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Triple positive solution to the one-dimensional *p*-Laplacian equation with delay

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Abstract. We obtain sufficient conditions for the existence of at least three positive solutions to the second-order nonlinear delay differential equation with one-dimensional p-Laplacian

$$\begin{cases} (\phi_p(x'(t)))' + w(t)f(t, x(t), x(t-\tau), x'(t)) = 0, & t \in (0,1),\ \tau > 0, \\ x(t) = 0, & -\tau \le t \le 0, \\ x(1) = 0, \end{cases}$$

where $\phi_p(s)$ is the p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$. The arguments are based upon a new fixed point theorem in a cone introduced by Avery and Peterson.

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1. Introduction

In this paper we study the existence of multiple positive solutions for the second-order nonlinear delay differential equations with one-dimensional p-Laplacian

$$\begin{cases} (\phi_p(x'(t)))' + w(t)f(t, x(t), x(t-\tau), x'(t)) = 0, & t \in (0,1),\ \tau > 0, \\ x(t) = 0, & -\tau \le t \le 0, \\ x(1) = 0. \end{cases} \tag{1.1}$$

Here $\phi_p(s)$ is the p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$ and $\frac{1}{p} + \frac{1}{q} = 1$; $w(t)$ is a nonnegative continuous function defined on $(0,1)$ and

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$f(t,u,v,\mu)$ is a nonnegative continuous function defined on $[0,1]\times[0,+\infty)\times[0,+\infty)\times R$.

In recent years many papers deal with the existence of positive solutions of the second-order delay differential equations; see [1], [4], [7], [9]–[17], [19], [20], [22]. For proving the existence of solutions either fixed point theorems in Banach space or the nonlinear alternative of Leray–Schauder are used. In particular, we mention some results of Jiang [9], and Wang and Ge [22]. In [9], Jiang considered the following second-order boundary value problem (BVP) described by delay differential equations of the form

$$\begin{cases} x'' + f\big(t, x(t-\tau)\big) = 0, & t \in (0,1), \\ x(t) = 0, \quad -\tau \le t \le 0, \\ x(1) = 0, \end{cases} \tag{1.2}$$

where $f \in C\big([0,1]\times[0,+\infty),[0,+\infty)\big)$. By using fixed point index theory in a cone, Jiang established the existence of multiple positive solutions of BVP (1.2).

By applying the Guo–Krasnoselskii fixed point theorem in a cone, Wang and Ge [22] established the existence of positive solutions to the problem

$$\begin{cases} \big(\phi_p(x'(t))\big)' + \lambda q(t) f\big(t, x(t-\tau)\big) = 0, & t \in (0,1),\ \tau > 0, \\ x(t) = 0, \quad -\tau \le t \le 0, \\ x(1) = 0, \end{cases}$$

where $\lambda > 0$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $q \in C[0,1] \cap L^1[0,1]$ with $q(t) > 0$ on $(0,1)$.

However, the literature on the multiplicity of positive solutions of the second-order nonlinear delay differential equations seems to be rather limited. This applies also to the case when the nonlinear term is involved in the first-order derivative explicitly.

This paper aims to fill this gap by improving and generalizing results mentioned in the references. We shall prove that BVP (1.1) possesses at least three positive solutions.

The following hypotheses are adopted throughout this paper:

(H_1) $f \in C\big([0,1]\times[0,+\infty)\times[0,+\infty)\times R,[0,+\infty)\big)$;

(H_2) $w \in C[0,1] \cap L^1[0,1]$ with $w(t) > 0$ on $(0,1)$.

Our main results will depend on an application of a fixed point theorem due to Avery and Peterson which deals with fixed points of a cone-preserving operator defined on an ordered Banach space. The emphasis here is that the nonlinear term is involved explicitly in the first-order derivative.

2. Preliminaries

In this section we provide some background material from the theory of cones in Banach spaces, and we then state the triple fixed-point theorem for a cone preserving operator. The following definitions can be found in the book by Deimling [6] as well as in the book by Guo and Lakshmikantham [8].

Definition 2.1. Let E be a Banach space over $\mathbb{R}$. A nonempty closed set $P \subset E$ is said to be a cone provided that

(i) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0$, $b \geq 0$,

(ii) $u, -u \in P$ implies $u = 0$.

Every cone $P \subset E$ induces an ordering on E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. The map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha\big(tx + (1-t)y\big) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, we say the map γ is a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\gamma : P \to [0, \infty)$ is continuous and

$$\gamma\big(tx + (1-t)y\big) \leq t\gamma(x) + (1-t)\gamma(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let γ and θ be nonnegative continuous convex functionals on P, let α be a nonnegative continuous concave functional on P, and let ψ be a nonnegative continuous functional on P. Then for positive real numbers a, b, c, and d, we define the following convex sets

$$\begin{aligned}
&P(\gamma, d) = \{x \in P \mid \gamma(x) < d\}, \\
&P(\gamma, \alpha, b, d) = \{x \in P \mid b \leq \alpha(x), \gamma(x) \leq d\}, \\
&P(\gamma, \theta, \alpha, b, c, d) = \{x \in P \mid b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d\},
\end{aligned}$$

and a closed set

$$R(\gamma, \psi, a, d) = \{x \in P \mid a \le \psi(x), \gamma(x) \le d\}.$$

The following fixed point theorem due to Avery and Peterson is fundamental in the proofs of our main results.

Themrem 2.1 ([3]). *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, let α be a nonnegative continuous concave functional on P, and let ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \le \lambda\psi(x)$ for $0 \le \lambda \le 1$ such that for some positive numbers M and d,*

$$\alpha(x) \le \psi(x), \qquad \|x\| \le M\gamma(x) \tag{2.1}$$

for all $x \in \overline{P(\gamma, d)}$. Suppose that $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers a, b, and c with $a < b$ such that

(S_1) *$\{x \in P(\gamma, \theta, \alpha, b, c, d) \mid \alpha(x) > b\} \ne \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\gamma, \theta, \alpha, b, c, d)$;*

(S_2) *$\alpha(Tx) > b$ for $x \in P(\gamma, \alpha, b, d)$ with $\theta(Tx) > c$;*

(S_3) *$0 \notin R(\gamma, \psi, a, d)$ and $\psi(Tx) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.*

Then T has at least three positive solutions $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that

$$\gamma(x_i) \le d \quad \text{for } i = 1, 2, 3,$$
$$b < \alpha(x_1),$$
$$a < \psi(x_2) \quad \text{with } \alpha(x_2) < b$$

and

$$\psi(x_3) < a.$$

3. Main results

In this section we impose growth conditions on f which allow us to apply Theorem 2.1 to establish the existence of triple positive solutions of BVP (1.1).

Let $E = C^1[-\tau, 1]$ be a Banach space with the maximum norm

$$\|x\| = \max\Big\{\max_{t\in[-\tau,1]} |x(t)|, \max_{t\in[-\tau,1]} |x'(t)|\Big\}.$$

From the fact $\big(\phi_p(x'(t))\big)' = -w(t)f\big(t, x(t), x(t-\tau), x'(t)\big) \le 0$, we know that x is concave on $[0, 1]$. So define a cone $P \subset E$ by

$$P = \{x \in E \mid x(t) \geq 0, t \in [-\tau, 1], x(t) = 0, t \in [-\tau, 0], \\ x(1) = 0, x \text{ is concave on } [0, 1]\}.$$

For $x \in P$ we define

$$u(t) := \int_0^t \phi_p^{-1}\left(\int_s^t w(r) f\big(r, x(r), x(r-\tau), x'(r)\big)\, dr\right) ds \\ - \int_t^1 \phi_p^{-1}\left(\int_t^s w(r) f\big(r, x(r), x(r-\tau), x'(r)\big)\, dr\right) ds$$

where $0 < t < 1$. Clearly, $u(t)$ is continuous and strictly increasing in $(0, 1)$ and $u(0^+) < 0 < u(1^-)$. Thus, $u(t)$ has unique zero in $(0, 1)$. Let $t_0 = t_x$ (i.e., t_0 is dependent on x) be the zero of $u(t)$ in $(0, 1)$. Then

$$\int_0^{t_0} \phi_p^{-1}\left(\int_s^{t_0} w(r) f\big(r, x(r), x(r-\tau), x'(r)\big)\, dr\right) ds \\ = \int_{t_0}^1 \phi_p^{-1}\left(\int_{t_0}^s w(r) f\big(r, x(r), x(r-\tau), x'(r)\big)\, dr\right) ds. \tag{3.1}$$

For the sake of applying Theorem 2.1, define the nonnegative continuous concave functional α_1, the nonnegative continuous convex functionals θ_1, γ_1, and the nonnegative continuous functional ψ_1 on the cone P by

$$\alpha_1(x) = \min_{t \in [1/k, (k-1)/k]} |x(t)|,$$
$$\gamma_1(x) = \max_{t \in [0,1]} |x'(t)|,$$
$$\psi_1(x) = \theta_1(x) = \max_{t \in [0,1]} |x(t)|,$$

where $x \in P$ and k is a natural number with $k \geq 3$.

In our main results we will make use of the following lemmas.

Lemma 3.1. *For $x \in P$, there exists a constant $M > 0$ such that*

$$\max_{t \in [-\tau, 1]} |x(t)| \leq M \max_{t \in [-\tau, 1]} |x'(t)|.$$

Proof. By Lemma 3.1 of [5] we have

$$\max_{t \in [0,1]} |x(t)| \leq M \max_{t \in [0,1]} |x'(t)|. \tag{3.2}$$

Since $x(t) = 0$ for $t \in [-\tau, 0]$, it is clear that

$$\max_{t\in[-\tau,0]} |x(t)| \le M \max_{t\in[-\tau,0]} |x'(t)|. \tag{3.3}$$

Now (3.2) and (3.3) yield that

$$\max_{t\in[-\tau,1]} |x(t)| \le M \max_{t\in[-\tau,1]} |x'(t)|. \qquad \square$$

With Lemma 3.1 and the concavity of x, for all $x \in P$, the functionals defined above satisfy

$$\begin{aligned} &\frac{1}{k}\theta_1(x) \le \alpha_1(x) \le \theta_1(x), \\ &\quad \|x\| = \max\{\theta_1(x), \gamma_1(x)\} \le M\gamma_1(x), \\ &\alpha_1(x) \le \psi_1(x). \end{aligned} \tag{3.4}$$

Therefore, condition (2.1) of Theorem 2.1 is satisfied.

By (3.1), we define an operator $T : P \to P$ by

$$Tx(t) := \begin{cases} \int_0^t \phi_p^{-1}\left(\int_s^{t_0} w(r) f\left(r, x(r), x(r-\tau), x'(r)\right) dr\right) ds, & 0 \le t \le t_0, \\ \int_t^1 \phi_p^{-1}\left(\int_{t_0}^s w(r) f\left(r, x(r), x(r-\tau), x'(r)\right) dr\right) ds, & t_0 \le t \le 1, \\ 0, \quad -\tau \le t \le 0, \end{cases} \tag{3.5}$$

where t_0 is defined by (3.1). By (3.5), it is well known that BVP (1.1) has a positive solution x if and only if $x \in P$ is a fixed point of T.

Lemma 3.2. *Suppose that* (H_1) *and* (H_2) *hold. Then* $TP \subset P$ *and* $T : P \to P$ *is completely continuous.*

Proof. By (3.5), we have

$$Tx(t) \ge 0,\ t \in [-\tau, 1], \qquad Tx(t) = 0,\ t \in [-\tau, 0], \qquad Tx(1) = 0 \tag{3.6}$$

for $x \in P$. Moreover, $Tx(t_0)$ is the maximum value of Tx on $[0, 1]$ since

$$(Tx)'(t) := \begin{cases} \phi_p^{-1}\left(\int_t^{t_0} w(r) f\left(r, x(r), x(r-\tau), x'(r)\, dr\right) ds \ge 0, & 0 \le t \le t_0, \\ -\phi_p^{-1}\left(\int_{t_0}^t w(r) f\left(r, x(r), x(r-\tau), x'(r)\, dr\right) ds \le 0, & t_0 \le t \le 1, \\ 0, \quad -\tau \le t \le 0, \end{cases} \tag{3.7}$$

is continuous and nonincreasing in $[0, 1]$ and $(Tx)'(t_0) = 0$. So Tx is concave on $[0, 1]$, which together with (3.6) shows that $T(P) \subset P$ and each fixed point of T is a solution of BVP (1.1). By similar arguments as in [5], one can show that $T : P \to P$ is completely continuous. $\square$

In addition, for $x \in P$, we can prove the following result:

$$\min_{t \in [1/k,(k-1)/k]} Tx(t) \geq \frac{1}{k} \max_{t \in [0,1]} Tx(t). \tag{3.8}$$

In fact, from (3.5) we have

$$Tx(t) \geq \begin{cases} \frac{Tx(t_0)}{t_0} t \geq \max\limits_{t \in [0,1]} Tx(t)t & \text{for } 0 \leq t \leq t_0, \\ \frac{Tx(t_0)}{1-t_0}(1-t) \geq \max\limits_{t \in [0,1]} Tx(t)(1-t) & \text{for } t_0 \leq t \leq 1, \end{cases}$$

which implies that (3.8) holds.

Let

$$\delta = \min\left\{ \int_{1/k}^{1/2} \phi_p^{-1}\left(\int_s^{1/2} w(r)\,dr \right) ds, \int_{1/2}^{(k-1)/k} \phi_p^{-1}\left(\int_{1/2}^{s} w(r)\,dr \right) ds \right\},$$

$$\rho = \phi_p^{-1}\left(\int_0^1 w(r)\,dr \right),$$

$$N = \max\left\{ \int_0^{1/2} \phi_p^{-1}\left(\int_s^{1/2} w(r)\,dr \right) ds, \int_{1/2}^{1} \phi_p^{-1}\left(\int_{1/2}^{s} w(r)\,dr \right) ds \right\}.$$

We are now ready to apply the Avery–Peterson fixed point theorem to the operator T to give sufficient conditions for the existence of at least three positive solutions to BVP (1.1).

Theorem 3.1. *Assume that* (H_1) *and* (H_2) *hold. Let* $0 < a < b \leq \frac{d}{k}$ *and suppose that* f *satisfies the following conditions:*

(A_1) $f(t,u,v,\mu) \leq \phi_p(d/\rho)$ *for* $(t,u,v,\mu) \in [0,1] \times [0,d/2] \times [0,d/2] \times [-d,d]$;

(A_2) $f(t,u,v,\mu) > \phi_p(kb/\delta)$ *for* $(t,u,v,\mu) \in \left[\frac{1}{k},\frac{k-1}{k}\right] \times [b,kb] \times [0,kb] \times [-d,d]$;

(A_3) $f(t,u,v,\mu) < \phi_p(a/N)$ *for* $(t,u,v,\mu) \in [0,1] \times [0,a] \times [0,kb] \times [-d,d]$.

Then BVP (1.1) *has at least three positive solutions* x_1, x_2 *and* x_3 *such that*

$$\max_{t \in [0,1]} |x_i'(t)| \leq d, \qquad i = 1,2,3,$$

$$b < \min_{t \in [1/k,(k-1)/k]} |x_1(t)|, \qquad a < \max_{t \in [0,1]} |x_2(t)|$$

with

$$\min_{t \in [1/k,(k-1)/k]} |x_2(t)| < b,$$

and

$$\max_{t\in[0,1]} |x_3(t)| < a.$$

Proof. BVP (1.1) has a solution $x = x(t)$ if and only if x solves the operator equation $x = Tx$. Thus we set out to verify that the operator T satisfies the Avery–Peterson fixed point theorem, which then implies the existence of three fixed points of T.

For $x \in \overline{P(\gamma_1, d)}$, we have $\gamma_1(x) = \max_{t\in[0,1]} |x'(t)| \le d$, and, by Lemma 3.1, $\max_{t\in[0,1]} |x(t)| \le Md$ for $t \in [0,1]$. Then condition (A$_1$) implies that $f\big(t, x(t), x(t-\tau), x'(t)\big) \le \phi_p(d/\rho)$. On the other hand, $x \in P$ implies that $Tx \in P$, so Tx is concave on $[0,1]$ and $\max_{t\in[0,1]} |(Tx)'(t)| = \max\{|(Tx)'(0)|, |(Tx)'(1)|\}$. Thus

$$\begin{aligned}
\gamma_1(Tx) &= \max_{t\in[0,1]} |(Tx)'(t)| \\
&= \max\left\{\phi_p^{-1}\left(\int_0^{t_0} w(r) f\big(r, x(r), x(r-\tau), x'(r)\big)\,dr\right) ds,\right.\\
&\qquad\left.\phi_p^{-1}\left(\int_{t_0}^{1} w(r) f\big(r, x(r), x(r-\tau), x'(r)\big)\,dr\right) ds\right\}\\
&\le \frac{d}{\rho}\phi_p^{-1}\left(\int_0^1 w(r)\,dr\right) = \frac{d}{\rho}\rho = d.
\end{aligned}$$

Therefore, $T : \overline{P(\gamma_1, d)} \to \overline{P(\gamma_1, d)}$.

To check condition (S$_1$) of Theorem 2.1, we choose

$$x_0(t) = -4k^2 b\left(t - \frac{1}{2k}\right)^2 + kb, \qquad 0 \le t \le 1.$$

It is easy to see that $x_0 \in P(\gamma_1, \theta_1, \alpha_1, b, kb, d)$ and $\alpha_1(x_0) > b$, and so

$$\{x \in P(\gamma_1, \theta_1, \alpha_1, b, kb, d) \mid \alpha_1(x) > b\} \ne \emptyset.$$

Hence for $t \in [1/k, (k-1)/k]$, $x(t) \in P(\gamma_1, \theta_1, \alpha_1, b, kb, d)$ we have

$$b \le x(t) \le kb, \qquad |x'(t)| \le d.$$

Thus, for $t \in [1/k, (k-1)/k]$, it follows from condition (A$_2$) that

$$f\big(t, x(t), x(t-\tau), x'(t)\big) > \phi_p(kb/\delta).$$

By definition of α_1 and P, we have by (3.8)

$$\begin{aligned}
\alpha_1(Tx) &= \min_{t\in[1/k,(k-1)/k]} |(Tx)(t)| \ge \frac{1}{k}\max_{t\in[0,1]} Tx(t) = \frac{1}{k}(Tx)(t_0)\\
&= \frac{1}{k}\int_0^{t_0} \phi_p^{-1}\Big(\int_s^{t_0} w(r)f\big(r,x(r),x(r-\tau),x'(r)\big)\,dr\Big)\,ds\\
&= \frac{1}{k}\int_{t_0}^{1} \phi_p^{-1}\Big(\int_{t_0}^{s} w(r)f\big(r,x(r),x(r-\tau),x'(r)\big)\,dr\Big)\,ds\\
&\ge \frac{1}{k}\min\Big\{\int_0^{1/2} \phi_p^{-1}\Big(\int_s^{1/2} w(r)f\big(r,x(r),x(r-\tau),x'(r)\big)\,dr\Big)\,ds,\\
&\qquad \int_{1/2}^{1} \phi_p^{-1}\Big(\int_{1/2}^{s} w(r)f\big(r,x(r),x(r-\tau),x'(r)\big)\,dr\Big)\,ds\Big\}\\
&\ge \frac{1}{k}\min\Big\{\int_{1/k}^{1/2} \phi_p^{-1}\Big(\int_s^{1/2} w(r)f\big(r,x(r),x(r-\tau),x'(r)\big)\,dr\Big)\,ds,\\
&\qquad \int_{1/2}^{(k-1)/k} \phi_p^{-1}\Big(\int_{1/2}^{s} w(r)f\big(r,x(r),x(r-\tau),x'(r)\big)\,dr\Big)\,ds\Big\}\\
&> \frac{1}{k}\frac{kb}{\delta}\delta = b,
\end{aligned}$$

i.e., $\alpha_1(Tx) > b$ for all $x \in P(\gamma_1,\theta_1,\alpha_1,b,kb,d)$.

This shows that condition (S_1) of Theorem 2.1 is satisfied.

Moreover, by (3.4), we have

$$\alpha_1(Tx) \ge \frac{1}{k}\theta_1(Tx) > \frac{1}{k}kb = b. \tag{3.9}$$

for all $x \in P(\gamma_1,\alpha_1,b,d)$ with $\theta_1(Tx) > kb$. Thus condition (S_2) of Theorem 2.1 is satisfied.

Finally, we show that condition (S_3) of Theorem 2.1 holds as well. Clearly, $0 \notin R(\gamma_1,\psi_1,a,d)$ since $\psi_1(0) = 0 < a$. Suppose that $x \in R(\gamma_1,\psi_1,a,d)$ with $\psi_1(x) = a$. Then, by condition (A_3), we obtain that

$$\begin{aligned}
\psi_1(Tx) &= \max_{t\in[0,1]} |(Tx)(t)| = (Tx)(t_0)\\
&= \int_0^{t_0} \phi_p^{-1}\Big(\int_s^{t_0} w(r)f\big(r,x(r),x(r-\tau),x'(r)\big)\,dr\Big)\,ds\\
&= \int_{t_0}^{1} \phi_p^{-1}\Big(\int_{t_0}^{s} w(r)f\big(r,x(r),x(r-\tau),x'(r)\big)\,dr\Big)\,ds
\end{aligned}$$

$$\begin{aligned}
&\leq \max\Big\{\int_0^{1/2}\phi_p^{-1}\Big(\int_s^{1/2} w(r)f\big(r,x(r),x(r-\tau),x'(r)\big)\,dr\Big)\,ds,\\
&\qquad \int_{1/2}^{1}\phi_p^{-1}\Big(\int_{1/2}^{s} w(r)f\big(r,x(r),x(r-\tau),x'(r)\big)\,dr\Big)\,ds\Big\}\\
&\leq \frac{a}{N}\max\Big\{\int_0^{1/2}\phi_p^{-1}\Big(\int_s^{1/2} w(r)\Big)\,ds,\int_{1/2}^{1}\phi_p^{-1}\Big(\int_{1/2}^{s} w(r)\,dr\Big)\,ds\Big\}\\
&< a. \qquad (3.10)
\end{aligned}$$

Hence, from (3.10), we have

$$\psi_1(Tx) = \max_{t\in[0,1]} |Tx(t)| < a.$$

So condition (S_3) of Theorem 2.1 is satisfied.

Since (3.4) holds for $x \in P$, all conditions of Theorem 2.1 are satisfied. Therefore BVP (1.1) has at least three positive solutions x_1, x_2 and x_3 such that

$$\max_{t\in[0,1]} |x_i'(t)| \leq d, \qquad i = 1,2,3,$$

$$b < \min_{t\in[1/k,(k-1)/k]} \{|x_1(t)|\}, \qquad a < \max_{t\in[0,1]} |x_2(t)|$$

with

$$\min_{t\in[1/k,(k-1)/k]} |x_2(t)| < b,$$

and

$$\max_{t\in[0,1]} |x_3(t)| < a.$$

The proof is complete. □

To illustrate how our main results can be used in practice we present an example.

Example 3.1. Consider the boundary value problem

$$\begin{cases} (|x'|x')' + f\big(t,x(t),x(t-\tau),x'(t)\big) = 0, \quad 0 < t < 1,\\ x(0) = 0,\ t \in [-\tau,0]\\ x(1) = 0, \end{cases} \qquad (3.11)$$

where

$$f(t,u,v,\mu) = \begin{cases} \frac{e^t}{4} + 2306u^{10} + \frac{v}{15000} + \left(\frac{\mu}{30000}\right)^3, & u \le 4, \\ \frac{e^t}{4} + 2306(5-u)u^{10} + \frac{v}{15000} + \left(\frac{\mu}{30000}\right)^3, & 4 \le u \le 5, \\ \frac{e^t}{4} + 2306(u-5)u^{10} + \frac{v}{15000} + \left(\frac{\mu}{30000}\right)^3, & 5 \le u \le 6, \\ \frac{e^t}{4} + 2306 \cdot 6^{10} + \frac{v}{15000} + \left(\frac{\mu}{30000}\right)^3, & u \ge 6. \end{cases}$$

Choose $a = 1/2$, $b = 1$, $k = 4$, $d = 30000$. We note that $\delta = \frac{1}{12}$, $\rho = 1$, $N = \frac{\sqrt{2}}{6}$. Consequently, $f(t,u,v,\mu)$ satisfies

$$f(t,u,v,\mu) < \phi_3\left(\frac{a}{N}\right) = 4.5$$

for $0 \le t \le 1$, $0 \le u \le 1/2$, $0 \le v \le 4$, $-30000 \le \mu \le 30000$,

$$f(t,u,v,\mu) > \phi_3\left(\frac{4b}{\delta}\right) = 2304$$

for $1/4 \le t \le 3/4$, $1 \le u \le 4$, $0 \le v \le 4$, $-30000 \le \mu \le 30000$, and

$$f(t,u,v,\mu) < \phi_3\left(\frac{d}{\rho}\right) = 9 \times 10^8$$

for $0 \le t \le 1$, $0 \le u \le 15000$, $0 \le v \le 15000$ and $-30000 \le \mu \le 30000$.

Then all conditions of Theorem 3.1 are satisfied. Thus, problem (3.11) has at least three positive solutions x_1, x_2 and x_3 such that

$$\max_{t \in [0,1]} |x_i'(t)| \le 30000, \qquad i = 1,2,3,$$

$$1 < \min_{t \in [1/4,3/4]} \{|x_1(t)|\}, \qquad \frac{1}{2} < \max_{t \in [0,1]} |x_2(t)|$$

with

$$\min_{t \in [1/4,3/4]} |x_2(t)| < 1,$$

and

$$\max_{t \in [0,1]} |x_3(t)| < \frac{1}{2}.$$

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