\big) \le f(t,a) \le \sup_{t\in[0,1]} f(t,a) \le (aA)^{p-1}, \qquad t\in[0,1], \tag{9}$$

$$f\big(t,w(t)\big) \ge f(t,\delta^2 b) \ge \inf_{t\in[\delta,1-\delta]} f(t,\delta^2 b) \ge (bB)^{p-1}, \qquad t\in[\delta,1-\delta]. \tag{10}$$

Therefore, for $w \in P[b,a]$, on the one hand, by (10) we have, when $\sigma_w \in [\delta, 1-\delta]$,

$$\begin{aligned}
2\|Tw\| &= 2(Tw)(\sigma_w)\\
&= \frac{\sum_{i=1}^{q-1} \gamma_i \int_0^{\delta_i} \phi_p^{-1}\big(\int_s^{\sigma_w} q(r) f\big(r,w(r)\big)\,dr\big)\,ds}{1-\sum_{i=1}^{q-1}\gamma_i}\\
&\quad + \int_0^{\sigma_w} \phi_p^{-1}\Big(\int_s^{\sigma_w} q(r) f\big(r,w(r)\big)\,dr\Big)\,ds\\
&\quad + \frac{\sum_{i=1}^{q-1} \gamma_i \int_0^{\delta_i} \phi_p^{-1}\big(\int_s^{\sigma_w} q(r) f\big(r,w(r)\big)\,dr\big)\,ds}{1-\sum_{i=1}^{q-1}\gamma_i}\\
&\quad + \int_0^{1} \phi_p^{-1}\Big(\int_s^{\sigma_w} q(r) f\big(r,w(r)\big)\,dr\Big)\,ds + \int_{\sigma_w}^{1} \phi_p^{-1}\Big(\int_{\sigma_w}^{s} q(r) f\big(r,w(r)\big)\,dr\Big)\,ds\\
&= (Tw)(0) + \int_0^{\sigma_w} \phi_p^{-1}\Big(\int_s^{\sigma_w} q(r) f\big(r,w(r)\big)\,dr\Big)\,ds\\
&\quad + (Tw)(1) + \int_{\sigma_w}^{1} \phi_p^{-1}\Big(\int_{\sigma_w}^{s} q(r) f\big(r,w(r)\big)\,dr\Big)\,ds
\end{aligned}$$

$$\begin{aligned}
&\geq \int_0^{\sigma_w} \phi_p^{-1}\Big(\int_s^{\sigma_w} q(r)f\big(r,w(r)\big)\,dr\Big)\,ds + \int_{\sigma_w}^{1} \phi_p^{-1}\Big(\int_{\sigma_w}^{s} q(r)f\big(r,w(r)\big)\,dr\Big)\,ds\\
&\geq \int_\delta^{\sigma_w} \phi_p^{-1}\Big(\int_s^{\sigma_w} q(r)f\big(r,w(r)\big)\,dr\Big)\,ds + \int_{\sigma_w}^{1-\delta} \phi_p^{-1}\Big(\int_{\sigma_w}^{s} q(r)f\big(r,w(r)\big)\,dr\Big)\,ds\\
&\geq bB\Big[\int_\delta^{\sigma_w} \phi_p^{-1}\Big(\int_s^{\sigma_w} q(r)\,dr\Big)\,ds + \int_{\sigma_w}^{1-\delta} \phi_p^{-1}\Big(\int_{\sigma_w}^{s} q(r)\,dr\Big)\,ds\Big]\\
&= bBy(\sigma_w) \geq 2b.
\end{aligned}$$

Thus, $\|Tw\| \geq b$. When $\sigma_w \in [0,\delta]$,

$$\begin{aligned}
\|Tw\| &= (Tw)(\sigma_w)\\
&\geq (Tw)(\delta)\\
&= \frac{\sum_{i=1}^{q-1} \gamma_i \int_0^{\delta_i} \phi_p^{-1}\big(\int_s^{\sigma_w} q(r)f\big(r,w(r)\big)\,dr\big)\,ds}{1-\sum_{i=1}^{q-1}\gamma_i} + \int_0^1 \phi_p^{-1}\Big(\int_s^{\sigma_w} q(r)f\big(r,w(r)\big)\,dr\Big)\,ds\\
&\quad + \int_\delta^1 \phi_p^{-1}\Big(\int_{\sigma_w}^{s} q(r)f\big(r,w(r)\big)\,dr\Big)\,ds\\
&= (Tw)(1) + \int_\delta^1 \phi_p^{-1}\Big(\int_{\sigma_w}^{s} q(r)f\big(r,w(r)\big)\,dr\Big)\,ds\\
&\geq \int_\delta^1 \phi_p^{-1}\Big(\int_\delta^{s} q(r)f\big(r,w(r)\big)\,dr\Big)\,ds\\
&\geq \int_\delta^{1-\delta} \phi_p^{-1}\Big(\int_\delta^{s} q(r)f\big(r,w(r)\big)\,dr\Big)\,ds\\
&= bBy(\delta) \geq b;
\end{aligned}$$

and when $\sigma_w \in [1-\delta,1]$,

$$\begin{aligned}
\|Tw\| &= (Tw)(\sigma_w)\\
&\geq (Tw)(1-\delta)\\
&= \frac{\sum_{i=1}^{q-1} \gamma_i \int_0^{\delta_i} \phi_p^{-1}\big(\int_s^{\sigma_w} q(r)f\big(r,w(r)\big)\,dr\big)\,ds}{1-\sum_{i=1}^{q-1}\gamma_i} + \int_0^{1-\delta} \phi_p^{-1}\Big(\int_s^{\sigma_w} q(r)f\big(r,w(r)\big)\,dr\Big)\,ds
\end{aligned}$$

$$= (Tw)(0) + \int_0^{1-\delta} \phi_p^{-1}\Big(\int_s^{\sigma_w} q(r)f\big(r, w(r)\big)\,dr\Big)\,ds$$
$$\geq \int_0^{1-\delta} \phi_p^{-1}\Big(\int_s^{\sigma_w} q(r)f\big(r, w(r)\big)\,dr\Big)\,ds$$
$$\geq \int_\delta^{1-\delta} \phi_p^{-1}\Big(\int_s^{1-\delta} q(r)f\big(r, w(r)\big)\,dr\Big)\,ds$$
$$= bBy(1-\delta) \geq b.$$

On the other hand, by (9), we have

$$\|Tw\| = \|(Tw)(\sigma_w)\|$$
$$= \frac{\sum_{i=1}^{q-1}\gamma_i\int_0^{\delta_i}\phi_p^{-1}\big(\int_s^{\sigma_w} q(r)f\big(r,x(r)\big)\,dr\big)\,ds}{1-\sum_{i=1}^{q-1}\gamma_i} + \int_0^{\sigma_w}\phi_p^{-1}\Big(\int_s^{\sigma_w} q(r)f\big(r,x(r)\big)\,dr\Big)\,ds$$
$$\leq \frac{\sum_{i=1}^{q-1}\gamma_i\int_0^{\delta_i}\phi_p^{-1}\big(\int_0^{1} q(r)f\big(r,x(r)\big)\,dr\big)\,ds}{1-\sum_{i=1}^{q-1}\gamma_i} + \int_0^{1}\phi_p^{-1}\Big(\int_0^{1} q(r)f\big(r,x(r)\big)\,dr\Big)\,ds$$
$$\leq (aA)\frac{\sum_{i=1}^{q-1}\gamma_i\int_0^{\delta_i}\phi_p^{-1}\big(\int_0^{1} q(r)\,dr\big)\,ds}{1-\sum_{i=1}^{q-1}\gamma_i} + (aA)\int_0^{1}\phi_p^{-1}\Big(\int_0^{1} q(r)\,dr\Big)\,ds$$
$$= (aA)\frac{1-\sum_{i=1}^{q-1}\gamma_i+\sum_{i=1}^{q-1}\gamma_i\delta_i}{1-\sum_{i=1}^{q-1}\gamma_i}\phi_p^{-1}\Big(\int_0^{1} q(r)\,dr\Big)$$
$$= a.$$

Altogether, we get $b \leq \|Tw\| \leq a$ for $w \in P[b,a]$, which means that $TP[b,a] \subset P[b,a]$.

Let $w_0(t) = a$, $t \in [0,1]$, then $w_0 \in P[b,a]$. Let $w_1 = Tw_0$, then $w_1 \in P[b,a]$. We denote

$$w_{n+1} = Tw_n = T^{n+1}w_0, \qquad n = 1,2,\ldots. \tag{11}$$

Since $TP[b,a] \subset P[b,a]$, we have $w_n \in P[b,a]$, $n = 0,1,2,\ldots.$ By Lemma 2.5, T is compact, so $\{w_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ and there exists $w^* \in P[b,a]$ such that $w_{n_k} \to w^*$.

Now, since $w \in P[b, a]$, we have

$$0 \le w_1(t) \le \|w_1\| \le a = w_0(t).$$

By the definition of w_1 and w_2, we have

$$\begin{cases} \left(\phi_p(w_1')\right)' + q(t)f(t, w_0) = 0, \quad 0 < t < 1 \\ w_1(0) = \sum\limits_{i=1}^{q-1} \gamma_i w_1(\delta_i),\ w_1(1) = \sum\limits_{i=1}^{m-1} \eta_i w_1(\xi_i), \end{cases} \tag{12}$$

$$\begin{cases} \left(\phi_p(w_2')\right)' + q(t)f(t, w_1) = 0, \quad 0 < t < 1 \\ w_2(0) = \sum\limits_{i=1}^{q-1} \gamma_i w_2(\delta_i),\ w_2(1) = \sum\limits_{i=1}^{m-1} \eta_i w_2(\xi_i). \end{cases} \tag{13}$$

Combining (12), (13), the fact that $f(t, x) : I \times [0, a] \to [0, +\infty)$ is nondecreasing in x, and $w_1 \le w_0$, we obtain that

$$\begin{cases} -\left(\phi_p\left(w_1'(t)\right)\right)' + \left(\phi_p\left(w_2'(t)\right)\right)' \ge 0, \quad 0 < t < 1, \\ w_1(0) = \sum\limits_{i=1}^{q-1} \gamma_i w_1(\delta_i),\ w_1(1) = \sum\limits_{i=1}^{m-1} \eta_i w_1(\xi_i), \\ w_2(0) = \sum\limits_{i=1}^{q-1} \gamma_i w_2(\delta_i),\ w_2(1) = \sum\limits_{i=1}^{m-1} \eta_i w_2(\xi_i). \end{cases}$$

By Lemma 2.2, we know that $w_2(t) \le w_1(t)$, $0 \le t \le 1$.

By induction, it follows that $w_{n+1}(t) \le w_n(t)$, $0 \le t \le 1$, $n = 0, 1, 2, \ldots$. Hence we see that $w_n \to w^*$. Letting $n \to \infty$ in (11), we obtain that $Tw^* = w^*$ since T is continuous. Because $\|w^*\| \ge b > 0$ and w^* is a nonnegative concave function on $[0, 1]$, we conclude that $w^*(t) > 0$, $t \in (0, 1)$. Therefore, w^* is a positive solution of BVP (2). □

Corollary 3.1. *Assume that* (H_1) *and* (H_2) *hold. If there exists a constant* $\delta \in \left(0, \frac{1}{2}\right)$ *such that*

$(1')$ $f(t, x) : I \times [0, +\infty) \to [0, +\infty)$ *is non-decreasing in x;*

$(2')$ $\overline{\lim}_{l \to 0} \inf_{t \in [\delta, 1-\delta]} \frac{f(t,l)}{l^{p-1}} > \left(\frac{B}{\delta^2}\right)^{p-1}$ *and* $\underline{\lim}_{l \to +\infty} \inf_{t \in I} \frac{f(t,l)}{l^{p-1}} < A^{p-1}$ (*in particular,* $\lim_{l \to 0} \inf_{t \in [\delta, 1-\delta]} \frac{f(t,l)}{l^{p-1}} = +\infty$ *and* $\lim_{l \to +\infty} \inf_{t \in I} \frac{f(t,l)}{l^{p-1}} = 0$), *where A, B are defined by* (8).

Then there exist two constants $a > 0$ *and* $b > 0$ *such that BVP* (2) *has at least one positive solution* $w^* \in P$ *with* $b \le \|w^*\| \le a$ *and* $\lim_{n \to +\infty} T^n w_0 = w^*$, *where* $w_0(t) \equiv a$, $t \in [0, 1]$.

Proof. Condition (H_4) of Theorem 3.1 can be obtained by condition ($2'$). Then the proof is obvious and we omit it. □

Example 3.1. Consider the existence of solutions for

$$(|u'|^{-2/3}u')'(t) + u^{1/4}(t) + \ln[u^{1/4}(t) + 1] = 0, \qquad t \in (0,1), \tag{14}$$

subject to the boundary conditions

$$u(0) = \frac{1}{2}u\left(\frac{1}{2}\right), \qquad u(1) = \frac{1}{3}u\left(\frac{1}{3}\right). \tag{15}$$

To solve BVP (14)–(15), we will use Theorem 3.1. Here $p = \frac{4}{3}$ and $f(t,u) = u^{1/4} + \ln(u^{1/4} + 1)$. Choose $\delta = \frac{1}{4}$. Then it is easy to compute that

$$y(x) = \frac{1}{2}x^4 - x^3 + \frac{15}{16}x^2 - \frac{7}{16}x + \frac{41}{512}, \qquad x \in \left[\frac{1}{4}, \frac{3}{4}\right],$$

and so $\min_{x\in[1/4,3/4]} y(x) = \frac{1}{2\times 4^4}$. Therefore we have

$$A = \frac{2}{3}, \qquad B = 4^5,$$

and we may choose two positive numbers a and b such that

$$b = \left(\frac{1}{4}\right)^{35} < a = 3^{12}.$$

Condition (H_4) of Theorem 3.1 is satisfied.

By the results of Theorem 3.1, we obtain not only the existence but also an iterative method for a positive solution of BVP (14)–(15).

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