Portugal. Math. (N.S.)
Vol. 65, Fasc. 1, 2008, 95–120

**Portugaliae Mathematica**


# Existence of solutions for degenerated problems in $L^1$ having lower order terms with natural growth

L. Aharouch, E. Azroul and A. Benkirane

(Communicated by Miguel Ramos)

**Abstract.** We prove the existence of a solution for a strongly nonlinear degenerated problem associated to the equation

$$Au + g(x, u, \nabla u) = f,$$

where $A$ is a Leray–Lions operator from the weighted Sobolev space $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^*)$. While $g(x, s, \xi)$ is a nonlinear term having natural growth with respect to $\xi$ and no growth with respect to $s$, it satisfies a sign condition on $s$, i.e., $g(x, s, \xi) \cdot s \geq 0$ for every $s \in \mathbb{R}$. The right-hand side $f$ belongs to $L^1(\Omega)$.

**Mathematics Subject Classification (2000).** 35J15, 35J20, 35J70.
**Keywords.** Weighted Sobolev spaces, boundary value problems, truncations.

## 1. Introduction and Basic assumptions

Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$. Let $p$ be a real number with $1 < p < \infty$ and let $p'$ be its Hölder conjugate (i.e., $\frac{1}{p} + \frac{1}{p'} = 1$). By $w = \{w_i(x) \mid i = 0, \ldots, N\}$ we denote a collection of weight functions on $\Omega$. Consider the following nonlinear elliptic degenerated problem

$$\begin{gathered} -\operatorname{div}\big(a(x, u, \nabla u)\big) + g(x, u, \nabla u) = f \quad \text{ in } \Omega, \\ u = 0 \quad \text{ on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following assumptions:

$(\mathrm{H}_1')$ (Growth, monotonicity and degeneracy)

$$|a_i(x, s, \xi)| \leq w_i^{1/p}(x)\Big[k(x) + \sigma^{1/p'}|s|^{q/p'} + \sum_{j=1}^{N} w_j^{1/p'}(x)|\xi_j|^{p-1}\Big], \quad 1 \leq i \leq N, \tag{1.2}$$

$$[a(x,s,\xi) - a(x,s,\eta)](\xi - \eta) > 0 \qquad \text{for all } \xi \neq \eta \in \mathbb{R}^N, \tag{1.3}$$

$$a(x,s,\zeta)\zeta \geq \alpha \sum_{i=1}^{N} w_i(x)|\zeta_i|^p, \tag{1.4}$$

for a.e. $x$ in $\Omega$, all $s \in \mathbb{R}$ and all $\zeta \in \mathbb{R}^N$, where $k(x)$ is a positive function in $L^{p'}(\Omega)$, $\alpha$ is some constant strictly positive and where $\sigma(x)$ and $q$ are the so-called Hardy parameters (cf. hypotheses $(H_0)$ below).

$(H_2)$ (Sign condition and growth)
$g(x,s,\xi)$ is a Carathéodory function satisfying

$$g(x,s,\xi) \cdot s \geq 0, \tag{1.5}$$

$$|g(x,s,\xi)| \leq b(|s|)\Big(c(x) + \sum_{i=1}^{N} w_i(x)|\xi_i|^p\Big), \tag{1.6}$$

where $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive increasing function and $c(x)$ is a positive function in $L^1(\Omega)$.

We will be concerned with some existence result for the solutions of (1.1).

We begin by recalling some previous works on nonlinear elliptic equations.

In the variational case (i.e., $f$ belongs to $W^{-1,p'}(\Omega, w^*)$) it is well known (see [2]) that there exists a weak solution $u$ of (1.1) with $u \in W_0^{1,q}(\Omega, w)$.

The case where $f$ is a function in $L^1(\Omega)$ is investigated in [3], but under the following additional assumption on $g$,

$$|g(x,s,\xi)| \geq \gamma \sum_{i=1}^{N} w_i|\xi_i|^p \qquad \text{for } |s| \text{ sufficiently large}, \tag{1.7}$$

and this implies that such a solution belongs to $W_0^{1,p}(\Omega, w)$.

Note that the results of [2] and [3] are given under the following integrability of the Hardy function $\sigma$,

$$\sigma^{1-q'} \in L^1_{\text{loc}}(\Omega), \qquad 1 < q < \infty. \tag{1.8}$$

Let us recall this integrability condition that has been used in order to prove the existence of a solution for the approximate problem, while (1.7) plays a crucial role in the a priori estimates. More precisely, we have proved that the solutions $u_n$ of the approximate problem are bounded in $W_0^{1,p}(\Omega, w)$.

We are pleased to refer to [13] where the author establishes an existence result for problem (1.1) in the non-degenerated case without assuming any coercivity condition on the nonlinearity $g$. Moreover the solution $u$ belongs to $W_0^{1,q}(\Omega)$ for all $q < \frac{N(p-1)}{N-1}$, which implies that $a(x, u, \nabla u) \in L^1(\Omega)$ so that it is then possible to find a weak solution of (1.1); see [13].

The aim of this paper is to prove the existence of a solution for (1.1) in weighted Sobolev spaces without assuming the conditions (1.7) and (1.8). For this we will approximate $f$ with regular functions $f_n$ and the nonlinearity $g$ by the sequence

$$g_n(x, s, \xi) = n \frac{g(x, s, \xi)}{1 + \frac{1}{n}|g(x, s, \xi)|} T_{1/n}(\sigma^{1/q}).$$

We have considered the following approximate problem

$$\begin{gathered} u_n \in W_0^{1,p}(\Omega, w), \\ \int_\Omega a(x, u_n, \nabla u_n) \nabla v \, dx + \int_\Omega g_n(x, u_n, \nabla u_n) v \, dx = \int_\Omega f_n v \, dx, \\ v \in W_0^{1,p}(\Omega, w) \end{gathered} \tag{1.9}$$

and studied the possibility to find a solution of (1.1) as limit of a subsequence $(u_n)_n$ of solutions to (1.9). We are going to prove the existence of $u_n$ by using the techniques of pseudo-monotonicity.

In our framework, the boundedness of $u_n$ in a weighted Sobolev space is not guaranteed because of the non-existence of the imbedding theorems and the violation of condition (1.7). (But in the non-weighted case it is known that $u_n$ is bounded in $W_0^{1,q}(\Omega)$ for all $q < \frac{N(p-1)}{N-1}$; see [13].) However an a priori estimate of the sequence $\left(T_k(u_n)\right)_n$ is always available, and by adapting the same techniques introduced in [4], we can show that $u_n$ converges almost everywhere to some function $u$ in $\Omega$. Thus we can prove the strong convergence of $T_k(u_n)$.

Note that the existence results for a weak solution of problem (1.1) in weighted Sobolev space appear in the literature only under slightly stronger conditions. For completeness we prove in Theorem 3.1 an existence result in the setting of our hypotheses. Then the solution constructed via approximation methods is not necessarily in $W^{1,1}(\Omega; w)$ and has not necessarily a gradient in the usual sense. In order to resolve this difficulty we argue as in [4] and seek as solution a new space $\mathscr{T}_0^{1,p}(\Omega; w)$. This leads to the notion of entropy solution.

Our main result (Theorem 3.1) can be viewed as a continuation of the analogous result in [3] in the sense of a non-coercive perturbation term and free Hardy weight.

The present paper is organized as follows. In Section 2 we begin with some preliminaries results. In Section 3 we present and prove our main existence result.

To conclude this section, let us mention that if we take $w \equiv 1$ in our present work, we obtain an existence result for problem (1.1) in the non-weighted case as in [13]. The approach is different from that in [13] and allows for some more general coercivity of type (3.1). However, when $w \not\equiv 1$, we do not know how to extend our approach in the case where the weaker coercivity (1.4) is assumed instead of the stronger one (3.1); see Remarks 3.3, 3.4, 4.1 and 4.2 below.

## 2. Preliminaries

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$. Let $p$ be a real number such that $1 < p < \infty$ and let $w = \{w_i(x); i = 1, \ldots, N\}$ be a vector of weight functions, i.e., every component $w_i(x)$ is a measurable function which is strictly positive almost everywhere in $\Omega$. Further, we suppose in all our considerations that

$$w_i \in L^1_{\mathrm{loc}}(\Omega) \tag{2.1}$$

and

$$w_i^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\Omega) \qquad \text{for } 0 \leq i \leq N. \tag{2.2}$$

We define the weighted space with weight $\gamma$ in $\Omega$ as

$$L^p(\Omega, \gamma) = \{u(x), u\gamma^{1/p} \in L^p(\Omega)\}$$

endowed with the norm

$$\|u\|_{p,\gamma} = \Big(\int_\Omega |u(x)|^p \gamma(x)\, dx\Big)^{1/p}.$$

We denote by $W^{1,p}(\Omega, w)$ the weighted Sobolev space of all real-valued functions $u \in L^p(\Omega, w_0)$ such that the derivatives in the sense of distributions satisfy

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i) \qquad \text{for all } i = 1, \ldots, N.$$

Then $W^{1,p}(\Omega, w)$ is a Banach space with respect to the norm

$$\|u\|_{1,p,w} = \Big(\int_\Omega |u(x)|^p w_0\, dx + \sum_{i=1}^N \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^p w_i(x)\, dx\Big)^{1/p}. \tag{2.3}$$

$W_0^{1,p}(\Omega, w)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.3). Note that $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, w)$ and $\big(W_0^{1,p}(\Omega, w), \|\cdot\|_{1,p,w}\big)$ is a reflexive Banach space.

We recall that the dual of the weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \{w_i^* = w_i^{1-p'}\}$, $i = 1, \dots, N$. For more details the reader is referred to [9].

Now we introduce the truncature operator. For a given constant $k > 0$ we define the cut function $T_k : \mathbb{R} \to \mathbb{R}$ by

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k \operatorname{sign}(s) & \text{if } |s| > k. \end{cases} \tag{2.4}$$

For a function $u = u(x)$, $x \in \Omega$, we define the truncated function $T_k u = T_k(u)$ pointwise: for every $x \in \Omega$ the value of $(T_k u)$ at $x$ is just $T_k\big(u(x)\big)$. We now introduce the functional space that will need in our work:

$$\mathcal{T}_0^{1,p}(\Omega, w) = \{u : \Omega \to \mathbb{R} \text{ measurable} \mid T_k(u) \in W_0^{1,p}(\Omega, w) \text{ for all } k > 0\}.$$

The following lemma is a generalization of [4], Lemma 2.1, to weighted Sobolev spaces (its proof is a slight modification of the original proof [4]).

**Lemma 2.1.** *For every $u \in \mathcal{T}_0^{1,p}(\Omega, w)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}} \text{ almost everywhere in } \Omega \text{ for every } k > 0.$$

*We will define the gradient of $u$ as the function $v$ and denote it by $v = \nabla u$.*

**Lemma 2.2.** *Let $\lambda \in \mathbb{R}$ and let $u$ and $v$ be two measurable functions which are finite almost everywhere and which belong to $\mathcal{T}_0^{1,p}(\Omega, w)$. Then*

$$\nabla(u + \lambda v) = \nabla u + \lambda \nabla v \text{ a.e. in } \Omega,$$

*where $\nabla u$, $\nabla v$ and $\nabla(u + \lambda v)$ are the gradients of $u$, $v$ and $u + \lambda v$ introduced in Lemma* 2.1.

The proof is similar to the proof of [7], Lemma 2.12, in the non-weighted case.

**Definition 2.1.** Let $Y$ be a reflexive Banach space. A bounded operator $B$ from $Y$ to its dual $Y^*$ is called *pseudo-monotone* if for any sequence $u_n \in Y$ with $u_n \rightharpoonup u$ weakly in $Y$ and $\limsup_{n\to+\infty} \langle Bu_n, u_n - u\rangle \leq 0$, we have

$$\liminf_{n\to+\infty} \langle Bu_n, u_n - v\rangle \geq \langle Bu, u - v\rangle \qquad \text{for all } v \in Y.$$

Now we state the following assumption.

($H_0$) The expression

$$\|u\| = \Big(\sum_{i=1}^{N}\int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^p w_i(x)\,dx\Big)^{1/p} \tag{2.5}$$

is a norm on $W_0^{1,p}(\Omega)$ equivalent to the norm (2.3).

There exists a weight function $\sigma$ strictly positive a.e. in $\Omega$ and a parameter $q$, $1 < q < \infty$, such that the Hardy inequality

$$\Big(\int_\Omega |u|^q \sigma(x)\,dx\Big)^{1/q} \le C\Big(\sum_{i=1}^{N}\int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^p w_i(x)\,dx\Big)^{1/p} \tag{2.6}$$

holds for every $u \in W_0^{1,p}(\Omega, w)$ with a constant $C > 0$ independent of $u$. Moreover, the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega, \sigma) \tag{2.7}$$

determined by the inequality (2.6) is compact.

Note that $\big(W_0^{1,p}(\Omega, w), \|u\|\big)$ is a uniformly convex (and thus reflexive) Banach space.

**Remark 2.1.** Assume that $w_0(x) = 1$ and, in addition, the integrability condition holds: there exists $\nu \in \big]\frac{N}{p}, \infty\big[ \cap \big[\frac{1}{p-1}, \infty\big[$ such that $w_i^{-\nu} \in L^1(\Omega)$ for all $i = 1, \dots, N$ (which is stronger than (2.2)). Then

$$\|u\| = \Big(\sum_{i=1}^{N}\int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^p w_i(x)\,dx\Big)^{1/p}$$

is a norm defined on $W_0^{1,p}(\Omega, w)$. It is equivalent to (2.3). Moreover,

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega)$$

for all $1 \le q < p_1^*$ if $p\nu < N(\nu+1)$ and for all $q \ge 1$ if $p\nu \ge N(\nu+1)$, where $p_1 = \frac{p\nu}{\nu+1}$ and $p_1^* = \frac{Np_1}{N-p_1} = \frac{Np\nu}{N(\nu+1)-p\nu}$ is the Sobolev conjugate of $p_1$ (see [9]). Thus hypothesis ($H_0$) is satisfied for $\sigma \equiv 1$.

**Remark 2.2.** We use the special weight functions $w$ and $\sigma$ expressed in terms of the distance to the boundary $\partial\Omega$. Denote $d(x) = \operatorname{dist}(x, \partial\Omega)$ and set

$$w(x) = d^\lambda(x), \qquad \sigma(x) = d^\mu(x).$$

In this case the Hardy inequality reads

$$\Big(\int_\Omega |u|^q d^\mu(x)\,dx\Big)^{1/q} \le C\Big(\int_\Omega |\nabla u|^p d^\lambda(x)\,dx\Big)^{1/p}.$$

(i) For $1 < p \le q < \infty$,

$$\lambda < p-1, \qquad \frac{N}{q} - \frac{N}{p} + 1 \ge 0, \qquad \frac{\mu}{q} - \frac{\lambda}{p} + \frac{N}{q} - \frac{N}{p} + 1 > 0. \tag{2.8}$$

(ii) For $1 \le q < p < \infty$,

$$\lambda < p-1, \qquad \frac{\mu}{q} - \frac{\lambda}{p} + \frac{1}{q} - \frac{1}{p} + 1 > 0. \tag{2.9}$$

The conditions (2.8) or (2.9) are sufficient for the compact imbedding (2.7) to hold; see, e.g., [8], Example 1, [9], Example 1.5, p. 34, and [16], Theorems 19.17 and 19.22.

Now we give the following technical lemmas which are needed later.

**Lemma 2.3** (cf. [3, 15]). *Let $g \in L^r(\Omega,\gamma)$ and let $g_n \in L^r(\Omega,\gamma)$, with $\|g_n\|_{\Omega,\gamma} \le c$, $1 < r < \infty$. If $g_n(x) \to g(x)$ a.e. in $\Omega$, then $g_n \rightharpoonup g$ weakly in $L^r(\Omega,\gamma)$.*

**Lemma 2.4** (cf. [3], [11]). *Assume that $(H_0)$ holds. Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $u \in W_0^{1,p}(\Omega,w)$. Then $F(u) \in W_0^{1,p}(\Omega,w)$. Moreover, if the set $D$ of discontinuity points of $F'$ is finite, then*

$$\frac{\partial(F\circ u)}{\partial x_i} = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} \text{ a.e. in } \{x\in\Omega \mid u(x)\notin D\},\\ 0 \text{ a.e. in } \{x\in\Omega \mid u(x)\in D\}.\end{cases}$$

From the previous lemma we deduce the following.

**Lemma 2.5.** *Assume that $(H_0)$ holds. Let $u \in W_0^{1,p}(\Omega,w)$ and let $T_k(u)$ be the usual truncation, $k \in \mathbb{R}^+$. Then $T_k(u) \in W_0^{1,p}(\Omega,w)$. Moreover,*

$$T_k(u) \to u \text{ strongly in } W_0^{1,p}(\Omega,w).$$

## 3. Main results

First we define a variant of assumption $(\mathrm{H}_1')$.

($H_1$) This is the same as condition ($H_1'$), with (1.4) is replaced by: there exist $v_0 \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$ and $\delta \in L^1(\Omega)$ such that

$$a(x,s,\xi)(\xi - \nabla v_0) \geq \alpha \sum_{i=1}^{N} w_i(x)|\xi_i|^p - \delta(x). \tag{3.1}$$

The main result of the paper is the following existence theorem.

**Theorem 3.1.** *Assume that* ($H_0$)–($H_2$) *hold and let* $f \in L^1(\Omega)$. *Then there exists at least one solution of the following problem:*

$$\begin{gathered} u \in \mathcal{T}_0^{1,p}(\Omega, w), \qquad g(x,u,\nabla u) \in L^1(\Omega), \\ \int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \\ \leq \int_\Omega f T_k(u-v)\,dx \quad \text{for all } v \in W_0^{1,p}(\Omega,w) \cap L^\infty(\Omega) \text{ and all } k > 0. \end{gathered} \tag{3.2}$$

**Remark 3.1.** Theorem 3.1 has not yet been proved for classical Sobolev spaces; see, however, [13] for the case where $v_0 = 0$.

The following lemma plays an important rôle in the proof of our main result.

**Lemma 3.1** (cf. [3], [15]). *Assume that* ($H_0$) *and* ($H_1$) *hold, and let* $(u_n)_n$ *be a sequence in* $W_0^{1,p}(\Omega, w)$ *such that* $u_n \rightharpoonup u$ *weakly in* $W_0^{1,p}(\Omega, w)$ *and*

$$\int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\,dx \to 0.$$

*Then*

$$u_n \to u \quad \text{in } W_0^{1,p}(\Omega, w).$$

**3.1. Study of approximate problem.** Put

$$g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1 + \frac{1}{n}|g(x,s,\xi)|}\theta_n(x), \tag{3.3}$$

with $\theta_n(x) = nT_{1/n}(\sigma^{1/q}(x))$.

Note that $g_n(x,s,\xi)$ satisfies the following conditions:

$$g_n(x,s,\xi)s \geq 0, \qquad |g_n(x,s,\xi)| \leq |g(x,s,\xi)| \quad \text{and} \quad |g_n(x,s,\xi)| \leq n.$$

We define an operator $G_n : W_0^{1,p}(\Omega, w) \to W^{-1,p'}(\Omega, w^*)$ by

$$\langle G_n u, v\rangle = \int_\Omega g_n(x,u,\nabla u)v\,dx$$

and

$$\langle Au, v\rangle = \int_\Omega a(x,u,\nabla u)\nabla v\,dx.$$

Due to the Hölder inequality we have

$$\begin{aligned}\Big|\int_\Omega g_n(x,u,\nabla u)v\,dx\Big| &\le \Big(\int_\Omega |g_n(x,u,\nabla u)|^{q'}\sigma^{-q'/q}\,dx\Big)^{1/q'}\Big(\int_\Omega |v|^q\sigma\,dx\Big)^{1/q}\\ &\le n\Big(\int_\Omega \sigma^{q'/q}\sigma^{-q'/q}\,dx\Big)^{1/q'}\|v\|_{q,\sigma}\\ &\le C_n\|v\|. \end{aligned} \tag{3.4}$$

for all $u \in W_0^{1,p}(\Omega, w)$ and all $v \in W_0^{1,p}(\Omega, w)$.

The last inequality follows from (2.5) and (2.6).

**Proposition 3.1.** *The operator $B_n = A + G_n$ defined from $W_0^{1,p}(\Omega, w)$ into $W^{-1,p'}(\Omega, w^*)$ is pseudo-monotone. Moreover, $B_n$ is coercive in the following sense:*

$$\frac{\langle B_n v, v - v_0\rangle}{\|v\|} \to +\infty \quad \text{if} \quad \|v\| \to +\infty, \quad v \in W_0^{1,p}(\Omega, w).$$

**3.1.1. Proof of Proposition 3.1.** From Hölder's inequality and the growth conditions (1.2), we can show that $A$ is bounded, and by using (3.4), we have that $B_n$ is bounded. The coercivity follows from (1.4), (1.5) and (3.4). It remains to show that $B_n$ is pseudo-monotone.

Let $(u_k)_k \in W_0^{1,p}(\Omega, w)$ be a sequence such that $u_k \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, w)$ and

$$\limsup_{k\to+\infty} \langle B_n u_k, u_k - u\rangle \le 0. \tag{3.5}$$

Let $v \in W_0^{1,p}(\Omega, w)$. We will show that

$$\liminf_{k\to+\infty} \langle B_n u_k, u_k - v\rangle \ge \langle B_n u, u - v\rangle.$$

Since $(u_k)_k$ is a bounded sequence in $W_0^{1,p}(\Omega, w)$, we deduce that $\big(a(x,u_k,\nabla u_k)\big)_k$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega, w_i^{1-p'})$. Then there exists a function $h \in \prod_{i=1}^N L^{p'}(\Omega, w_i^{1-p'})$ such that

$$a(x, u_k, \nabla u_k) \rightharpoonup h \text{ weakly in } \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'}).$$

Similarly, it is easy to see that $\left(g_n(x, u_k, \nabla u_k)\right)_k$ is bounded in $L^{q'}(\Omega, \sigma^{1-q'})$. So there exists a function $\rho_n \in L^{q'}(\Omega, \sigma^{1-q'})$ such that

$$g_n(x, u_k, \nabla u_k) \rightharpoonup \rho_n \text{ weakly in } Ł^{q'}(\Omega, \sigma^{1-q'}).$$

It is clear that

$$\begin{aligned}\liminf_{k\to+\infty} \langle B_n u_k, u_k - v\rangle &= \liminf_{k\to+\infty} \langle A u_k, u_k\rangle - \int_\Omega h \nabla v\, dx + \int_\Omega \rho_n(u - v)\, dx \\ &= \liminf_{k\to+\infty} \int_\Omega a(x, u_k, \nabla u_k)\nabla u_k\, dx \\ &\quad - \int_\Omega h\nabla v\, dx + \int_\Omega \rho_n(u - v)\, dx. \end{aligned} \tag{3.6}$$

On the other hand, by condition (1.3) we have

$$\int_\Omega \big(a(x, u_k, \nabla u_k) - a(x, u_k, \nabla u)\big)(\nabla u_k - \nabla u)\, dx \geq 0,$$

which implies that

$$\begin{aligned}\int_\Omega a(x, u_k, \nabla u_k)\nabla u_k\, dx \geq -\int_\Omega a(x, u_k, \nabla u)\nabla u\, dx + \int_\Omega a(x, u_k, \nabla u_k)\nabla u\, dx \\ + \int_\Omega a(x, u_k, \nabla u)\nabla u_k\, dx.\end{aligned}$$

Hence

$$\liminf_{k\to\infty} \int_\Omega a(x, u_k, \nabla u_k)\nabla u_k\, dx \geq \int_\Omega h\nabla u\, dx. \tag{3.7}$$

Combining (3.6) and (3.7) we get

$$\liminf_{k\to+\infty} \langle B_n u_k, u_k - v\rangle \geq \langle h + \rho_n, u - v\rangle. \tag{3.8}$$

Now, since $v$ is arbitrary and $\lim_{k\to+\infty} \langle G_n u_k, u_k - u\rangle = 0$, we have by using (3.5) and (3.8)

$$\lim_{k\to+\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla(u_k-u)\,dx=0.$$

Consequently, we get

$$\lim_{k\to+\infty}\int_\Omega \big(a(x,u_k,\nabla u_k)-a(x,u_k,\nabla u)\big)\nabla(u_k-u)\,dx=0.$$

In view of Lemma 3.1, we have that $\nabla u_k\to\nabla u$ a.e. in $\Omega$, which by (3.8) yields that

$$\liminf_{k\to+\infty}\langle B_nu_k,u_k-v\rangle\geq\langle B_nu,u-v\rangle.$$

This completes the proof of the proposition.

**Remark 3.2.** The approximation (3.3) appears necessary to prove the boundedness of $\big(g_n(x,u_k,\nabla u_k)\big)_k$ in $L^{q'}(\Omega,\sigma^{1-q'})$.

**Remark 3.3.** In the case where $\sigma$ satisfies the integrability condition $\sigma^{1-q'}\in L^1_{\mathrm{loc}}(\Omega)$, it suffices to approximate the term $g(x,s,\xi)$ by some function involving $\chi_{\Omega_n}$. Here $\Omega_n$ is a sequence of compact subsets converging to the bounded open set $\Omega$ and $\chi_{\Omega_n}$ is a characteristic function, i.e., $g_n(x,s,\xi)=\frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}\chi_{\Omega_n}$.

Let us consider the approximate problem:

$$\begin{aligned}&u_n\in W_0^{1,p}(\Omega,w),\\ &\langle Au_n,v\rangle+\int_\Omega g_n(x,u_n,\nabla u_n)v\,dx\leq\int_\Omega f_nv\,dx\quad\text{for all } v\in W_0^{1,p}(\Omega,w),\end{aligned}\tag{3.9}$$

where $f_n$ is a regular function such that $f_n$ strongly converges to $f$ in $L^1(\Omega)$.

Applying Proposition 3.1, the problem (3.9) has a solution by the classical result of [15] (cf. Theorem 8.2 Chapter 2 of [15]).

## 4. Some principal propositions

**Proposition 4.1.** *Assume that* $(H_0)$–$(H_2)$ *hold true and let* $u_n$ *be a solution of the approximate problem* (3.9). *Then for all* $k>0$, *there exists a constant* $c(k)$ (*which does not depend on* $n$) *such that*

$$\int_\Omega\sum_{i=1}^N\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|w_i\leq c(k).$$

**4.1.1. Proof of Proposition 4.1.** Let $k > 0$ and let $\varphi_k(s) = se^{\gamma s^2}$, where $\gamma = \left(\frac{b(k)}{\alpha}\right)^2$.

It is well known that

$$\varphi_k'(s) - \frac{b(k)}{\alpha}|\varphi_k(s)| \geq \frac{1}{2} \qquad \text{for all } s \in \mathbb{R}. \tag{4.1}$$

Taking $\varphi_k\big(T_l(u_n - v_0)\big)$ as test function in (3.9), where $l = k + \|v_0\|_\infty$, we obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_l(u_n - v_0)\varphi_k'\big(T_l(u_n - v_0)\big)\,dx + \int_\Omega g_n(x, u_n, \nabla u_n)\varphi_k\big(T_l(u_n - v_0)\big)\,dx \leq \int_\Omega f_n\varphi_k\big(T_l(u_n - v_0)\big)\,dx.$$

Since $g_n(x, u_n, \nabla u_n)\varphi_k\big(T_l(u_n - v_0)\big) \geq 0$ for $\{x \in \Omega \,|\, |u_n(x)| > k\}$ it follows that

$$\int_{\{|u_n - v_0| \leq l\}} a(x, u_n, \nabla u_n)\nabla(u_n - v_0)\varphi_k'\big(T_l(u_n - v_0)\big)\,dx \leq \int_{\{|u_n| \leq k\}} |g_n(x, u_n, \nabla u_n)|\,\big|\varphi_k\big(T_l(u_n - v_0)\big)\big|\,dx + \int_\Omega f_n\varphi_k\big(T_l(u_n - v_0)\big)\,dx.$$

By using (1.6) and (3.1), we have

$$\begin{aligned}\alpha\int_{\{|u_n - v_0| \leq l\}} \sum_{i=1}^N w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^p \varphi_k'\big(T_l(u_n - v_0)\big)\,dx &\leq b(|k|)\int_\Omega \left(c(x) + \sum_{i=1}^N w_i(x)\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p\right)\big|\varphi_k\big(T_l(u_n - v_0)\big)\big|\,dx \\ &\quad + \int_\Omega \delta(x)\varphi_k'\big(T_l(u_n - v_0)\big)\,dx + \int_\Omega f_n\varphi_k\big(T_l(u_n - v_0)\big)\,dx.\end{aligned}$$

Since $\{x \in \Omega \,|\, |u_n(x)| \leq k\} \subseteq \{x \in \Omega \,|\, |u_n - v_0| \leq l\}$, $c, \delta \in L^1(\Omega)$ and $f_n$ is bounded in $L^1(\Omega)$, it follows that

$$\int_\Omega \sum_{i=1}^N w_i(x)\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p \varphi_k'\big(T_l(u_n - v_0)\big)\,dx \leq \frac{b(k)}{\alpha}\int_\Omega \sum_{i=1}^N w_i(x)\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p \big|\varphi_k\big(T_l(u_n - v_0)\big)\big|\,dx + C_k,$$

where $C_k$ is a positive constant depending on $k$. This implies that

$$\int_\Omega \sum_{i=1}^N w_i(x)\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p \left[\varphi_k'\big(T_l(u_n-v_0)\big) - \frac{b(k)}{\alpha}\left|\varphi_k\big(T_l(u_n-v_0)\big)\right|\right] dx \le C_k.$$

From (4.1) we deduce that

$$\int_\Omega \sum_{i=1}^N w_i(x)\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p dx \le 2C_k. \tag{4.2}$$

**Remark 4.1.** In the case where the Hardy parameters satisfy the condition (1.8) with $1 < q < p + p'$, the previous estimate can be proved easily by using the Hölder inequality.

Indeed, using $T_k(u_n)$ as test function in (3.9), we have by (1.5)

$$\int_\Omega a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\nabla T_k(u_n)\, dx \le Ck$$

and by (3.1)

$$\alpha\|T_k(u_n)\|^p \le \int_\Omega a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\nabla v_0\, dx + C_1(k).$$

From Hölder's inequality, the growth condition (1.2) and $1 < q < p + p'$ we deduce that

$$\|T_k(u_n)\| \le C_2(k).$$

**Proposition 4.2.** *Assume that* $(H_0)$–$(H_2)$ *are sattisfied, and let* $u_n$ *be a solution to the approximate problem* (3.9). *Then there exists a measurable function* $u$ *such that* (*for a subsequence still denote by* $u_n$)

1) $u_n \to u$ *almost every where in* $\Omega$,

2) $T_k(u_n) \rightharpoonup T_k(u)$ *weakly in* $W_0^{1,p}(\Omega)$ *for all* $k > 0$.

**4.1.2. Proof of Proposition 4.2.** Let $k_0 \ge \|v_0\|_\infty$ and $k > k_0$. Taking $v = T_k(u_n - v_0)$ as a test function in (3.9), we get

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v_0)\, dx + \int_\Omega g_n(x, u_n, \nabla u_n) T_k(u_n - v_0)\, dx \le \int_\Omega f_n T_k(u_n - v_0)\, dx. \tag{4.3}$$

Since $g_n(x, u_n, \nabla u_n) T_k(u_n - v_0) \ge 0$ for $\{x \in \Omega \mid |u_n(x)| > k_0\}$, (4.3) implies that

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v_0)\,dx \le k\int_{\{|u_n|\le k_0\}} |g_n(x,u_n,\nabla u_n)|\,dx + k\|f\|_{L^1(\Omega)},$$

which gives by using (1.6)

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v_0)\,dx \le kb(k_0)\Big[\int_\Omega c(x)\,dx + \int_\Omega \sum_{i=1}^N w_i(x)\Big|\frac{\partial T_{k_0}(u_n)}{\partial x_i}\Big|^p dx\Big] + kC. \tag{4.4}$$

Combining (4.2) and (4.4), we have

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v_0)\,dx \le k[C_{k_0}+C].$$

Due to (1.3) we obtain

$$\int_{\{|u_n-v_0|\le k\}} \sum_{i=1}^N w_i(x)\Big|\frac{\partial u_n}{\partial x_i}\Big|^p dx \le kC_1,$$

where $C_1$ is independent of $k$. Since $k$ is arbitrary, we have

$$\int_{\{|u_n|\le k\}} \sum_{i=1}^N w_i(x)\Big|\frac{\partial u_n}{\partial x_i}\Big|^p dx \le \int_{\{|u_n-v_0|\le k+\|v_0\|_\infty\}} \sum_{i=1}^N w_i(x)\Big|\frac{\partial u_n}{\partial x_i}\Big|^p dx \le kC_2,$$

hence

$$\int_\Omega \sum_{i=1}^N w_i(x)\Big|\frac{\partial T_k(u_n)}{\partial x_i}\Big|^p dx \le kC_2. \tag{4.5}$$

Now we follow the lines of the proof of [4] to show that $u_n$ converges to some function $u$ in measure (and therefore we can always assume that the convergence is almost everywhere after passing to a suitable subsequence). To prove this, we show that $u_n$ is a Cauchy sequence in measure.

Let $k > 0$ be large enough. Then

$$k\,\mathrm{meas}(\{|u_n|>k\}\cap B_R) = \int_{\{|u_n|>k\}\cap B_R} |T_k(u_n)|\,dx \le \int_{B_R} |T_k(u_n)|\,dx$$
$$\le \Big(\int_\Omega |T_k(u_n)|^p w_0\,dx\Big)^{1/p}\Big(\int_{B_R} w_0^{1-p'}\,dx\Big)^{1/q'}$$

$$\leq C_0\Big(\int_\Omega \sum_{i=1}^N \Big|\frac{\partial T_k(u_n)}{\partial x_i}\Big|^p w_i(x)\,dx\Big)^{1/p}$$
$$\leq C_1 k^{1/p}.$$

Thus

$$\text{meas}(\{|u_n| > k\} \cap B_R) \leq \frac{C_1}{k^{1-1/p}}. \tag{4.6}$$

We have, for every $\lambda > 0$,

$$\begin{aligned}\text{meas}(\{|u_n - u_m| > \lambda\}) &\leq \text{meas}(\{|u_n| > k\}) + \text{meas}(\{|u_m| > k\}) \\ &\quad + \text{meas}(\{|T_k(u_n) - T_k(u_m)| > \lambda\}) \\ &\leq \text{meas}(\{|u_n| > k\} \cap B_R) + \text{meas}(\{|u_m| > k\} \cap B_R) \\ &\quad + 2\,\text{meas}(\{|x| > R\}) \\ &\quad + \text{meas}(\{|T_k(u_n) - T_k(u_m)| > \lambda\}).\end{aligned} \tag{4.7}$$

Since $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega, w)$, there exists some $v_k \in W_0^{1,p}(\Omega, w)$ such that

$$\begin{aligned}&T_k(u_n) \rightharpoonup v_k \text{ weakly in } W_0^{1,p}(\Omega, w),\\ &T_k(u_n) \to v_k \text{ strongly in } L^q(\Omega, \sigma) \text{ and } a.e. \text{ in } \Omega.\end{aligned} \tag{4.8}$$

Consequently, we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in $\Omega$.

Let $\varepsilon > 0$. Then, by (4.6), (4.7) and the fact that $\text{meas}(\{x \in \Omega \,|\, |x| > R\}$ tends to 0 as $R \to +\infty$, there exists some $k(\varepsilon) > 0$ such that

$$\text{meas}(\{|u_n - u_m| > \lambda\}) < \varepsilon \qquad \text{for all } n, m \geq n_0\big(k(\varepsilon), \lambda\big).$$

This proves that $(u_n)_n$ is a Cauchy sequence in measure converging almost everywhere to some measurable function $u$, which together with (4.8) implies that

$$\begin{aligned}&T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega, w),\\ &T_k(u_n) \to T_k(u) \text{ strongly in } L^q(\Omega, \sigma) \text{ and } a.e. \text{ in } \Omega.\end{aligned} \tag{4.9}$$

**Proposition 4.3.** *Assume that* $(H_0)$–$(H_2)$ *hold true and let* $u_n$ *be a solution of the approximate problem* (3.9). *Then, for all* $k > 0$,

1) $T_k(u_n) \to T_k(u)$ *strongly in* $W_0^{1,p}(\Omega, w)$,
2) $\nabla u_n \to \nabla u$ *a.e. in* $\Omega$.

**4.1.3. Proof of Proposition 4.3.** Many ideas of this proof are inspired of [12] and [14].

Let $k > 0$. Since $(T_k(u_n))_n$ is bounded in $W_0^{1,p}(\Omega, w)$, the sequence $(a(x, T_k(u_n), \nabla T_k(u_n)))_n$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega, w_i^{1-p'})$ by (1.2). So, up to a subsequence still denoted by $u_n$, $a(x, T_k(u_n), \nabla T_k(u_n))$ converges weakly to some function $h_k \in \prod_{i=1}^N L^{p'}(\Omega, w_i^{1-p'})$ such that

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k \text{ weakly in } \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'}). \tag{4.10}$$

Fix $k > 0$, and let $w_{n,h} = T_{2k}(u_n - v_0 - T_h(u_n - v_0) + T_k(u_n) - T_k(u))$ and $w_h = T_{2k}(u - v_0 - T_h(u - v_0))$ with $h > 2k$.

Define the following function

$$v_{n,h} = \varphi_k(w_{n,h}). \tag{4.11}$$

By taking $v_{n,h}$ as test functions in (3.9), we get

$$\langle A(u_n), \varphi_k(w_{n,h})\rangle + \int_\Omega g_n(x, u_n, \nabla u_n)\varphi_k(w_{n,h})\,dx \le \int_\Omega f_n\varphi_k(w_{n,h})\,dx. \tag{4.12}$$

It follows that

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx + \int_\Omega g_n(x, u_n, \nabla u_n)\varphi_k(w_{n,h})\,dx \\ \le \int_\Omega f_n\varphi_k(w_{n,h})\,dx. \tag{4.13}$$

For any fixed value of $h$, denote by $\varepsilon_h^1(n), \varepsilon_h^2(n), \dots$ sequences of real numbers which converge to zero as $n$ tends to infinity. By the almost everywhere convergence of $u$ we have

$$\varphi_k(w_{n,h}) \rightharpoonup \varphi_k(w_h) \text{ weakly}^* \text{ in } L^\infty(\Omega) \text{ as } n \to +\infty. \tag{4.14}$$

Therefore,

$$\int_\Omega f_n\varphi_k(w_{n,h})\,dx \to \int_\Omega f\varphi_k(w_h)\,dx \text{ as } n \to +\infty. \tag{4.15}$$

On the set $\{x \in \Omega, |u_n(x)| > k\}$ we have $g(x, u_n, \nabla u_n)\varphi_k(w_{n,h}) \ge 0$. So by (4.13) and (4.15),

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx+\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,h})\,dx$$
$$\le \int_\Omega f\varphi_k(w_h)\,dx+\varepsilon_h^1(n). \tag{4.16}$$

Splitting the first integral on the left-hand side of (4.16) where $|u_n|\le k$ and $|u_n|>k$, we can write

$$\begin{aligned}\int_\Omega & a(x,u_n,\nabla u_n)\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx\\ &=\int_{\{|u_n|\le k\}} a(x,u_n,\nabla u_n)\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx\\ &\quad+\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx.\end{aligned} \tag{4.17}$$

The first term of the right-hand side of the last inequality can be written as

$$\begin{aligned}\int_{\{|u_n|\le k\}} & a(x,u_n,\nabla u_n)\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx\\ &\ge \int_\Omega a\big(x,T_k(u_n),\nabla T_k(u_n)\big)[\nabla T_k(u_n)-\nabla T_k(u)]\varphi_k'(w_{n,h})\,dx\\ &\quad-\varphi_k'(2k)\int_{\{|u_n|>k\}}\sum_{i=1}^N\big|a_i\big(x,T_k(u_n),0\big)\big|\left|\frac{\partial T_k(u)}{\partial x_i}\right|dx.\end{aligned} \tag{4.18}$$

Recall that, for $i=1,\dots,N$, $\big|a_i\big(x,T_k(u_n),0\big)\big|\chi_{\{|u_n|>k\}}$ converges to $\big|a\big(x,T_k(u),0\big)\big|\chi_{\{|u|>k\}}$ strongly in $L^{p'}(\Omega,w_i^{1-p'})$. Moreover, since $\left|\frac{\partial T_k(u)}{\partial x_i}\right|\in L^p(\Omega,w_i)$, it follows that

$$-\varphi_k'(2k)\int_{\{|u_n|>k\}}\sum_{i=1}^N\big|a_i\big(x,T_k(u_n),0\big)\big|\left|\frac{\partial T_k(u)}{\partial x_i}\right|dx=\varepsilon_h^2(n).$$

For the second term of the right-hand side of (4.17) we can write

$$\begin{aligned}\int_{\{|u_n|>k\}} & a(x,u_n,\nabla u_n)\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx\\ &=\int_{\{|u_n|>k,\,|w_{n,h}|\le 2k\}} a(x,u_n,\nabla u_n)\nabla\big(u_n-v_0-T_h(u_n-v_0)\big)\varphi_k'(w_{n,h})\,dx\\ &\quad-\int_{\{|u_n|>k,\,|w_{n,h}|\le 2k\}} a(x,u_n,\nabla u_n)\nabla T_k(u))\varphi_k'(w_{n,h})\,dx,\end{aligned} \tag{4.19}$$

which implies that

$$\begin{aligned}\int_{\{|u_n|>k\}} & a(x,u_n,\nabla u_n)\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx \\ &= \int_{\{|u_n|>k,\,|w_{n,h}|\le 2k,\,|u_n-v_0|>h\}} a(x,u_n,\nabla u_n)\nabla(u_n-v_0)\varphi_k'(w_{n,h})\,dx \\ &\quad - \int_{\{|u_n|>k,\,|w_{n,h}|\le 2k\}} a(x,u_n,\nabla u_n)\nabla T_k(u)\varphi_k'(w_{n,h})\,dx. \qquad (4.20)\end{aligned}$$

Since $\{x\in\Omega \mid |w_{n,h}(x)|\le 2k\}\subset\{x\in\Omega \mid |u_n(x)|\le 5k+h\}$, in view of (3.1) we obtain that

$$\begin{aligned}\int_{\{|u_n|>k\}} & a(x,u_n,\nabla u_n)\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx \\ &\ge -\varphi_k'(2k)\int_{\{|u_n|>k\}}\sum_{i=1}^{N}\big|a_i\big(x,T_{5k+h}(u_n),\nabla T_{5k+h}(u_n)\big)\big|\left|\frac{\partial T_k(u)}{\partial x_i}\right|dx \\ &\quad - \varphi'(2k)\int_{\{|u_n-v_0|>h\}}\delta(x)\,dx. \qquad (4.21)\end{aligned}$$

Since $\big(a_i\big(x,T_{5k+h}(u_n),\nabla T_{5k+h}(u_n)\big)\big)_n$ is bounded in $L^{p'}(\Omega,w_i^{1-p'})$ for $i=1,\dots,N$, the first term on the right-hand side of (4.21) tends to zero for every $h$ fixed.

On the other hand, since $\delta\in L^1(\Omega)$ it is easy to see that

$$-\varphi_k'(2k)\int_{\{|u_n-v_0|>h\}}\delta(x)\,dx = -\varphi_k'(2k)\int_{\{|u-v_0|>h\}}\delta(x)\,dx+\epsilon_h^3(n). \qquad (4.22)$$

Combining (4.18)–(4.22), we deduce that

$$\begin{aligned}\int_\Omega & a(x,u_n,\nabla u_n)\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx \\ &\ge \int_\Omega a\big(x,T_k(u_n),\nabla T_k(u_n)\big)[\nabla T_k(u_n)-\nabla T_k(u)]\varphi_k'(w_{n,h})\,dx \\ &\quad - \varphi_k'(2k)\int_{\{|u-v_0|>h\}}\delta(x)\,dx+\epsilon_h^4(n). \qquad (4.23)\end{aligned}$$

This implies that

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx\\
&\quad\geq \int_\Omega \big[a\big(x,T_k(u_n),\nabla T_k(u_n)\big)-a\big(x,T_k(u_n),\nabla T_k(u)\big)\big]\\
&\qquad\cdot[\nabla T_k(u_n)-\nabla T_k(u)]\varphi_k'(w_{n,h})\,dx\\
&\qquad+\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u)\big)[\nabla T_k(u_n)-\nabla T_k(u)]\varphi_k'(w_{n,h})\,dx\\
&\qquad-\varphi_k'(2k)\int_{\{|u-v_0|>h\}}\delta(x)\,dx+\epsilon_h^4(n). \qquad (4.24)
\end{aligned}$$

We claim that

$$\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u)\big)[\nabla T_k(u_n)-\nabla T_k(u)]\varphi_k'(w_{n,h})\,dx=\epsilon_h^5(n). \qquad (4.25)$$

Indeed, since $\{x\in\Omega \mid |u_n(x)|\leq k\}\subseteq\{x\in\Omega \mid |u_n-v_0|\leq h\}$ we have

$$\begin{aligned}
&\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u)\big)[\nabla T_k(u_n)-\nabla T_k(u)]\varphi_k'(w_{n,h})\,dx\\
&\quad=\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u)\big)\nabla T_k(u_n)\varphi_k'\big(T_k(u_n)-T_k(u)\big)\,dx\\
&\qquad-\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u)\big)\nabla T_k(u)\varphi_k'(w_{n,h})\,dx. \qquad (4.26)
\end{aligned}$$

By the continuity of the Nemytskii operator (see [9]), we have for all $i=1,\dots,N$,

$$a_i\big(x,T_k(u_n),\nabla T_k(u)\big)\varphi'\big(T_k(u_n)-T_k(u)\big)\to a_i\big(x,T_k(u),\nabla T_k(u)\big)\varphi'(0)$$

and

$$a_i\big(x,T_k(u_n),\nabla T_k(u)\big)\to a_i\big(x,T_k(u),\nabla T_k(u)\big)$$

strongly in $L^{p'}(\Omega,w_i^{1-p'})$, while $\frac{\partial(T_k(u_n))}{\partial x_i}\rightharpoonup\frac{\partial(T_k(u))}{\partial x_i}$ weakly in $L^p(\Omega,w_i)$, and $\frac{\partial(T_k(u))}{\partial x_i}\varphi'(w_{n,h})\to\frac{\partial(T_k(u))}{\partial x_i}\varphi'(0)$ strongly in $L^p(\Omega,w_i)$. Hence it follows that

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u)\big)\nabla T_k(u_n)\varphi_k'\big(T_k(u_n)-T_k(u)\big)\,dx\\
&\quad=\int_\Omega a\big(x,T_k(u),\nabla T_k(u)\big)\nabla T_k(u)\varphi'(0)\,dx \qquad (4.27)
\end{aligned}$$

and

$$\lim_{n\to\infty}\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u)\big)\nabla T_k(u)\varphi_k'(w_{n,h})\,dx$$
$$=\int_\Omega a\big(x,T_k(u),\nabla T_k(u)\big)\nabla T_k(u)\varphi'(0)\,dx. \tag{4.28}$$

Combining (4.27) and (4.28) we obtain (4.25), which proves the claim. From (4.24) and (4.25) it follows that

$$\int_\Omega a(x,u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(u)]\varphi'(w_{n,h})\,dx$$
$$\geq \int_\Omega \big[a\big(x,T_k(u_n),\nabla T_k(u_n)\big)-a\big(x,T_k(u_n),\nabla T_k(u)\big)\big]$$
$$\cdot[\nabla T_k(u_n)-\nabla T_k(u)]\varphi'(w_{n,h})\,dx$$
$$-\varphi_k'(2k)\int_{\{|u-v_0|>h\}}\delta(x)\,dx+\epsilon_h^6(n). \tag{4.29}$$

We now turn to the second term of the left-hand side of (4.16). Using (1.6) we have

$$\Big|\int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,h})\,dx\Big|$$
$$\leq b(k)\int_\Omega\Big(c(x)+\sum_{i=1}^N w_i\Big|\frac{\partial T_k(u_n)}{\partial x_i}\Big|^p\Big)|\varphi_k(w_{n,h})|\,dx$$
$$\leq b(k)\int_\Omega c(x)|\varphi_k(w_{n,h})|\,dx+\frac{b(k)}{\alpha}\int_\Omega\delta(x)|\varphi_k(w_{n,h})|$$
$$+\frac{b(k)}{\alpha}\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u_n)\big)\nabla T_k(u_n)|\varphi_k(w_n)|\,dx$$
$$-\frac{b(k)}{\alpha}\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u_n)\big)\nabla v_0|\varphi_k(w_{n,h})|\,dx.$$

Invoking (4.10) and (4.14) we get

$$\Big|\int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,h})\,dx\Big|$$
$$\leq\frac{b(k)}{\alpha}\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u_n)\big)\nabla T_k(u_n)|\varphi_k(w_{n,h})|\,dx$$
$$+b(k)\int_\Omega c(x)|\varphi_k(w_h)|\,dx+\frac{b(k)}{\alpha}\int_\Omega\delta(x)|\varphi_k(w_h)|\,dx$$
$$-\frac{b(k)}{\alpha}\int_\Omega h_k\nabla v_0|\varphi_k(w_h)|\,dx+\epsilon_h^7(n). \tag{4.30}$$

The first term of the right-hand side can be written in the form

$$\begin{aligned}
&\frac{b(k)}{\alpha}\int_\Omega \big[a\big(x,T_k(u_n),\nabla T_k(u_n)\big)-a(x,T_k(u_n),\nabla T_k(u)\big]\\
&\quad\cdot[\nabla T_k(u_n)-\nabla T_k(u)]|\varphi_k(w_{n,h})|\,dx\\
&\quad+\frac{b(k)}{\alpha}\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u)\big)[\nabla T_k(u_n)-\nabla T_k(u)]|\varphi_k(w_{n,h})|\,dx\\
&\quad+\frac{b(k)}{\alpha}\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u_n)\big)\nabla T_k(u)|\varphi_k(w_{n,h})|\,dx. \qquad (4.31)
\end{aligned}$$

From Lebesgue's theorem we conclude that

$$\nabla T_k(u)|\varphi_k(w_{n,h})|\to\nabla T_k(u)\big|\varphi_k\big(T_{2k}\big(u-v_0-T_h(u-v_0)\big)\big)\big|=0$$

strongly in $\prod_{i=1}^N L^p(\Omega,w_i)$. By (4.10) this implies that the third term of (4.31) tends to 0 as $n\to\infty$. By the same argument as in (4.25), the second term of (4.31) tends to 0 as $n\to\infty$.

From (4.30) and (4.31) we obtain

$$\begin{aligned}
&\Big|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,h})\,dx\Big|\\
&\quad\le\int_\Omega\big[a\big(x,T_k(u_n),\nabla T_k(u_n)\big)-a\big(x,T_k(u_n),\nabla T_k(u)\big)\big]\\
&\qquad\cdot[\nabla T_k(u_n)-\nabla T_k(u)]|\varphi_k(w_{n,h})|\,dx\\
&\qquad+b(k)\int_\Omega c(x)|\varphi_k(w_h)|\,dx+\frac{b(k)}{\alpha}\int_\Omega\delta(x)|\varphi_k(w_h)|\,dx\\
&\qquad-\frac{b(k)}{\alpha}\int_\Omega h_k\nabla v_0|\varphi_k(w_h)|\,dx+\epsilon_h^8(n). \qquad (4.32)
\end{aligned}$$

Combining (4.16), (4.29) and (4.32), we obtain

$$\begin{aligned}
&\int_\Omega\big[a\big(x,T_k(u_n),\nabla T_k(u_n)\big)-a\big(x,T_k(u_n),\nabla T_k(u)\big)\big]\\
&\qquad\cdot[\nabla T_k(u_n)-\nabla T_k(u)]\bigg(\varphi_k'(w_{n,h})-\frac{b(k)}{\alpha}|\varphi_k(w_{n,h})|\bigg)\,dx\\
&\quad\le b(k)\int_\Omega c(x)|\varphi_k(w_h|\,dx+\frac{b(k)}{\alpha}\int_\Omega\delta(x)|\varphi_k(w_h)|\,dx\\
&\qquad-\frac{b(k)}{\alpha}\int_\Omega h_k\nabla v_0|\varphi_k(w_h)|\,dx+\int_\Omega f(x)\varphi_k(w_h)\,dx+\epsilon_h^9(n). \qquad (4.33)
\end{aligned}$$

Then from (4.1) we have

$$\begin{aligned}\int_\Omega &\big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(u)\big)\big][\nabla T_k(u_n) - \nabla T_k(u)]\,dx \\ &\le 2b(k)\int_\Omega c(x)|\varphi_k(w_h)|\,dx + 2\frac{b(k)}{\alpha}\int_\Omega \delta(x)|\varphi_k(w_h)|\,dx \\ &\quad - 2\frac{b(k)}{\alpha}\int_\Omega h_k\nabla v_0|\varphi_k(w_h)|\,dx + 2\int_\Omega f(x)\varphi_k(w_h)\,dx + \epsilon_h^{10}(n). \end{aligned}\tag{4.34}$$

Hence, passing to the limit over $n$, we get

$$\begin{aligned}\limsup_{n\to\infty} \int_\Omega &\big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(u)\big)\big] \\ &\cdot [\nabla T_k(u_n) - \nabla T_k(u)]\,dx \\ &\le 2b(k)\int_\Omega c(x)|\varphi_k(w_h)|\,dx + 2\frac{b(k)}{\alpha}\int_\Omega \delta(x)|\varphi_k(w_h)|\,dx \\ &\quad - 2\frac{b(k)}{\alpha}\int_\Omega h_k\nabla v_0|\varphi_k(w_h)|\,dx + 2\int_\Omega f(x)\varphi_k(w_h)\,dx. \end{aligned}\tag{4.35}$$

Now, since $h(x)$, $\delta(x)$, $f(x)$ and $h_k\nabla v_0$ belong to $L^1(\Omega)$, by Lebesgue's dominated convergence theorem, all the terms on the right-hand side of the last inequality tend to 0 as $h \to +\infty$. Consequently,

$$\lim_{n\to\infty}\int_\Omega \big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(u)\big)\big][\nabla T_k(u_n) - \nabla T_k(u)]\,dx = 0.$$

Furthermore, due to Lemma 3.1, we get

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega, w) \qquad \text{for all } k > 0. \tag{4.36}$$

For $k > 0$ large enough, we have

$$\begin{aligned}\text{meas}(\{|\nabla u_n - \nabla u| > \lambda\}) &\le \text{meas}(\{|u_n| > k\}) + \text{meas}(\{|u| > k\}) \\ &\quad + \text{meas}(\{|\nabla T_k(u_n) - \nabla T_k(u)| > \lambda\}) \end{aligned}\tag{4.37}$$

for every $\lambda > 0$. Since $T_k(u_n)$ converges strongly in $W_0^{1,p}(\Omega, w)$, we can assume that $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ in measure in $\Omega$.

Let $\varepsilon > 0$. As in (4.7) there exists some $n_0(k, \lambda, \varepsilon) > 0$ such that $\text{meas}(\{|\nabla u_n - \nabla u| > \lambda\}) < \varepsilon$ for all $n, m \ge n_0(k, \lambda, \varepsilon)$. We then have for a subsequence

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega, \tag{4.38}$$

which yields

$$\begin{aligned} a(x, u_n, \nabla u_n) &\to a(x, u, \nabla u) \text{ a.e. in } \Omega, \\ g_n(x, u_n, \nabla u_n) &\to g(x, u, \nabla u) \text{ a.e. in } \Omega. \end{aligned} \tag{4.39}$$

**Remark 4.2.** The introduction of $v_0$ in the used test function allows us to get rid of the first term on the right-hand side of (4.20) (by using (3.1)), which does not converge to 0 when $n$ and $h$ converge to $+\infty$.

## 5. Proof of Theorem 3.1

*Step* 1. Equi-integrability of the nonlinearities.

We need to prove that

$$g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \tag{5.1}$$

In particular, it is enough to prove that of $g_n(x, u_n, \nabla u_n)$ is the equi-integrable. To this end we take $T_1\big(u_n - v_0 - T_h(u_n - v_0)\big)$ (with $h$ large enough) as test function in (3.9) and obtain

$$\int_{\{|u_n - v_0| > h+1\}} |g_n(x, u_n, \nabla u_n)|\, dx \le \int_{\{|u_n - v_0| > h\}} \big(|f_n| + \delta(x)\big)\, dx.$$

Let $\varepsilon > 0$. Then there exists $h(\varepsilon) \ge 1$ such that

$$\int_{\{|u_n - v_0| > h(\varepsilon)\}} |g(x, u_n, \nabla u_n)|\, dx < \varepsilon/2. \tag{5.2}$$

For any measurable subset $E \subset \Omega$, we have

$$\begin{aligned} &\int_E |g_n(x, u_n, \nabla u_n)|\, dx \\ &\quad \le \int_E b\big(h(\varepsilon) + \|v_0\|_\infty\big)\left(c(x) + \sum_{i=1}^N w_i \left|\frac{\partial T_{h(\varepsilon)+\|v_0\|_\infty}(u_n)}{\partial x_i}\right|^p\right) dx \\ &\qquad + \int_{\{|u_n - v_0| > h(\varepsilon)\}} |g(x, u_n, \nabla u_n)|\, dx. \end{aligned} \tag{5.3}$$

In view of (4.36) there exists $\eta(\varepsilon) > 0$ such that

$$\int_E b\big(h(\varepsilon) + \|v_0\|_\infty\big)\left(c(x) + \sum_{i=1}^N w_i \left|\frac{\partial T_{h(\varepsilon)+\|v_0\|_\infty}(u_n)}{\partial x_i}\right|^p\right) dx < \varepsilon/2 \tag{5.4}$$

for all $E$ such that $\operatorname{meas}(E) < \eta(\varepsilon)$.

Finally, combining (5.2), (5.3) and (5.4), one easily has

$$\int_E |g_n(x, u_n, \nabla u_n)|\, dx < \varepsilon \qquad \text{for all } E \text{ such that } \operatorname{meas}(E) < \eta(\varepsilon),$$

which implies (5.1).

*Step* 2. Passing to the limit.

Let $v \in K_\psi \cap L^\infty(\Omega)$. Take $T_k(u_n - v)$ as test function in (3.9). Then

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v)\, dx + \int_\Omega g_n(x, u_n, \nabla u_n) T_k(u_n - v)\, dx \\ \le \int_\Omega f_n T_k(u_n - v)\, dx. \tag{5.5}$$

This implies that

$$\int_{\{|u_n - v| \le k\}} a(x, u_n, \nabla u_n)\nabla(u_n - v_0)\, dx \\ + \int_{\{|u_n - v| \le k\}} a\big(x, T_{k+\|v\|_\infty}(u_n), \nabla T_{k+\|v\|_\infty}(u_n)\big)\nabla(v_0 - v)\, dx \\ + \int_\Omega g_n(x, u_n, \nabla u_n) T_k(u_n - v)\, dx \le \int_\Omega f_n T_k(u_n - v)\, dx. \tag{5.6}$$

By Fatou's Lemma and the fact that

$$a\big(x, T_{k+\|v\|_\infty}(u_n), \nabla T_{k+\|v\|_\infty}(u_n)\big) \rightharpoonup a\big(x, T_{k+\|v\|_\infty}(u), \nabla T_{k+\|v\|_\infty}(u)\big)$$

weakly in $\prod_{i=1}^N L^{p'}(\Omega, w_i^{1-p'})$ one easily sees that

$$\int_{\{|u - v| \le k\}} a(x, u, \nabla u)\nabla(u - v_0)\, dx \\ + \int_{\{|u - v| \le k\}} a\big(x, T_{k+\|v\|_\infty}(u), \nabla T_{k+\|v\|_\infty}(u)\big)\nabla(v_0 - v)\, dx \\ + \int_\Omega g(x, u, \nabla u) T_k(u - v)\, dx \le \int_\Omega f T_k(u - v)\, dx. \tag{5.7}$$

Hence

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \leq \int_\Omega f T_k(u-v)\,dx. \tag{5.8}$$

This proves Theorem 3.1.

## References

[1] L. Aharouch and Y. Akdim, Existence of solutions of degenerated unilateral problems with $L^1$ data. *Ann. Math. Blaise Pascal* **11** (2004), 47–66. Zbl 02207858 MR 2077238

[2] Y. Akdim, E. Azroul, and A. Benkirane, Existence of solutions for quasilinear degenerate elliptic equations. *Electron. J. Differential Equations* **2001** (2001), No. 71, 1–19. Zbl 0988.35065 MR 1872050

[3] Y. Akdim, E. Azroul, and A. Benkirane, Existence results for quasilinear degenerated equations via strong convergence of truncations. *Rev. Mat. Complut.* **17** (2004), 359–379. Zbl 02109940 MR 2083959

[4] P. Bénilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre, and J. L. Vázquez, An $L^1$-theory of existence and uniqueness of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **22** (1995), 241–273. Zbl 0866.35037 MR 1354907

[5] L. Boccardo, F. Murat, and J.-P. Puel, Existence of bounded solutions for nonlinear elliptic unilateral problems. *Ann. Mat. Pura Appl.* (4) **152** (1988), 183–196. Zbl 0687.35042 MR 980979

[6] A. Elmahi and D. Meskine, Unilateral elliptic problems in $L^1$ with natural growth terms. *J. Nonlinear Convex Anal.* **5** (2004), 97–112. Zbl 1097.35062 MR 2049060

[7] G. Dal Maso, F. Murat, L. Orsina, and A. Prignet, Renormalized solutions of elliptic equations with general measure data. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **28** (1999), 741–808. Zbl 0958.35045 MR 1760541

[8] P. Drábek, A. Kufner, and V. Mustonen, Pseudo-monotonicity and degenerated or singular elliptic operators. *Bull. Austral. Math. Soc.* **58** (1998), 213–221. Zbl 0913.35051 MR 1642031

[9] P. Drábek, A. Kufner, and F. Nicolosi, Non linear elliptic equations, singular and degenerate cases. University of West Bohemia, 1996.

[10] P. Drábek and F. Nicolosi, Existence of bounded solutions for some degenerated quasilinear elliptic equations. *Ann. Mat. Pura Appl.* (4) **165** (1993), 217–238. Zbl 0806.35047 MR 1271420

[11] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*. Grundlehren Math. Wiss. 224, Springer-Verlag, Berlin 1977. Zbl 0361.35003 MR 0473443

[12] C. Leone and A. Porretta, Entropy solutions for nonlinear elliptic equations in $L^1$. *Nonlinear Anal.* **32** (1998), 325–334. Zbl 01337963 MR 1610574

[13] A. Porretta, Existence for elliptic equations in $L^1$ having lower order terms with natural growth. *Portugal. Math.* **57** (2000), 179–190. Zbl 0963.35068 MR 1759814

[14] A. Porretta, Some remarks on the regularity of solutions for a class of elliptic equations with measure data. *Houston J. Math.* **26** (2000), 183–213. Zbl 0974.35032 MR 1814734

[15] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod; Gauthier-Villars, Paris 1969. Zbl 0189.40603 MR 0259693

[16] B. Opic and A. Kufner, *Hardy-type inequalities*. Pitman Res. Notes Math. Ser. 219, Longman Scientific & Technical, Harlow 1990. Zbl 0698.26007 MR 1069756

Received June 14, 2006

L. Aharouch, Université Ibn Zohr, Faculté Polydisciplinaire Ouarzazate, B.P. 638 Ouarzazate, Morocco

E. Azroul, A. Benkirane, Département de Mathématiques et Informatique, Faculté des Sciences Dhar-Mahraz, B.P 1796 Atlas Fès, Morocco

E-mail: l_aharouch@yahoo.fr