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Uniqueness of meromorphic functions sharing two sets with finite weight

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Abstract. In this paper we consider a problem initially posed by Yi [15]. We prove two uniqueness theorems on meromorphic functions which improve and extend results of Lahiri [9] and Fang and Lahiri [1]. We provide an example that shows that a condition in one of our results is sharp.

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1. Introduction definitions and results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities).

Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a \in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $E_f(S) = \bigcup_{a \in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$.

If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM.

We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \to \infty$, outside a possible exceptional set of finite linear measure.

We use I to denote any set of infinite linear measure of $0 < r < \infty$.

Gross [2] proved that there exist three finite sets S_j $(j = 1, 2, 3)$ such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$ must be identical.

In [3] Gross asked the following question: can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?

In response to this question Yi [15] proved for meromorphic functions the following result.

Theorem A ([15]). *Let $S = \{z \mid z^n + az^{n-m} + b = 0\}$ where n and m are two positive integers such that $m \geq 2$, $n \geq 2m + 7$ with n and m having no common factor, and a, b are two nonzero constants such that $z^n + az^{n-m} + b = 0$ has no multiple roots. If f and g are two non-constant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ then $f \equiv g$.*

In the same paper Yi [15] also asked the following question: what can be said if $m = 1$ in Theorem A?

In connection with this question Yi [15] proved the following theorem.

Theorem B ([15]). *Let $S = \{z \mid z^n + az^{n-1} + b = 0\}$ where n (≥ 9) is an integer and a, b are two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple roots. If f and g are two non-constant meromorphic functions such that $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$, then either $f \equiv g$ or $f \equiv \frac{-ah(h^{n-1}-1)}{h^n-1}$ and $g \equiv \frac{-a(h^{n-1}-1)}{h^n-1}$, where h is a non-constant meromorphic function.*

To provide an answer to the question of Yi and to find under which condition $f \equiv g$, Lahiri [5] proved the following result.

Theorem C ([5]). *Let $S = \{z \mid z^n + az^{n-1} + b = 0\}$ where n (≥ 8) is a positive integer and a, b are two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple roots. If f and g are two non-constant meromorphic functions having no simple poles such that $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$, then $f \equiv g$.*

Fang and Lahiri [1] improved Theorem C by replacing the range set with a smaller one and proved the following theorem.

Theorem D ([1]). *Let $S = \{z \mid z^n + az^{n-1} + b = 0\}$ where n (≥ 7) is a positive integer and a, b are two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple roots. If f and g are two non-constant meromorphic functions having no simple poles such that $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$, then $f \equiv g$.*

Let $S = \{z \mid z^7 - z^6 - 1 = 0\}$ and

$$f = \frac{e^z + e^{2z} + \cdots + e^{6z}}{1 + e^z + \cdots + e^{6z}}, \qquad g = \frac{1 + e^z + \cdots + e^{5z}}{1 + e^z + \cdots + e^{6z}}.$$

Obviously we have $f = e^z g$, $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$, but $f \not\equiv g$. So for the validity of Theorem D, f and g must not have any simple pole.

If two meromorphic functions f and g have no simple pole then clearly $\Theta(\infty; f) \geq \frac{1}{2}$ and $\Theta(\infty; g) \geq \frac{1}{2}$. To state the next theorem we require the following definition.

Definition 1.1 ([7], [8]). Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

Definition 1.2 ([7]). Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $E_f(S) = \bigcup_{a \in S} \{z \mid f(z) - a = 0\}$.

Clearly $E_f(S) = E_f(S, \infty)$ and $\bar{E}_f(S) = E_f(S, 0)$.

Improving Theorem D Lahiri [9] showed the following theorem.

Theorem E ([9]). *Let $S = \{z \mid z^n + az^{n-1} + b = 0\}$ where n (≥ 7) is an integer, a and b are two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple roots. If for two non-constant meromorphic functions f and g, $\Theta(\infty; f) + \Theta(\infty; g) > 1$, $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$, then $f \equiv g$.*

Considering all the above theorems it is natural to ask the following questions.

i) Is it possible to further relax the nature of sharing the set $\{\infty\}$ in Theorem E such that the obtained result is a generalization of it?

ii) What happens in Theorems D and E if $\Theta(\infty; f) + \Theta(\infty; g) < 1$?

Here we shall concentrate our attention on the above two questions and provide affirmative answers to both of them. The following two theorems are the main results of this paper.

Theorem 1.1. *Let $S = \{z \mid z^n + az^{n-1} + b = 0\}$ where n (≥ 7) is an integer and a, b are are two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple roots. If for two non-constant meromorphic functions f and g, $\Theta(\infty; f) + \Theta(\infty; g) >$*

$1 + \frac{29}{6nk+6n-5}$, $E_f(S,2) = E_g(S,2)$ *and* $E_f(\{\infty\},k) = E_g(\{\infty\},k)$ *with* $0 \le k < \infty$, *then* $f \equiv g$.

Remark 1.1. In Theorem E, since $E_f(\{\infty\},\infty) = E_g(\{\infty\},\infty)$, it follows that f, g share (∞,k) for all large k. Also since $\Theta(\infty;f) + \Theta(\infty;g) > 1$, for sufficiently large k we can have $\Theta(\infty;f) + \Theta(\infty;g) > 1 + \frac{29}{6nk+6n-5}$ and hence by Theorem 1.1 we get the conclusion of Theorem E. So Theorem E can be considered a special case of Theorem 1.1.

Theorem 1.2. *Let* $S = \{z \mid z^n + az^{n-1} + b = 0\}$ *where* n (≥ 8) *is an integer and a, b be are two nonzero constants such that* $z^n + az^{n-1} + b = 0$ *has no multiple roots. If for two non-constant meromorphic functions f and g,* $\Theta(\infty;f) + \Theta(\infty;g) > \frac{4}{n-1}$, $E_f(S,2) = E_g(S,2)$ *and* $E_f(\{\infty\},0) = E_g(\{\infty\},0)$, *then* $f \equiv g$.

The following example shows that the condition $\Theta(\infty;f) + \Theta(\infty;g) > \frac{4}{n-1}$ is sharp in Theorem 1.2.

Example 1.1 (Example 2, [10]). Let $f = -a\frac{1-h^{n-1}}{1-h^n}$ and $g = -ah\frac{1-h^{n-1}}{1-h^n}$, where $h = \frac{\alpha^2(e^z-1)}{e^z-\alpha}$, $\alpha = \exp\left(\frac{2\pi i}{n}\right)$ and n (≥ 3) is an integer.

Then $T(r,f) = (n-1)T(r,h) + O(1)$ and $T(r,g) = (n-1)T(r,h) + O(1)$.

Further we see that $h \ne \alpha, \alpha^2$ and a root of $h = 1$ is not a pole of f and g. Hence $\Theta(\infty;f) = \Theta(\infty;g) = \frac{2}{n-1}$. Clearly f and g share $(\infty;\infty)$. Also $E_f(S,\infty) = E_g(S,\infty)$ because $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$, but $f \not\equiv g$.

Although the standard definitions and notations of the value distribution theory are available in [4], we explain some terminology which is used in the paper.

Definition 1.3 ([6]). For $a \in \mathbb{C} \cup \{\infty\}$we denote by $N(r,a;f \mid =1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r,a;f \mid \le m)$ $\left(N(r,a;f \mid \ge m)\right)$ the counting function of those a-points of f whose multiplicities are not greater (less) than m, where each a-point is counted according to its multiplicity.

$\overline{N}(r,a;f \mid \le m)$ $\left(\overline{N}(r,a;f \mid \ge m)\right)$ is defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r,a;f \mid < m)$, $N(r,a;f \mid > m)$, $\overline{N}(r,a;f \mid < m)$ and $\overline{N}(r,a;f \mid > m)$ are defined analogously.

Definition 1.4 ([8]). We denote by $N_2(r,a;f)$ the sum $\overline{N}(r,a;f) + \overline{N}(r,a;f \mid \ge 2)$.

Definition 1.5. We denote by $\overline{N}(r,a;f \mid = k)$ the reduced counting function of those a-points of f whose multiplicities is exactly k, where $k \ge 2$ is an integer.

Definition 1.6. Let f and g be two non-constant meromorphic functions such that f and g share $(a, 2)$ for $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p and an a-point of g with multiplicity q. We denote by $\overline{N}_L(r, a; f)$ $(\overline{N}_L(r, a; g))$ the reduced counting function of those a-points of f and g where $p > q \geq 3$ $(q > p \geq 3)$. Also we denote by $\overline{N}_E^{(3}(r, a; f)$ the reduced counting function of those a-points of f and g where $p = q \geq 3$. Clearly $\overline{N}_E^{(3}(r, a; f) = \overline{N}_E^{(3}(r, a; g)$.

Definition 1.7 ([7], [8]). Let f, g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

Definition 1.8 ([11]). Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g = b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.

Definition 1.9 ([11]). Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g \neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not b-points of g.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be non-constant meromorphic functions defined in $\mathbb{C}$. We shall denote by H and V the following two functions:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right)$$

and

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}.$$

Lemma 2.1 ([8], Lemma 1). *If F, G share $(1, 1)$ and $H \not\equiv 0$ then*

$$N(r, 1; F \mid = 1) \leq N(r, \infty; H) + S(r, F) + S(r, G).$$

Lemma 2.2 ([12]). *If $N(r, 0; f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r, 0; f^{(k)} \mid f \neq 0) \leq k\overline{N}(r, \infty; f) + N(r, 0; f \mid < k) + k\overline{N}(r, 0; f \mid \geq k) + S(r, f).$$

Lemma 2.3 ([11], Lemma 4). *Let F, G share* $(1,0)$, $(\infty,0)$ *and* $H \not\equiv 0$. *Then*

$$N(r,H) \le \overline{N}(r,0;F\,|\ge 2) + \overline{N}(r,0;G\,|\ge 2) + \overline{N}_*(r,\infty;F,G) + \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'),$$

where $\overline{N}_0(r,0;F')$ *is the reduced counting function of those zeros of* F' *which are not the zeros of* $F(F-1)$, *and* $\overline{N}_0(r,0;G')$ *is similarly defined.*

Lemma 2.4. *Let F and G be two non-constant meromorphic functions sharing* $(1,2)$. *Then*

(i) $2\overline{N}_L(r,1;F) + 3\overline{N}_L(r,1;G) + 2\overline{N}_E^{(3}(r,1;F) + \overline{N}(r,1;F\,|=2) \le N(r,1;G) - \overline{N}_L(r,1;F)$;

(ii) $2\overline{N}_L(r,1;G) + 3\overline{N}_L(r,1;F) + 2\overline{N}_E^{(3}(r,1;G) + \overline{N}(r,1;G\,|=2) \le N(r,1;F) - \overline{N}(r,1;F)$.

Proof. We prove (i) only because (ii) can be proved similarly. Let z_0 be a 1-point of F of multiplicity p a 1-point of G of multiplicity q. We denote by $N_1(r)$, $N_2(r)$ and $N_3(r)$ the counting functions of those 1-points of F and G where $3 \le q < p$, $3 \le q = p$ and $3 \le p < q$, respectively, and each point in these counting functions is counted $q-2$ times.

Since F, G share $(1,2)$, we note that

$$N(r,1;G) - \overline{N}(r,1;G) = \overline{N}_E^{(3}(r,1;F) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}(r,1;F\,|=2) + N_1(r) + N_2(r) + N_3(r). \tag{2.1}$$

Also note that

$$N_1(r) \ge \overline{N}_L(r,1;F), \tag{2.2}$$

$$N_2(r) \ge \overline{N}_E^{(3}(r,1;F), \tag{2.3}$$

$$N_3(r) \ge 2\overline{N}_L(r,1;G). \tag{2.4}$$

Using (2.2)–(2.4) in (2.1) we deduce that

$$N(r,1;G) - \overline{N}(r,1;G) \ge 2\overline{N}_L(r,1;F) + 3\overline{N}_L(r,1;G) + 2\overline{N}_E^{(3}(r,1;F) + \overline{N}(r,1;F\,|=2).$$

This proves the lemma. □

Lemma 2.5. *Let F, G share* $(1,2)$. *Then*

$$\overline{N}_L(r,1;F) \le \frac{1}{3}\overline{N}(r,0;F) + \frac{1}{3}\overline{N}(r,\infty;F) - \frac{1}{3}N_0(r,0;F') + S(r,F),$$

where $N_0(r,0;F')$ *is the counting function of those zeros of* F' *which are not the zeros of* $F(F-1)$.

Proof. Using Lemma 2.2 we see that

$$\begin{aligned}\overline{N}_L(r,1;F) &\le \overline{N}(r,1;F\mid\ge 4)\\ &\le \frac{1}{3}N(r,0;F'\mid F=1)\\ &\le \frac{1}{3}N(r,0;F'\mid F\ne 0)-\frac{1}{3}N_0(r,0;F')\\ &\le \frac{1}{3}\overline{N}(r,0;F)+\frac{1}{3}\overline{N}(r,\infty;F)-\frac{1}{3}N_0(r,0;F')+S(r,F).\end{aligned}$$

This proves the lemma. □

Lemma 2.6 ([13], Lemma 2). *Let* f *be a non-constant meromorphic function and* $P(f)=a_0+a_1f+a_2f^2+\cdots+a_nf^n$, *where* $a_0,a_1,a_2\dots,a_n$ *are constants and* $a_n\ne 0$. *Then* $T\big(r,P(f)\big)=nT(r,f)+O(1)$.

Lemma 2.7 ([14]). *If* $H\equiv 0$ *then* $T(r,G)=T(r,F)+O(1)$. *Also if* $H\equiv 0$ *and*

$$\limsup_{r\to\infty,\,r\in I}\frac{\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)}{T(r,F)}<1,$$

then $F\equiv G$ *or* $F\cdot G\equiv 1$.

Lemma 2.8. *Let* $F=\frac{f^{n-1}(f+a)}{-b}$, $G=\frac{g^{n-1}(g+a)}{-b}$, *where* $n\ (\ge 7)$ *is an integer. If* $H\equiv 0$ *then* $f^{n-1}(f+a)g^{n-1}(g+a)\equiv b^2$ *or* $f^{n-1}(f+a)\equiv g^{n-1}(g+a)$.

Proof. Since

$$\begin{aligned}&\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)\\ &\quad=\overline{N}(r,0;f)+\overline{N}(r,0;f+a)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)\\ &\qquad+\overline{N}(r,0;g+a)+\overline{N}(r,\infty;g)\\ &\quad\le 3T(r,f)+3T(r,g)+O(1),\end{aligned}$$

using Lemmas 2.6 and 2.7 we get

$$\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)\le 6T(r,f)+S(r,f). \tag{2.5}$$

Again by Lemma 2.6,

$$T(r,F) = nT(r,f) + S(r,f). \tag{2.6}$$

So from (2.5) and (2.6) we obtain from Lemma 2.7 that

$$f^{n-1}(f+a)g^{n-1}(g+a) \equiv b^2 \quad \text{or} \quad f^{n-1}(f+a) \equiv g^{n-1}(g+a).$$

This proves the lemma. □

Lemma 2.9 ([9], Lemma 5). *If f, g share $(\infty, 0)$ then for n (≥ 2)*

$$f^{n-1}(f+a)g^{n-1}(g+a) \not\equiv b^2,$$

where a, b are finite nonzero constants.

Lemma 2.10 ([10], Lemma 9). *Let f, g be two non-constant meromorphic functions such that $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n-1}$, where n (≥ 4) is an integer. Then $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$ implies that $f \equiv g$, where a is a nonzero finite constant.*

Lemma 2.11 ([16], Lemma 7). *If F, G share $(\infty, 0)$ and $V \equiv 0$, then $F \equiv G$.*

Lemma 2.12. *Let $F = \frac{f^{n-1}(f+a)}{-b}$, $G = \frac{g^{n-1}(g+a)}{-b}$ and $V \not\equiv 0$. If f, g share (∞, k), where $0 \leq k < \infty$, and F, G share $(1, 2)$. Then the poles of F and G are zeros of V and*

$$\begin{aligned}
&(nk+n-1)\overline{N}(r,\infty; f \mid \geq k+1) \\
&\quad = (nk+n-1)\overline{N}(r,\infty; g \mid \geq k+1) \\
&\quad \leq \overline{N}(r,0;f) + \overline{N}(r,0;f+a) + \overline{N}(r,0;g) + \overline{N}(r,0;g+a) \\
&\qquad + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + S(r,f) + S(r,g).
\end{aligned}$$

Proof. Since f, g share $(\infty; k)$, it follows that F, G share $(\infty; nk)$ and so a pole of F with multiplicity p $(\geq nk+1)$ is a pole of G with multiplicity r $(\geq nk+1)$ and vice versa. We note that F and G have no pole of multiplicity q where $nk < q < nk+n$. Now, using the Milloux theorem [4], p. 55, and Lemma 2.6, we obtain from the definition of V that

$$m(r,V) = S(r,f) + S(r,g).$$

Hence

$$\begin{aligned}
&(nk+n-1)\overline{N}(r,\infty; f \mid \geq k+1) \\
&\quad = (nk+n-1)\overline{N}(r,\infty; g \mid \geq k+1) \\
&\quad = (nk+n-1)\overline{N}(r,\infty; F \mid \geq nk+n)
\end{aligned}$$

$$
\begin{aligned}
&\le N(r,0;V)\\
&\le T(r,V)+O(1)\\
&\le N(r,\infty;V)+m(r,V)+O(1)\\
&\le N(r,\infty;V)+S(r,f)+S(r,g)\\
&\le \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\le \overline{N}(r,0;f)+\overline{N}(r,0;f+a)+\overline{N}(r,0;g)+\overline{N}(r,0;g+a)\\
&\quad+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+S(r,f)+S(r,g).
\end{aligned}
$$

This proves the lemma. □

Lemma 2.13. *Let F, G share $(1,2)$, (∞,k) where $0\le k<\infty$ and $H\not\equiv 0$. Then*

i) $T(r,F)\le N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)-m(r,1;G)-\overline{N}_E^{(3}(r,1;F)-\overline{N}_L(r,1;G)+S(r,F)+S(r,G)$;

ii) $T(r,G)\le N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)-m(r,1;F)-\overline{N}_E^{(3}(r,1;G)-\overline{N}_L(r,1;F)+S(r,F)+S(r,G)$.

Proof. We prove only i) since the proof of ii) is similar.

By the second fundamental theorem we obtain

$$T(r,F)\le \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)-N_0(r,0;F')+S(r,F) \tag{2.7}$$

and

$$T(r,G)\le \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}(r,1;G)-N_0(r,0;G')+S(r,G) \tag{2.8}$$

Adding (2.7) and (2.8) we get

$$
\begin{aligned}
T(r,F)+T(r,G)\le{}& \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)\\
&+\overline{N}(r,1;F)+\overline{N}(r,1;G)-N_0(r,0;F')-N_0(r,0;G')\\
&+S(r,F)+S(r,G).
\end{aligned}\tag{2.9}
$$

Since

$$
\begin{aligned}
\overline{N}(r,1;F)+\overline{N}(r,1;G)\le{}& N(r,1;F\mid=1)+\overline{N}(r,1;F\mid=2)+\overline{N}_E^{(3}(r,1;F)\\
&+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,1;G),
\end{aligned}
$$

using Lemmas 2.1 and 2.3, in view of Definition 1.7 we obtain that

$$
\begin{aligned}
&\overline{N}(r,1;F)+\overline{N}(r,1;G)\\
&\quad\le \overline{N}(r,0;F\mid\ge 2)+\overline{N}(r,0;G\mid\ge 2)+\overline{N}_*(r,\infty;F,G)\\
&\qquad+2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}(r,1;F\mid=2)+\overline{N}(r,1;G).
\end{aligned}
$$

Now substituting the value of $\overline{N}(r,1;G)$ from Lemma 2.4 we obtain

$$\begin{aligned}
&\overline{N}(r,1;F)+\overline{N}(r,1;G)\\
&\quad\le \overline{N}(r,0;F\mid\ge 2)+\overline{N}(r,0;G\mid\ge 2)+\overline{N}_*(r,\infty;F,G)\\
&\qquad+2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}(r,1;F\mid=2)\\
&\qquad+T(r,G)-m(r,1;G)+O(1)-\overline{N}(r,1;F\mid=2)\\
&\qquad-\overline{N}_E^{(3}(r,1;F)-2\overline{N}_L(r,1;F)-3\overline{N}_L(r,1;G)\\
&\qquad+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\
&\quad\le \overline{N}(r,0;F\mid\ge 2)+\overline{N}(r,0;G\mid\ge 2)+\overline{N}_*(r,\infty;F,G)\\
&\qquad+T(r,G)-m(r,1;G)-\overline{N}_L(r,1;G)-\overline{N}_E^{(3}(r,1;F)\\
&\qquad+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \qquad (2.10)
\end{aligned}$$

In view of Definition 1.4 the lemma now follows from (2.9) and (2.10). □

3. Proofs of the theorems

Proof of Theorem 1.1. Let $F=\frac{f^{n-1}(f+a)}{-b}$ and $G=\frac{g^{n-1}(g+a)}{-b}$. Since $E_f(S,2)=E_g(S,2)$ and $E_f(\{\infty\},k)=E_g(\{\infty\},k)$ it follows that F, G share $(1,2)$ and (∞,nk). So $\overline{N}_*(r,\infty;F,G)\le\overline{N}(r,\infty;F\mid\ge nk+n)=\overline{N}(r,\infty;f\mid\ge k+1)$. If possible, suppose that $H\not\equiv 0$. Then $F\not\equiv G$. So from Lemma 2.11 we get $V\not\equiv 0$. Hence from Lemmas 2.5, 2.6, 2.12 and 2.13 we obtain for $\varepsilon\ (>0)$

$$\begin{aligned}
nT(r,f)&\le N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\
&\quad+\overline{N}(r,\infty;F\mid\ge nk+n)-\overline{N}_L(r,1;G)+S(r,F)+S(r,G)\\
&\le 2\overline{N}(r,0;f)+N_2(r,0;f+a)+2\overline{N}(r,0;g)+N_2(r,0;g+a)\\
&\quad+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f\mid\ge k+1)\\
&\quad-\overline{N}_L(r,1;G)+S(r,f)+S(r,g)\\
&\le 3T(r,f)+3T(r,g)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\
&\quad+\frac{1}{nk+n-1}[\overline{N}(r,0;f)+\overline{N}(r,0;f+a)+\overline{N}(r,0;g)\\
&\quad+\overline{N}(r,0;g+a)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)]\\
&\quad-\overline{N}_L(r,1;G)+S(r,f)+S(r,g)
\end{aligned}$$

$$\begin{aligned}
&\le \left[3+\frac{2}{nk+n-1}+\frac{2}{3(nk+n-1)}\right]T(r,f)\\
&\quad+\left[3+\frac{2}{nk+n-1}\right]T(r,g)+\left[1+\frac{1}{6(nk+n-1)}\right]\\
&\quad\cdot\big(1-\Theta(\infty;f)+\varepsilon\big)T(r,f)+\left[1+\frac{1}{6(nk+n-1)}\right]\\
&\quad\cdot\big(1-\Theta(\infty;g)+\varepsilon\big)T(r,g)+S(r,f)+S(r,g)\\
&\le \left[7+\frac{nk+n+4}{nk+n-1}-\frac{6nk+6n-5}{6(nk+n-1)}\{\Theta(\infty;f)+\Theta(\infty;g)-2\varepsilon\}\right]T(r)\\
&\quad+S(r,f)+S(r,g). \qquad (3.1)
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
nT(r,g) &\le \left[7+\frac{nk+n+4}{nk+n-1}-\frac{6nk+6n-5}{6(nk+n-1)}\{\Theta(\infty;f)+\Theta(\infty;g)-2\varepsilon\}\right]T(r)\\
&\quad+S(r,f)+S(r,g). \qquad (3.2)
\end{aligned}$$

Combining (3.1) and (3.2) we obtain

$$\left[n-7-\frac{nk+n+4}{nk+n-1}+\frac{6nk+6n-5}{6(nk+n-1)}\{\Theta(\infty;f)+\Theta(\infty;g)-2\varepsilon\}\right]T(r)\le S(r). \qquad (3.3)$$

Since $\varepsilon > 0$ (3.3) leads to a contradiction. Hence $H \equiv 0$ and the theorem follows from Lemmas 2.8, 2.9 and Lemma 2.10. This completes the proof of the theorem. □

Proof of Theorem 1.2. Let F and G be defined as in Theorem 1.1. Then F, G share $(1,2)$ and $(\infty,0)$. So $\overline{N}_*(r,\infty;F,G) \le \overline{N}(r,\infty;F) = \overline{N}(r,\infty;G)$.

If possible, suppose $H \equiv 0$. We obtain from Lemmas 2.6 and 2.13

$$\begin{aligned}
T(r,F)+T(r,G) &\le 2N_2(r,0;F)+2N_2(r,0;G)+3\overline{N}(r,\infty;F)\\
&\quad+3\overline{N}(r,\infty;G)-\overline{N}_L(r,1;F)-\overline{N}_L(r,1;G)\\
&\quad+S(r,F)+S(r,G).
\end{aligned}$$

Hence

$$\begin{aligned}
nT(r,f)+nT(r,g) &\le 4\overline{N}(r,0;f)+2N_2(r,0;f+a)+3\overline{N}(r,\infty;f)\\
&\quad+4\overline{N}(r,0;g)+2N_2(r,0;g+a)+3\overline{N}(r,\infty;g)\\
&\quad-\overline{N}_L(r,1;F)-\overline{N}_L(r,1;G)+S(r,f)+S(r,g)
\end{aligned}$$

$$\leq 6T(r,f) + 6T(r,g) + 3\overline{N}(r,\infty;f) + 3\overline{N}(r,\infty;g) - \overline{N}_L(r,1;F) - \overline{N}_L(r,1;G) + S(r,f) + S(r,g). \quad (3.4)$$

Since $H \not\equiv 0$, $F \not\equiv G$. So from Lemma 2.11 we get $V \not\equiv 0$ Hence using Lemma 2.12 for $k = 0$ we get from (3.4)

$$\begin{aligned} nT(r,f) + nT(r,g) &\leq 6T(r,f) + 6T(r,g) + \frac{6}{n-1}\{\overline{N}(r,0;f) + \overline{N}(r,0;f+a)\} \\ &\quad + \frac{6}{n-1}\{\overline{N}(r,0;g) + \overline{N}(r,0;g+a)\} + \frac{6}{n-1}\overline{N}_L(r,1;F) \\ &\quad + \frac{6}{n-1}\overline{N}_L(r,1;G) - \overline{N}_L(r,1;F) - \overline{N}_L(r,1;G) \\ &\quad + S(r,f) + S(r,g) \\ &\leq \left\{6 + \frac{12}{n-1}\right\}T(r,f) + \left\{6 + \frac{12}{n-1}\right\}T(r,g) \\ &\quad + S(r,f) + S(r,g). \end{aligned}$$

Thus

$$\left(n - 6 - \frac{12}{n-1}\right)T(r,f) + \left(n - 6 - \frac{12}{n-1}\right)T(r,g) \leq S(r,f) + S(r,g),$$

which is a contradiction for any integer n (≥ 8). Hence $H \equiv 0$ and so the theorem follows from Lemmas 2.8, 2.9 and 2.10. □

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References

[1] M.-L. Fang and I. Lahiri, Unique range set for certain meromorphic functions. *Indian J. Math.* **45** (2003), 141–150. Zbl 1048.30014 MR 2035902

[2] F. Gross, On the distribution of values of meromorphic functions. *Trans. Amer. Math. Soc.* **131** (1968), 199–214; addendum *ibid.* **136** (1969), 547. Zbl 0157.12903 Zbl 0175.07704 MR 0220938

[3] F. Gross, Factorization of meromorphic functions and some open problems. In *Complex analysis* (Proc. Conf., Univ. Kentucky, Lexington, Ky., 1976), Lecture Notes in Math. 599, Springer-Verlag, Berlin 1977, 51–67. Zbl 0357.30007 MR 0450529

[4] W. K. Hayman, *Meromorphic functions*. Oxford Mathematical Monographs, Clarendon Press, Oxford 1964. Zbl 0115.06203 MR 0164038

[5] I. Lahiri, The range set of meromorphic derivatives. *Northeast. Math. J.* **14** (1998), 353–360. Zbl 0934.30027 MR 1685269

[6] I. Lahiri, Value distribution of certain differential polynomials. *Internat. J. Math. Math. Sci.* **28** (2001), 83–91. Zbl 0999.30023 MR 1885054

[7] I. Lahiri, Weighted sharing and uniqueness of meromorphic functions. *Nagoya Math. J.* **161** (2001), 193–206. Zbl 0981.30023 MR 1820218

[8] I. Lahiri, Weighted value sharing and uniqueness of meromorphic functions. *Complex Variables Theory Appl.* **46** (2001), 241–253. Zbl 1025.30027 MR 1869738

[9] I. Lahiri, On a question of Hong Xun Yi. *Arch. Math.* (*Brno*) **38** (2002), 119–128. Zbl 1087.30028 MR 1909593

[10] I. Lahiri and A. Banerjee, Uniqueness of meromorphic functions with deficient poles. *Kyungpook Math. J.* **44** (2004), 575–584. Zbl 1070.30015 MR 2108462

[11] I. Lahiri and A. Banerjee, Weighted sharing of two sets. *Kyungpook Math. J.* **46** (2006), 79–87. Zbl 1103.30017 MR 2214802

[12] I. Lahiri and S. Dewan, Value distribution of the product of a meromorphic function and its derivative. *Kodai Math. J.* **26** (2003), 95–100. Zbl 1077.30025 MR 1966685

[13] C. C. Yang, On deficiencies of differential polynomials. II. *Math. Z.* **125** (1972), 107–112. Zbl 0217.38402 MR 0294642

[14] H.-X. Yi, Meromorphic functions that share one or two values. *Complex Variables Theory Appl.* **28** (1995), 1–11. Zbl 0841.30027 MR 1700258

[15] H.-X. Yi, Unicity theorems for meromorphic or entire functions. II. *Bull. Austral. Math. Soc.* **52** (1995), 215–224. Zbl 0844.30022 MR 1348480

[16] H.-X. Yi, Meromorphic functions that share three sets. *Kodai Math. J.* **20** (1997), 22–32. Zbl 0882.30019 MR 1443362

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