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Generalized Dieudonné and Hill criteria

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Abstract. Let A be a reduced abelian p-group with a nice subgroup G. It is proved that if A/G is simply presented, then A is simply presented precisely when G is strongly simply presented in A. Moreover, the same type theorem for the class of $\aleph_1$-Σ-cyclic p-groups is also established without the niceness of G in A. Some analogous assertions for other exotic sorts of abelian groups are also considered.

The results obtained strengthen previous results due to J. A. Dieudonné (Portugal. Math., 1952), P. D. Hill (Trends in Math., 1999), and some other authors.

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I. Introduction

Throughout the present paper, A always denotes an abelian p-group, written additively, and G is a subgroup of A.

In [12] Jean Dieudonné generalized the classical Kulikov criterion for an abelian p-group to be a direct sum of cyclic groups proved in [22]. This generalization has numerous applications; for instance it leads to the first example of a separable $p^{\omega+1}$-projective group which is not a direct sum of cyclic groups. This is because it ensures a connection between a property of the whole group and the same property for a certain subgroup combined with the factor group modulo this subgroup.

We refer to this result by Dieudonné as *Dieudonné's criterion*. It has many refinements; see e.g. [3], [4], [5], [8], [9]. In [3] we prove it for the class of σ-summable p-groups. In [8] we extend Dieudonné's criterion to summable groups of countable lengths, totally projective groups of countable lengths, and Σ-groups, respectively. For the valuated version of these three group classes the interested reader may consult [4], [5] and [9]. In addition, in [6] we consider the situation for $p^{\omega+n}$-projective p-groups when $n \in \mathbb{N}$.

Recently, in [17] Hill generalized Kulikov's criterion (see also the joint paper [19] with Ullery) to primary simply presented groups of arbitrary length by giving a description in terms of height-finite subgroups similar to that of Kulikov. We denote this result of Hill's *Hill's criterion*. In the sequel, we shall refine Hill's necessary and sufficient condition in the sense of Dieudonné.

The purpose of this article is to continue the work in [8] by investigating for which classes of torsion abelian groups a Dieudonné-type theorem can be obtained. In addition, we shall also consider concrete examples of some other exotic group classes for which this type of result is not preserved in general. Therefore we shall present only sufficient conditions in this case.

The significance of Dieudonné-type results is amply illustrated by their valuable applications with respect to the inner structure of the abelian groups; see, e.g., [13].

The notation is standard and follows essentially [13] or the works cited in the bibliography. For instance, for an arbitrary element a from A, the symbol $|a|_A$ will always denote the height of a in A.

II. Main results

1. Simply presented p-groups

The definition of a p-torsion *simply presented group* can be found in [13], vol. II, p. 95; see also [17]. For the reader's convenience we shall list the following necessary and sufficient condition due to Hill [17]; before doing that we need some conventions.

Definition 1.1 (Hill). Two subgroups C and H of the abelian p-group A are called *compatible* in A if each $x \in C + H$ can be written as $x = c + h$, where $c \in C$ and $h \in H$ such that $|c|_A = |h|_A = |c + h|_A$, i.e., in other words, the elements c and h have as much height in A as the sum $c + h$.

Definition 1.2 (Hill). A family of subgroups $\{A_i\}_{i \in I}$ of A is said to be *compatible* in A if for each pair of subsets J and M of the index set I, the subgroups $\langle A_j \rangle_{j \in J}$ and $\langle A_m \rangle_{m \in M}$ are compatible in A.

Now we come to the crucial

Criterion (Hill, 1999). *An abelian p-group A is simply presented if and only if $A = \bigcup_{i \in I} A_i$, where $\{A_i\}_{i \in I}$ is a compatible family of height-finite (in A) subgroups.*

In this way, the subgroup G of A is known to be *strongly simply presented* in A provided that $G = \bigcup_{i \in I} G_i$, where $\{G_i\}_{i \in I}$ is a compatible family in A of height-finite (in A) subgroups, or, equivalently, in the reduced case there exists a composition series from 1 to G consisting of nice subgroups of A.

Note that if G is strongly simply presented in A, it is not necessarily itself simply presented; however when it is an isotype subgroup this is true.

We can now prove the following result.

Theorem 1.1. *Let A be an abelian reduced p-group with a nice subgroup G so that A/G is simply presented. Then A is simply presented if and only if G is strongly simply presented in A.*

Proof. $\Rightarrow$: By the above criterion we write $A = \bigcup_{i \in I} A_i$, where $\{A_i\}_{i \in I}$ is a compatible family of subgroups height-finite in A. Therefore $G = \bigcup_{i \in I}(A_i \cap G) = \bigcup_{i \in I} G_i$ where we put $G_i = A_i \cap G$. It is easy to see that G_i is height-finite in A for each index $i \in I$. That $\{G_i\}_{i \in I}$ is a compatible family in A follows easily from the definition and the fact that $\{A_i\}_{i \in I}$ is a compatible family in A.

$\Leftarrow$: First of all, since A/G is simply presented, by virtue of [13], vol. II, "Characteristic Theorem", pp. 99–100, there is a nice composition series from 1 to A/G,

$$1 = A_0/G, \ldots, A_\alpha/G, \ldots, A_\lambda/G = A/G$$

where $A_{\alpha+1}/G/A_\alpha/G \cong A_{\alpha+1}/A_\alpha$ is cyclic of order p and A_α/G is nice in A/G for all $\alpha < \lambda$, and $A_\lambda = A$. Since G is nice in A, it follows from [13], vol. II, Lemma 79.3, p. 74, that all A_α are nice in A.

Thus, we extract a nice composition series in A from G to A, namely

$$G = A_0, \ldots, A_\alpha, \ldots, A_\lambda = A.$$

On the other hand, since G is strongly simply presented in A, it follows that there exists a nice composition series in A from 1 to G,

$$1 = G_0, \ldots, G_\beta, \ldots, G_\mu = G,$$

where $G_{\beta+1}/G_\beta$ is cyclic of order p and G_β is nice in A for all $\beta \leq \mu$; we put $G_{\mu+1} = A_1$, etc.

Finally, we obtain a smooth well-ordered ascending tower of nice subgroups of A,

$$1 = C_0, \ldots, C_\nu, \ldots, C_\lambda = A,$$

such that $C_{\nu+1}/C_\nu$ is cyclic of order p for all $\nu < \lambda$. Therefore A is simply presented, as claimed. □

As a direct consequence, we obtain the following assertion regardless of the lengths both of A and G. (Compare with [8] for groups with countable length.)

Corollary ([8]). *Let A be an abelian reduced p-group with a nice subgroup G so that A/G is simply presented. If $G = \bigcup_{n<\omega} G_n$, where $G_n \subseteq G_{n+1} \leq G$ is height-finite in A for each $n \geq 1$ (in particular, if G is countable), then A is simply presented.*

When A possesses countable length, the converse implication is also satisfied.

2. C_λ-p-groups, λ an ordinal

Following Megibben [23] an abelian p-group A is said to be a *C_λ-group* provided that $A/p^\alpha A$ is simply presented (i.e., totally projective) for each $\alpha < \lambda$.

Similarly the subgroup G of A is called a *strongly C_λ-subgroup* of A if $G/p^\alpha G$ is strongly simply presented (i.e., strongly totally projective) in $A/p^\alpha G$ for all $\alpha < \lambda$.

The following technicality is folklore, but for the sake of completeness we include a proof.

Lemma 2.1. *Assume that G is a balanced (i.e., nice and isotype) subgroup of the abelian p-group A. Then, for each ordinal number α, the following hold:*

(a) *$(p^\alpha A + G)/p^\alpha A$ is balanced in $A/p^\alpha A$;*

(b) *$p^\alpha G$ is nice in A;*

(c) *$G/p^\alpha G$ is balanced in $A/p^\alpha G$.*

Proof. (a) First we show that the isotypeness of G in A implies the same property for $(p^\alpha A + G)/p^\alpha A$ in $A/p^\alpha A$.

For each ordinal $\delta < \alpha$, it follows from the modular law that

$$\begin{aligned}
[(p^\alpha A + G)/p^\alpha A] \cap p^\delta(A/p^\alpha A) &= [(p^\alpha A + G)/p^\alpha A] \cap (p^\delta A/p^\alpha A) \\
&= [(p^\alpha A + G) \cap p^\delta A]/p^\alpha A \\
&= (p^\alpha A + G \cap p^\delta A)/p^\alpha A \\
&= (p^\alpha A + p^\delta G)/p^\alpha A \\
&\subseteq \big(p^\alpha A + p^\delta(G + p^\alpha A)\big)/p^\alpha A \\
&\subseteq p^\delta\big((p^\alpha A + G)/p^\alpha A\big),
\end{aligned}$$

whence the desired equality.

Secondly, concerning the niceness, it is equivalent to show by [13], vol. II, Lemma 79.3, that $p^\alpha A + G$ is nice in A. For any limit ordinal number β, we distinguish two cases.

Case 1. $\beta \leq \alpha$: then $\bigcap_{\tau<\beta}(p^\alpha A + G + p^\tau A) = \bigcap_{\tau<\beta}(G + p^\tau A) = G + p^\beta A = p^\alpha A + G + p^\beta A$.

Case 2. $\beta > \alpha$: then $\bigcap_{\tau<\beta}(p^\alpha A + G + p^\tau A) = \bigcap_{\tau<\alpha}(p^\tau A + G) \cap \bigcap_{\alpha\le\tau<\beta}(p^\alpha A + G) = \bigcap_{\tau<\alpha}(p^\tau A + G) \cap (p^\alpha A + G) = p^\alpha A + G = p^\alpha A + G + p^\beta A$, as required.

(b) The niceness of G in A ensures that $\bigcap_{\tau<\beta}(G + p^\tau A) = G + p^\beta A$ for each limit ordinal β. Thus, by the modular law, for each limit ordinal $\beta > \alpha$ we have $\bigcap_{\tau<\beta}(p^\alpha G + p^\tau A) = \bigcap_{\tau<\alpha}(p^\alpha G + p^\tau A) \cap \bigcap_{\alpha\le\tau<\beta}(p^\alpha G + p^\tau A) \subseteq (G + p^\beta A) \cap p^\alpha A = p^\beta A + (G \cap p^\alpha A) = p^\beta A + p^\alpha G$, as required. For $\beta \le \alpha$ we have $\bigcap_{\tau<\beta}(p^\alpha G + p^\tau A) = \bigcap_{\tau<\beta} p^\tau A = p^\beta A = p^\beta A + p^\alpha G$. That is why, in both cases, $\bigcap_{\tau<\beta}(p^\alpha G + p^\tau A) = p^\beta A + p^\tau G$, as required.

By applying the same technique, one can show that if C is a nice subgroup of the balanced subgroup G of A, then C is nice in A; note that $p^\alpha G$ is always nice in G.

(c) The niceness follows directly from [13], vol. II, Lemma 79.3. As for the isotypeness, for every ordinal δ, by the modular law together with (b), it follows that $(G/p^\alpha G) \cap p^\delta(A/p^\alpha G) = [G \cap (p^\delta A + p^\alpha G)]/p^\alpha G = (p^\alpha G + p^\delta G)/p^\alpha G = p^\delta(G/p^\alpha G)$, as required.

Notice that (c) follows also directly from [13], vol. II, property (c), p. 78. □

It is worth noting that $p^n A + G$ is always nice in A for each $n \in \mathbb{N}$, independent of the subgroup G of A. Indeed, for each limit ordinal β, $\bigcap_{\tau<\beta}(p^n A + G + p^\tau A) = \bigcap_{n\le\tau<\beta}(p^n A + G) = p^n A + G = p^n A + G + p^\beta A$.

Now we prove the following result.

Theorem 2.1. *Let G be a balanced subgroup of the abelian p-group A so that A/G is a C_λ-group. Then A is a C_λ-group if and only if G is a C_λ-group.*

Proof. Observe that $(p^\alpha A + G)/p^\alpha A \cong G/(G \cap p^\alpha A) = G/p^\alpha G$. Moreover, $A/G/p^\alpha(A/G) = A/G/(p^\alpha A + G)/G \cong A/(p^\alpha A + G) \cong A/p^\alpha A/(p^\alpha A + G)/p^\alpha A$ is simply presented. But, by Lemma 2.1, $(p^\alpha A + G)/p^\alpha A$ is balanced in $A/p^\alpha A$ for all ordinals α. Furthermore, it is well known that

$$A/p^\alpha A \cong [G/p^\alpha G] \oplus [A/G/p^\alpha(A/G)].$$

Consequently, $A/p^\alpha A$ must be totally projective for all $\alpha < \lambda$ provided that so is $G/p^\alpha G$. Thus A is a C_λ-group.

Conversely, if $A/p^\alpha A$ is simply presented, then so is $G/p^\alpha G$ as an isomorphic copy of its direct summand (see, e.g., [13], vol. II, Lemma 81.5, p. 84, and "Characteristic Theorem", pp. 99–100). □

Note. Since it is well known that any p-group is a σ-summable C_λ-group only when it is totally projective of length cofinal with ω, one may derive, in accordance with [3] and Theorem 2.1, once again Theorem 1.1 for length cofinal with ω.

3. Pillared p-groups

The notion of "pillared groups" was introduced, to my knowledge, by P. Hill; see [14], Definition 3.4, p. 261. These are abelian p-groups whose first Ulm factor is Σ-cyclic (i.e., a direct sum of cyclic groups).

Definition 3.1. The abelian p-group A is called *pillared* if $A/p^{\omega}A$ is Σ-cyclic.

Moreover, the subgroup G of an abelian p-group A is called *strongly pillared* in A provided that $G/p^{\omega}G$ is *strongly* Σ*-cyclic* in $A/p^{\omega}G$ (for the terminology see [3] and [8]).

It is well known that the following inclusions hold:

$$\begin{aligned}\{\text{simply presented } p\text{-groups}\} &\subseteq \{C_{\lambda}\text{-groups}, \lambda > \omega\}\\ &\subseteq \{\text{pillared groups} = C_{\omega+1}\text{-groups}\}\\ &\subseteq \{\Sigma\text{-groups}\}.\end{aligned}$$

Theorem 3.1. *Let A be an abelian p-group with a nice subgroup G such that A/G is pillared. If G is strongly pillared in A, then A is pillared. The converse holds provided that $G \cap p^{\omega}A = p^{\omega}G$.*

In particular, when G is in addition pure in A, the group A is pillared if and only if the group G is pillared.

Proof. Observe that $A/G/p^{\omega}(A/G) = A/G/(p^{\omega}A + G)/G \cong A/(p^{\omega}A + G) \cong A/p^{\omega}A/(p^{\omega}A + G)/p^{\omega}A$ is Σ-cyclic. By hypothesis, $G/p^{\omega}G = \bigcup_{i<\omega}(G_i/p^{\omega}G)$ so that $p^{\omega}G \subseteq G_i \subseteq G_{i+1} \leq G$ and, for every $i \geq 1$, $G_i \cap p^iA = p^{\omega}G$. Consequently, $G = \bigcup_{i<\omega} G_i$ and so $(p^{\omega}A + G)/p^{\omega}A = \bigcup_{i<\omega}[(p^{\omega}A + G_i)/p^{\omega}A]$. Moreover, by modularity, we have $[(p^{\omega}A + G_i)/p^{\omega}A] \cap p^i(A/p^{\omega}A) = [(p^{\omega}A + G_i)/p^{\omega}A] \cap (p^iA/p^{\omega}A) = [(p^{\omega}A + G_i) \cap p^iA]/p^{\omega}A = (p^{\omega}A + G_i \cap p^iA)/p^{\omega}A = 0$. Thus $(p^{\omega}A + G)/p^{\omega}A$ is strongly Σ-cyclic in $A/p^{\omega}A$. Hence, the aforementioned criterion of Dieudonné in [12] works to conclude that $A/p^{\omega}A$ is Σ-cyclic, as asserted.

Let now $A/p^{\omega}A$ be Σ-cyclic. Then, being a subgroup, $A/p^{\omega}A \supseteq (G + p^{\omega}A)/p^{\omega}A$ is strongly Σ-cyclic in $A/p^{\omega}A$. Thus $(G + p^{\omega}A)/p^{\omega}A = \bigcup_{i<\omega}(C_i/p^{\omega}A)$, where $p^{\omega}A \subseteq C_i \subseteq C_{i+1} \leq G + p^{\omega}A$ and $C_i \cap p^iA = p^{\omega}A$ for each $i \geq 1$. Hence $G = \bigcup_{i<\omega}(C_i \cap G)$. On the other hand, $(G + p^{\omega}A)/p^{\omega}A \cong G/(G \cap p^{\omega}A) = G/p^{\omega}G$ are valuated isomorphic, thus preserve heights, and $G/p^{\omega}G = \bigcup_{i<\omega}[(C_i \cap G)/p^{\omega}G]$. Hence it follows that $C_i \cap G \cap p^iA = p^{\omega}A \cap G = p^{\omega}G$. Thus $G/p^{\omega}G$ is strongly Σ-cyclic in $A/p^{\omega}G$, as claimed.

The final assertion of the theorem is an elementary consequence of the first two statements. □

4. $\aleph_1$-Σ-cyclic p-groups

The abelian p-group A is called $\aleph_1$-Σ-*cyclic* if each of its countable subgroups is Σ-cyclic. Obviously such a group is reduced since the divisible groups $\mathbb{Z}(p^\infty)$ are countable.

The subgroup G of A is said to be *strongly* $\aleph_1$-Σ-*cyclic* in A if every countable subgroup of G is strongly Σ-cyclic in A. Because a strongly Σ-cyclic subgroup is Σ-cyclic, it is clear that a strongly $\aleph_1$-Σ-cyclic subgroup is itself $\aleph_1$-Σ-cyclic.

It is also known for a long time that, in virtue of the second theorem due to Prüfer (see e.g. [13], vol. I, Theorem 17.3, p. 88), each Σ-cyclic p-group is $\aleph_1$-Σ-cyclic, while the converse is wrong. In fact, let $A = \bigcup_{\alpha<\Omega} A_\alpha$, $A_\alpha \subseteq A_{\alpha+1} \leq A$, where A_α is a countable separable p-group for all $\alpha < \Omega$. Then, by Prüfer's second theorem, A_α is a Σ-cyclic group. Clearly, $|A| = |\Omega| = \aleph_1$ and for each $H \leq A$ with $|H| = \aleph_0$ there is an index $\gamma < \Omega$ such that $H \subseteq A_\gamma$, whence H is Σ-cyclic, as required. Nevertheless A need not be Σ-cyclic although it is even separable (i.e., without elements of infinite heights $\Leftrightarrow p^\omega A = 0$) when almost all subgroups A_α are not pure and nice in A, hence it is not Σ-cyclic, as expected, which proves our claim (see also [13] for more details). Similar constructions were considered in [15].

An abelian p-group A is called an m-*group* if $|A| = m$ and, for every $H \leq A$, the inequality $|H| < m$ implies that H is Σ-cyclic. The first example of a separable m-group which is not Σ-cyclic was constructed by Nunke; see [25], where $m = \aleph_n$ for $n \in \mathbb{N}$. Hill showed in [16] that the limitation on n cannot be ignored by proving that each $\aleph_\omega$-group is indeed Σ-cyclic. In the literature there also appear the so-called m-Σ-*cyclic groups*, i.e., groups which are not Σ-cyclic, but any subgroup of cardinal strictly less than m is Σ-cyclic. It is immediate that each $\aleph_k$-Σ-cyclic group is $\aleph_n$-Σ-cyclic whenever $k \geq n$. Notice that $\aleph_1$-Σ-cyclic p-groups are precisely the separable ones.

We are now in a position to show the following result.

Theorem 4.1. *Suppose that A is an abelian p-group with a subgroup G so that A/G is $\aleph_1$-Σ-cyclic. Then A is $\aleph_1$-Σ-cyclic if and only if G is strongly $\aleph_1$-Σ-cyclic in A.*

Proof. The necessity is obvious since a subgroup of a Σ-cyclic group is strongly Σ-cyclic in it.

To prove sufficiency, take $B \leq A$ with $|B| = \aleph_0$. Since $(B+G)/G \cong B/(B\cap G)$ we distinguish two cases.

Case 1. $B/(B\cap G)$ is finite. Then $|B| = |B\cap G|$ and so G being $\aleph_1$-Σ-cyclic assures that $B\cap G$ is Σ-cyclic. Then so is B by [13], vol. I, Proposition 18.3, p. 92.

Case 2. $B/(B\cap G)$ is countable. The hypothesis on A/G ensures that $B/(B\cap G)$ is Σ-cyclic. If $B\cap G$ is countable, the conditions on G force $B\cap G$ to be strongly Σ-cyclic in A, whence in B. Otherwise, if $B\cap G$ is finite, then it can be imbedded in a countable subgroup of G and so in a subgroup which is strongly

Σ-cyclic in A. Thus it follows that $B \cap G$ is strongly Σ-cyclic in A, hence in B. Finally, Dieudonné's criterion implies that B is Σ-cyclic, as required. □

As a direct consequence we obtain the following result.

Corollary 4.1. *Let A be an abelian p-group with a subgroup G such that A/G is bounded. Then A is $\aleph_1$-Σ-cyclic if and only if G is $\aleph_1$-Σ-cyclic.*

Proof. It is trivial to check that subgroups of $\aleph_1$-Σ-cyclic groups are themselves $\aleph_1$-Σ-cyclic, so the necessity follows.

To establish sufficiency, observe that $p^n A \subseteq G$ for some $n \in \mathbb{N}$, hence $p^\omega A = p^\omega G$. Now everything follows from the proof of the previous theorem.

Let us, however, give the following independent confirmation. Let B be a countable subgroup of A. Then $(B+G)/G \cong B/(B \cap G)$ is bounded. Being a subgroup of G, $B \cap G$ is separable (finite or infinite countable) and so it follows easily that B is separable as well. Hence B is a direct sum of cyclic groups by Prüfer's second theorem. □

In closing this section, we emphasize that if $G \leq A$ is a nice subgroup of the abelian p-group A, then A is separable if and only if G is strongly separable in A (i.e., $G \cap p^\omega A = 0$) and A/G is separable; in particular, when G is a pure and nice subgroup, A is separable if and only if both G and A/G are separable. If A/G is taken to be separable a priori, the condition on niceness follows as a corollary.

5. Torsion-complete *p*-groups

For the definition of a *torsion-complete p-group* see [13], vol. II, p. 15. We need the following criterion due to Kulikov [22] and Papp [26]; see also [13], vol. II, Theorem 68.4, p. 17.

Theorem (Kulikov, 1941; Papp, 1958). *Let A be a reduced abelian p-group. Then A is torsion-complete if and only if A is a direct summand of each abelian p-group in which it is a pure subgroup.*

Before proving the main result of this section, we briefly sketch the history.

Fuchs asked whether a separable abelian p-group A is torsion-complete if it contains a torsion-complete subgroup G such that A/G is torsion-complete (see [13], vol. II, Problem 54, p. 55). Almost thirty years ago, Hill and Megibben [18] answered this question in the negative. In other words, a separable extension of a torsion-complete group by another torsion-complete group need not be again torsion-complete. By imposing restrictions one can, however, obtain a positive answer to Fuchs' question. We show in the sequel that if the subgroup is chosen to be pure, then Fuchs' problem has a positive solution: if G is a pure subgroup of an abelian p-group A and G and A/G are torsion-complete, then so is A.

We mention some further results in this direction. If $G \leq A$ with G torsion-complete and A/G bounded, then A is torsion-complete. This is mainly due to D. O. Cutler; see also [13], vol. II, Ex. 8(b), p. 20. Moreover, if A is separable containing a bounded subgroup G such that A/G is torsion-complete, then A is torsion-complete (see [28], Lemma 2, and [1], Theorem 4.1). This is in contrast to Σ-cyclic groups by an example due to Dieudonné given in [12]. It was also shown in [1] that if G is a bounded and closed subgroup of the separable p-group A (thus A/G is separable), then A is torsion-complete if and only if A/G is torsion-complete. It is worth noting that the necessity was preliminary known; see [20], Lemma 4.23. Here it is shown that if G is bounded and A is torsion-complete such that A/G is separable, then A/G is torsion-complete. So Theorem 4.1 in [1] has a predecessor, but the proof given there is of some interest.

It is well known that a nice subgroup of a torsion-complete p-group is torsion-complete as well. This is perhaps not true in general for direct sums of torsion-complete p-groups, but it is correct for fully invariant (i.e., completely characteristic) subgroups. In fact, a nice fully invariant subgroup of a direct sum of torsion-complete p-groups is a direct sum of torsion-complete p-groups. Indeed, write $G = \bigoplus_{i \in I} G_i$, where all G_i are torsion-complete, and let N be nice in G. Then $N = \bigoplus_{i \in I} N_i$, where $N_i = N \cap G_i$. Now it is well known that N is nice in G if and only if N_i is nice in G_i for each $i \in I$. Thus every N_i is torsion-complete, hence N is a direct sum of torsion-complete groups.

Theorem 5.1. *Suppose that A is an abelian p-group which contains a pure and nice subgroup G. Then A is torsion-complete if and only if both G and A/G are torsion-complete.*

Proof. First we prove the necessity. It is well known that a nice subgroup of a torsion-complete group is again torsion-complete, which is actually a consequence of [20], Lemma 4.22, or [13], vol. II, Corollary 68.7, p. 18. Thus G is torsion-complete if A is torsion-complete. Since G is pure in the torsion-complete group A, A/G must be a direct sum of a divisible group and a torsion-complete group by [20] or [13], vol. II, Proposition 68.8, p. 19. But G being nice in the reduced group A assures that A/G is reduced, hence A/G is torsion-complete.

To prove sufficiency, we first note that A is reduced since G and A/G are reduced (this is true even without G being nice and pure in A). Furthermore, assume that A is a pure subgroup of the abelian p-group K. Then A/G is pure in K/G, and by Kulikov–Papp's result together with the hypothesis on A/G we obtain that $K/G = (A/G) \oplus (L/G)$ for some subgroup L of K containing G. Thus it is a routine matter to see that $K = A + L$ with $A \cap L = G$. But since G is pure in A, the transitivity of the pureness implies that G is pure in K, hence in L. Consequently, the hypothesis on G combined with the Kulikov–Papp criterion yields that $L = G \oplus L_1$ for some $L_1 \leq L$. Furthermore, $K = A \oplus L_1$ since $A \cap L_1 = A \cap L \cap L_1 = G \cap L_1 = 0$. Finally, we again impose the theorem of Kulikov–Papp to deduce that A is torsion-complete, as required. □

We recall that the abelian p-group A is *semi-complete* if it is the direct sum of a torsion-complete group and a direct sum of cyclic groups.

As an immediate consequence, we have the following.

Corollary 5.1. *Let A be an abelian p-group with a pure subgroup G such that A/G is semi-complete. If G is torsion-complete, then A is semi-complete.*

Proof. Write $A/G = (T/G) \oplus (C/G)$, where the first factor is torsion-complete and the second one is Σ-cyclic. Then $A = T + C$ with $T \cap C = G$. Since G is pure in A, it follows that G is pure both in T and C. Thus, as in the proof of the sufficiency in Theorem 5.1, one shows that T is torsion-complete.

On the other hand, referring to a theorem of Kulikov [22] (see also [13], vol. I, Theorem 28.2, p. 120), we may write $C = G \oplus C_1$ where $C_1 \cong C/G$ is Σ-cyclic. Finally, we conclude that $A = T \oplus C_1$ because $T \cap C_1 = T \cap C \cap C_1 = G \cap C_1 = 0$. So A is semi-complete. □

Problem 5.1. Determine whether or not A is a semi-complete p-group if it contains a pure, semi-complete subgroup G such that A/G is semi-complete.

Problem 5.2. Decide whether or not A is a semi-complete p-group if it contains a pure subgroup G such that G is Σ-cyclic and A/G is torsion-complete. (Notice that if G is bounded, by a theorem of Prüfer–Kulikov (see [22] or [13], vol. I, Theorem 27.5, p. 118), $A \cong G \oplus A/G$ is torsion-complete, hence semi-complete.)

We now show that Problem 5.1 has a positive answer if Problem 5.2 can be answered in the affirmative.

Indeed, as in the proof of Corollary 5.1, $A = T \oplus C_1$, where C_1 is Σ-cyclic. It is enough to show that T is semi-complete if T/G is torsion-complete and G is semi-complete. Write $G = U \oplus V$, where U is torsion-complete and V is Σ-cyclic. Since $T/G \cong T/U/G/U$ is torsion-complete, G/U is pure in T/U because G is pure in A, whence in T, and $G/U \cong V$ is Σ-cyclic. By our assumption that Problem 5.2 has a positive solution we conclude that T/U is semi-complete. Write $T/U = (X/U) \oplus (Y/U)$, where the first summand is torsion-complete and the latter one is Σ-cyclic. As above, $T = X + Y$ with $X \cap Y = U$. Moreover, U being pure in G implies that it is pure in A, hence in Y. As above, $Y = U \oplus Y_1$, where Y_1 is Σ-cyclic, whence Y is semi-complete. Thus $T = X \oplus Y_1$ since $X \cap Y_1 = X \cap Y \cap Y_1 = U \cap Y_1 = 0$. Now Theorem 5.1 ensures that X is torsion-complete since U, being pure in A, is pure in X. Thus T is semi-complete, and we are done.

6. Thick p-groups and essentially finitely indecomposable p-groups

For the definition of such groups we refer to [1]. We recollect the following necessary and sufficient condition from [1].

Criterion (Benabdallah–Wilson, 1978). *The abelian p-group A is thick if and only if there exists $m \in \mathbb{N}$ such that $(p^m A)[p] \subseteq K$ for all $K \leq A$ with A/K is Σ-cyclic.*

Analogously the subgroup G of the abelian p-group A is called *strongly thick* in A if and only if there exists $t \in \mathbb{N}$ such that $(p^t A)[p] \subseteq K$ for all $K \leq G$ with G/K is Σ-cyclic.

It is an easy exercise to show that strongly thick subgroups are thick.

It is well known that the class of all thick groups properly contains the torsion-complete groups (see [1], Corollary 3.4). Also the divisible p-groups are thick, and the Σ-cyclic groups are thick precisely when they are bounded.

The following statement is our major tool.

Proposition 6.1. *Assume that A is a thick abelian p-group with a pure subgroup G. Then A/G is thick.*

Proof. Choose a subgroup M/G of A/G such that $A/G/M/G \cong A/M$ is Σ-cyclic. Then, by the hypothesis on A and the Benabdallah–Wilson criterion, there is $m \in \mathbb{N}$ such that $(p^m A)[p] \subseteq M$. Therefore $(p^m A)[p] + G \subseteq M$ or, equivalently, $\big((p^m A)[p] + G\big)/G \subseteq M/G$. Since G is pure in A it follows that $(p^m A + G)[p] = (p^m A)[p] + G[p]$, hence $\big((p^m A + G)[p] + G\big)/G \subseteq M/G$. Moreover, G being pure in $p^m A + G \subseteq A$ gives that $\big(p^m(A/G)\big)[p] = \big((p^m A + G)/G\big)[p] = \big((p^m A + G)[p] + G\big)/G$. Hence $\big(p^m(A/G)\big)[p] \subseteq M/G$, and by the Benabdallah–Wilson criterion the assertion follows. □

We note that if G is bounded, the converse holds. For in view of [13], vol. I, Theorem 27.5, we may write $A \cong G \oplus A/G$, so Proposition 6.3 below will work.

We now show the following result (for the special case when A/G is bounded a priori, see [10]).

Proposition 6.2. *Suppose that A is an abelian p-group with a pure subgroup G such that A/G is Σ-cyclic. Then the following conditions are equivalent:*

(a) *A is thick;*

(b) *G is strongly thick in A;*

(c) *G is thick and A/G is bounded.*

Proof. (a) $\Rightarrow$ (b): Let G/H be Σ-cyclic for an arbitrary subgroup H of G. Since G/H is obviously pure in A/H, it is strongly Σ-cyclic in it. On the other hand,

$A/G \cong A/H/G/H$ is also Σ-cyclic. Thus, by Dieudonné's criterion, A/H is Σ-cyclic, and a result due to Kulikov [22] (see [13], vol. I, Theorem 28.2) is applicable to deduce that $A/H \cong (G/H) \oplus (A/G)$ is Σ-cyclic. By the hypothesis on A this implies that $(p^m A)[p] \subseteq H$ for some natural number m. Finally, $(p^m G)[p] \subseteq H$.

(b) $\Rightarrow$ (a): Let A/H be Σ-cyclic for some arbitrary but fixed $H \leq A$. Then $(G+H)/H \cong G/(G \cap H)$ is Σ-cyclic as well. The hypothesis on G then yields that there exists a positive integer t such that $(p^t A)[p] \subseteq G \cap H \subseteq H$.

(b) $\Leftrightarrow$ (c): To check the second equivalence, we first observe by Proposition 6.1 that A thick implies that A/G is thick, hence the limitation on this quotient to be Σ-cyclic implies that it is bounded. Moreover, if A/G is bounded there is $t \in \mathbb{N}$ so that $p^t A \subseteq G$. Hence it is easy to show that the strong thickness of G in A and the ordinary thickness of G are, really, equivalent.

Finally, what remains to prove is that G strongly thick in A implies that A/G must be bounded if G is pure in A. Indeed, since $G/G \cong 1$ is always a Σ-cyclic group, there exists a natural number t such that $(p^t A)[p] \subseteq G$. But the purity of G in A ensures that $(p^t A)[p] = (p^t G)[p]$. Since $p^t G$ is a pure subgroup of $p^t A$, it follows that $p^t A = p^t G$, hence $p^t A \subseteq G$ and so A/G is bounded, as required. □

Remark 6.1. Since A/G is Σ-cyclic and G is pure in A, it follows by a result of Kulikov [22] (see [13], vol. I, Theorem 28.2) that $A \cong G \oplus A/G$. Thus, if G is thick then A is the direct sum of a thick group and a Σ-cyclic group; however A need not be thick or, in other words, the direct sum of a thick group and a Σ-cyclic group is not necessarily thick; for Σ-cyclic groups are thick if and only if they are bounded.

Moreover, we note that if A is thick and A/G is Σ-cyclic, then there is $t \in \mathbb{N}$ so that $(p^t A)[p] \subseteq G$. Thus, as shown above, the purity of G in A gives that A/G must be bounded. This is another confirmation of the above result.

By analogy with the case of torsion-complete groups, we state the following.

Problem 6.1. Let A be an abelian p-group with a pure (and/or nice) subgroup G such that A/G is thick. Does it follow that A is thick if and only if G is thick?

In accordance with Proposition 6.1, the necessity of the problem can be reformulated as whether a pure (and/or nice) subgroup of a thick group is again thick. In fact, the condition on A/G to be thick does not give any extra information since it follows from the thickness of A whenever G is pure in A. We conjecture that the question has a negative answer. In this direction we note that it follows from [27], Theorem 5.7, that thick groups are closed under pure extensions of thick groups by thick groups, i.e., if G is pure in A such that both G and A/G are thick, then A is thick. Thus the sufficiency holds.

By definition, an abelian p-group is *essentially finitely indecomposable* (abbreviated to e.f.i.) if it has no unbounded Σ-cyclic direct summand. It is a well-known fact and an elementary matter to verify that all thick groups are e.f.i. Irwin conjectured that the converse holds as well; however, these two classes of groups do not coincide. But by a result of Irwin (see [1], Corollary 3.3) pure-complete e.f.i. groups are thick. In this light, it is straightforward that direct summands of e.f.i. groups are again e.f.i. and finite direct sums of e.f.i. groups are also e.f.i. (see e.g. [20], Theorem 4.3). These two assertions have the corresponding analogue for thick groups.

Proposition 6.3. *Finite direct sums of thick p-groups are thick p-groups.*

Proof. It is sufficient to consider the sum of two thick p-groups, B and C, and the proof that $A = B \oplus C$ is thick. The statement then follows by induction.

Let A/K be Σ-cyclic for some $K \leq A$. Then $(B+K)/K \cong B/(B \cap K)$ and $(C+K)/K \cong C/(C \cap K)$ are both Σ-cyclic as being subgroups of A/K. Thus the hypotheses on B and C combined with the Benabdallah–Wilson criterion imply that there are integers r and s such that $(p^r B)[p] \subseteq B \cap K \subseteq K$ and $(p^s C)[p] \subseteq C \cap K \subseteq K$. Therefore $(p^t A)[p] = (p^t B)[p] \oplus (p^t C)[p] \subseteq (p^r B)[p] \oplus (p^s C)[p] \subseteq K$ where $t = \max(r, s)$. By invoking again the Benabdallah–Wilson criterion the claim follows. □

Notice that Proposition 6.3 fails for infinite direct sums: the unbounded Σ-cyclic groups provide a counterexample.

Proposition 6.4. *A direct summand of a thick p-group is a thick p-group.*

Proof. Write $A = B \oplus C$ where A is thick. By the Benabdallah–Wilson criterion it suffices to prove that there exists a positive integer t such that $(p^t B)[p] \subseteq K$ whenever B/K is a Σ-cyclic group. If B/K is Σ-cyclic, then so is $A/(K \oplus C) = (B \oplus C)/(K \oplus C) \cong B/K$. Hence, by the Benabdallah–Wilson criterion, there is $t \in \mathbb{N}$ such that $(p^t A)[p] \subseteq K \oplus C$. Thus by modularity we obtain that $(p^t B)[p] \subseteq (K \oplus C) \cap B = K \oplus (B \cap C) = K$, as required. □

Note that there exist subgroups of thick p-groups which are not necessarily direct summands, but which are thick groups too. In fact, if P is a pure subgroup of the thick p-group T such that $T/(P + T^1)$ is Σ-cyclic, then it is thick. For it is clear that $(P + T^1)/T^1$ is pure in T/T^1 with $T/T^1/(P + T^1)/T^1 \cong T/(P + T^1)$ Σ-cyclic. By Kulikov's theorem [22] (see also [13], vol. I, Theorem 28.2) we have $T/T^1 \cong (P + T^1)/T^1 \oplus T/(P + T^1) \cong (P/P^1) \oplus \big(T/(P + T^1)\big)$. We know that a p-group M is thick if and only if M/M^1 is thick. So it follows from Proposition 6.4 that P is thick.

Question 6.1. Are the balanced subgroups of thick p-groups thick as well?

We now show the validity of the corresponding analogue of Proposition 6.1.

Proposition 6.5. *Suppose that A is an e.f.i. abelian p-group with a pure subgroup G. Then A/G is e.f.i.*

Proof. Write $A/G = (M/G) \oplus (L/G)$ where L/G is Σ-cyclic. Then $A = M + L$ and $M \cap L = G$. From G being pure in A, whence in L, it follows by Kulikov's theorem [22] or [13], vol. I, Theorem 28.2, that $L = G \oplus L_1$. Thus $A = M \oplus L_1$ for $M \cap L_1 = M \cap (L \cap L_1) = (M \cap L) \cap L_1 = G \cap L_1 = 0$. Since $L_1 \cong L/G$ is Σ-cyclic, it follows that it is bounded. □

Notice that when G is bounded, the converse is true. In fact, applying [13], vol. I, Theorem 27.5, one can write $A \cong G \oplus A/G$ and so [20], Theorem 4.3, works.

We are now concerned with the following assertion which is parallel to Proposition 6.2.

Proposition 6.6. *Let A be an abelian p-group with a pure subgroup G such that A/G is Σ-cyclic. Then A is e.f.i. if and only if G is e.f.i. and A/G is bounded.*

Proof. By virtue to Kulikov's theorem [22] or [13], vol. I, Theorem 28.2, one can write that $A \cong G \oplus A/G$. Furthermore, Proposition 6.5 ensures that A/G is e.f.i. Σ-cyclic, hence bounded. Then apply that finite direct sums of e.f.i. groups are e.f.i. and a direct summand of e.f.i. groups is again e.f.i. □

We can replace the condition of pureness with full invariance.

Proposition 6.7. *Suppose G is an e.f.i. fully invariant subgroup of the abelian p-group A. If A/G is e.f.i., then A is e.f.i.*

Proof. Write $A = K \oplus P = K \oplus \bigoplus_{n=0}^{\infty} \bigoplus_{\alpha_n} \langle p^n \rangle$ where $P = \bigoplus_{n=0}^{\infty} \bigoplus_{\alpha_n} \langle p^n \rangle$ is Σ-cyclic and α_n is a cardinal. Then $G = (G \cap K) \oplus (G \cap P) = (G \cap K) \oplus \bigoplus_{n=0}^{\infty} \bigoplus_{\alpha_n} (G \cap \langle p^n \rangle)$. Thus $G \cap P = \bigoplus_{n=0}^{\infty} \bigoplus_{\alpha_n} (G \cap \langle p^n \rangle)$ and $(P + G)/G \cong P/(P \cap G) = \bigoplus_{n=0}^{\infty} \bigoplus_{\alpha_n} \langle p^n \rangle / \bigoplus_{n=0}^{\infty} \bigoplus_{\alpha_n} (G \cap \langle p^n \rangle) \cong \bigoplus_{n=0}^{\infty} \bigoplus_{\alpha_n} [\langle p^n \rangle / (G \cap \langle p^n \rangle)]$ are Σ-cyclic too. Now $A/G = [(K + G)/G] \oplus [(P + G)/G]$ since by modularity $(K + G) \cap (P + G) = G + K \cap (P + G) = G + K \cap (P + G \cap K) = G + G \cap K + K \cap P = G$. But $(P + G)/G$ being Σ-cyclic implies that it is bounded, hence so is P since $P \cap G$ is bounded. □

Problem 6.2. Prove the same result for thick groups. Moroever, if G is a fully invariant thick (respectively a fully invariant e.f.i.) subgroup of A, does it follow that A is thick (respectively e.f.i.) if and only if A/G is thick (respectively e.f.i.)? Are e.f.i. groups closed under pure extensions?

We also mention the following result which is of independent interest.

Proposition 6.8. *Any thick (respectively e.f.i.) abelian p-group is semi-complete if only if it is torsion-complete.*

Proof. Recall that thick groups are e.f.i. Since e.f.i. groups have only bounded direct summands which are Σ-cyclic groups, the torsion-completeness follows with the aid of [20] or [13], vol. II, Corollary 68.6, p. 18, or Exercise 8(b), p. 20. □

To end this section, we state the following weaker version of the above-mentioned Irwin conjecture.

Problem 6.3. Are the weakly ω_1-separable e.f.i. p-groups thick and, in particular, bounded?

In this direction, we note that it is a result by Eklof–Mekler that under $(\mathrm{MA} + \neg\mathrm{CH})$ any unbounded weakly ω_1-separable p-group of cardinality $\aleph_1$ is C-decomposable, i.e., it possesses an unbounded direct summand of cyclic groups of final rank equal to that of the whole group; thus it is not e.f.i.

(For the class of weakly ω_1-separable p-groups, see the next section.)

7. (Weakly) ω_1-separable p-groups

The precise definition of a p-primary *weakly ω_1-separable group* is given in [24]. They are necessarily separable. For convenience of the reader and for further use, we shall quote the following necessary and sufficient condition due to Megibben [24].

Criterion (Megibben, 1987). *Let A be a separable abelian p-group A. Then the following conditions are equivalent:*

(a) *A is weakly ω_1-separable;*

(b) *for all $C \leq A$: if $|C| = \aleph_0$ then $|\bigcap_{i<\omega}(p^i A + C)| = \aleph_0$;*

(c) *$|p^\omega(A/C)| \leq \aleph_0$ whenever $C \leq A$ with $|C| = \aleph_0$.*

The following statement contrasts [11] where G is weakly ω_1-separable while A/G is countable.

Theorem 7.1. *Suppose that G is a countable nice subgroup of the separable p-group A. Then A is weakly ω_1-separable if and only if A/G is weakly ω_1-separable.*

Proof. $\Rightarrow$: First observe that A/G is separable since G is nice in the separable group A. Let $T/G \leq A/G$ with $|T/G| = \aleph_0$. Then $|T| = \aleph_0$ and Megibben's criterion gives that $|\bigcap_{i<\omega}(p^iA + T)| = \aleph_0$. But $\aleph_0 = |T/G| \leq |\bigcap_{i<\omega}(p^i(A/G) + T/G)| = |\bigcap_{i<\omega}[(p^iA + T)/G]| = |[\bigcap_{i<\omega}(p^iA + T)]/T| \leq |\bigcap_{i<\omega}(p^iA + T)| = \aleph_0$. Consequently, $|\bigcap_{i<\omega}(p^i(A/G) + T/G)| = \aleph_0$. By Megibben's criterion A/G is weakly ω_1-separable.

$\Leftarrow$: Let C be a countable subgroup of A. Since A/G is separable, G is nice in A and $p^\omega A \subseteq G$. We consider two basic cases:

Case 1. $(C + G)/G$ is finite. Then it is nice in A/G and because of the niceness of G in A (see [13], vol. II, Lemma 79.3) we infer that $C + G$ is nice in A. Thus $\bigcap_{i<\omega}(p^iA + C + G) = p^\omega A + C + G = C + G$, whence $|\bigcap_{i<\omega}(p^iA + C + G)| = |C + G| = \aleph_0$ and so $|\bigcap_{i<\omega}(p^iA + C)| = \aleph_0$ since $C \subseteq \bigcap_{i<\omega}(p^iA + C)$, as required.

Case 2. $(C + G)/G$ is countable. Then by Megibben's criterion, it follows that $|\bigcap_{i<\omega}[p^i(A/G) + (C + G)/G]| = |\bigcap_{i<\omega}[(p^iA + G)/G + (C + G)/G]| = |\bigcap_{i<\omega}[(p^iA + C + G)/G]| = |[\bigcap_{i<\omega}(p^iA + C + G)]/G| = \aleph_0$ and so $|\bigcap_{i<\omega}(p^iA + C + G)| = |G|$. Since $C \subseteq \bigcap_{i<\omega}(p^iA + C)$ and $|C| = |G| = \aleph_0$, it follows that $|\bigcap_{i<\omega}(p^iA + C)| = \aleph_0$.

Finally in both situations, we employ again Megibben's criterion to obtain the desired statement. □

Problem 7.1. Let A be an abelian separable p-group with a subgroup G so that A/G is weakly ω_1-separable. Then A is weakly ω_1-separable $\Leftrightarrow$ G is strongly ω_1-separable in A $\Leftrightarrow$ for all $H \leq G : |H| = \aleph_0 \Rightarrow |\bigcap_{i<\omega}(p^iA + H)| = \aleph_0$.

Note that if G is *strongly* ω_1*-separable* in A, it is weakly ω_1-separable since $|\bigcap_{i<\omega}(p^iA + H)| = \aleph_0$ and $|H| = \aleph_0$ force that $|\bigcap_{i<\omega}(p^iG + H)| = \aleph_0$.

The following criterion is of independent interest; it is a connection between Sections 5 and 7.

Proposition 7.1. *Any weakly* ω_1*-separable torsion-complete p-group with a countable basic subgroup is countably bounded. In particular, the weakly* ω_1*-separable p-groups are direct sums of torsion-complete groups with components each of which possesses a countable basic subgroup only when they are direct sums of cyclic groups.*

Proof. Denote by B a basic subgroup of such a group A. Since A/B is always divisible and $|B| = \aleph_0$, by Megibben's criterion A/B is also countable. Thus it is obvious that A must be countable. So supposing that A is unbounded, by [13], vol. II, Exercise 7, p. 20, we find that $|A| = |B|^{\aleph_0} = \aleph_0^{\aleph_0} > \aleph_0$, which is a contradiction. Therefore, A is bounded.

As for second part, it is well-known (see [24]) that a subgroup of a weakly ω_1-separable p-group is a weakly ω_1-separable p-group. Thus the claim follows from the first part of the proposition. □

Conjecture 7.1. We conjecture that Proposition 7.1 does not hold true in general, i.e., the limitation on the countability of the basic subgroup cannot be generally removed. Nevertheless, we believe that it is consistent in (ZFC), with (CH) eventually, that every torsion-complete p-group of cardinal $\aleph_1$ is weakly ω_1-separable precisely when it is bounded; for cardinalities strictly greater $\aleph_1$ the result probably fails. Notice that with ($\neg$CH) each torsion-complete group of cardinality $\leq \aleph_1$ is bounded; compare with the proof of Proposition 7.1. (See also the above-mentioned result by Eklof–Mekler in (MA + $\neg$CH).)

The abelian p-group A is said to be ω_1-*separable* if each of its countable subgroups is contained in a countable Σ-cyclic direct summand of A. Such a group is necessarily separable. In other words, by Prüfer's second theorem, an abelian separable p-group A is ω_1-separable if and only if its countable subgroups can be embedded in countable direct summands of A.

Proposition 7.2. *Let G be a countable nice subgroup of the abelian p-group A. If A is ω_1-separable, then A/G is ω_1-separable.*

Proof. Since A is separable, it follows easily that A/G is separable. Choose $K/G \leq A/G$ with $|K/G| = \aleph_0$. Then $|K| = \aleph_0$ and thus $K \subseteq P$ such that $|P| = \aleph_0$ and $A = P \oplus A_1$. Consequently, $A/G = (P/G) \oplus \big((A_1 + G)/G\big)$ since by modularity we have $P \cap (A_1 + G) = G + (P \cap A_1) = G$. But $K/G \subseteq P/G$ and $|P/G| = \aleph_0$ because $\aleph_0 = |K/G| \leq |P/G| \leq |P| = \aleph_0$. That P/G is Σ-cyclic follows from Prüfer's second theorem. □

We note that if G is a pure and bounded subgroup of the abelian p-group A, then A/G being ω_1-separable implies that A is ω_1-separable. In fact, by [13], vol. I, Theorem 27.5, one may write $A \cong G \oplus A/G$.

The following question arises naturally.

Problem 7.2. Assume that G is a countable subgroup of the separable abelian p-group A. If A/G is ω_1-separable, does it follow that A is ω_1-separable?

The next Wallace-type problem is also of interest (for more details see [11]).

Problem 7.3. Assume that G is a subgroup of the separable abelian p-group A so that A/G is at most countable. If G is ω_1-separable, does it follow that A is ω_1-separable?

8. Q-p-groups

For completeness we recollect the definition of a p-group to be a Q-group which is stated in [21] in all generality for arbitrary groups.

Definition (Irwin–Richman, 1965). The separable abelian p-group A is said to be a *Q-group* if for each infinite subgroup C of A the equality $|\bigcap_{i<\omega}(p^i A + C)| = |C|$ holds.

It is self-evident that every Q-group is weakly ω_1-separable and that every weakly ω_1-separable group of cardinality $\aleph_1$ is a Q-group. So for groups of cardinality $\aleph_1$ these two classes of groups coincide.

The following result is dual to [11] where G is a Q-group and A/G is countable.

Theorem 8.1. *Suppose that G is a countable nice subgroup of the separable p-group A. Then A is a Q-group if and only if A/G is a Q-group.*

Proof. $\Rightarrow$: First we see that A being separable with a nice subgroup G implies that A/G is separable as well. Take $T/G \le A/G$ with $|T/G| \ge \aleph_0$. Thus $|T| = |T/G| \ge \aleph_0$ and by definition $|\bigcap_{i<\omega}(p^i A + T)| = |T|$. On the other hand, $|T/G| \le \left|\bigcap_{i<\omega}\left(p^i(A/G) + T/G\right)\right| = |\bigcap_{i<\omega}[(p^i A + T)/G]| = |[\bigcap_{i<\omega}(p^i A + T)]/T| \le |\bigcap_{i<\omega}(p^i A + T)| = |T| = |T/G|$. Therefore, $\left|\bigcap_{i<\omega}\left(p^i(A/G) + T/G\right)\right| = |T/G|$. Now, by definition, A/G is a Q-group.

$\Leftarrow$: The separability of A/G yields that G is nice in A and that $p^\omega A \subseteq G$. Take an arbitrary infinite subgroup C of A. We distinguish two cases.

Case 1. $(C+G)/G$ is finite. Then $(C+G)/G$ is nice in A/G and thus $C+G$ is nice in A. Furthermore, $\bigcap_{i<\omega}(p^i A + C + G) = p^\omega A + C + G = C + G$, whence $|C| \le |\bigcap_{i<\omega}(p^i A + C)| \le |\bigcap_{i<\omega}(p^i A + C + G)| = |C+G| = |C|$ and thus $|\bigcap_{i<\omega}(p^i A + C)| = |C|$, as required.

Case 2. $(C+G)/G \cong C/(C \cap G)$ is infinite. Consequently, $|\bigcap_{i<\omega}[p^i(A/G) + (C+G)/G]| = |\bigcap_{i<\omega}[(p^i A + G)/G + (C+G)/G]| = |\bigcap_{i<\omega}[(p^i A + G + C)/G]| = |[\bigcap_{i<\omega}(p^i A + G + C)]/G| = |(C+G)/G| = |C/(C \cap G)| = |C|$. Therefore $|\bigcap_{i<\omega}(p^i A + G + C)| = |C|$ and so $|\bigcap_{i<\omega}(p^i A + C)| = |C|$.

The assertion now follows from the definition. □

Problem 8.1. Let A be an abelian separable p-group with a subgroup G such that A/G is a Q-group. Does the following hold: A is a Q-group $\Leftrightarrow$ G is a strong Q-group in A $\Leftrightarrow$ for all $H \le G : |H| \ge \aleph_0 \Rightarrow |\bigcap_{i<\omega}(p^i A + H)| = |H|$?

Notice that if G is a *strong Q-group* in A, it must be a Q-group itself.

9. *p*-groups with a nice basis

These groups were defined in [2]. We recall the definition and some basic facts obtained in [7].

Definition 9.1. The abelian p-group A is said to have a *nice basis* if it can be represented as a countable ascending union of nice Σ-cyclic subgroups. More precisely, $A = \bigcup_{n<\omega} A_n$, $A_n \subseteq A_{n+1} \leq A$ and, for each $n \geq 1$, A_n is nice in A and is Σ-cyclic.

This class of groups is quite large and properly contains the separable p-groups, the simply presented p-groups (in particular the direct sums of countable p-groups), and the $p^{\omega+n}$-projective p-groups for any natural n.

In [7] it was established that if G has a nice basis and is balanced in A such that A/G is at most countable, then A has a nice basis as well. Moreover, it was shown there that A possesses a nice basis only if its large subgroup possesses a nice basis.

The following classical statement from [12] is of interest.

Theorem (Dieudonné, 1952). *Let G be a strongly Σ-cyclic subgroup in A and let A/G be Σ-cyclic. Then A is Σ-cyclic.*

Since every Σ-cyclic group possesses a nice basis (see, e.g., [7]), we present the following generalization.

Proposition 9.1. *Let A be an abelian p-group with a nice subgroup G so that A/G has a nice basis. If G is strongly Σ-cyclic in A, then A has a nice basis.*

Proof. Write $A/G = \bigcup_{m<\omega}(A_m/G)$, where $A_m/G \subseteq A_{m+1}/G \leq A/G$ are nice Σ-cyclic subgroups in A/G. Then $A = \bigcup_{m<\omega} A_m$, $A_m \subseteq A_{m+1} \leq A$. Moreover, by [13], vol. II, Lemma 79.3, we deduce that A_m is nice in A for all $m \geq 1$.

Next we show that each A_m is Σ-cyclic. Consider A_m/G for an arbitrary but fixed natural number m. Since $G = \bigcup_{i<\omega} G_i$, $G_i \subseteq G_{i+1} \leq G$ and, for every $i \geq 1$, $G_i \cap p^i A = 0$, we obtain that $G_i \cap p^i A_m = 0$, whence G is strongly Σ-cyclic in A_m. From A_m/G being Σ-cyclic we obtain by Dieudonné's criterion that each A_m is Σ-cyclic, as requested. □

The next consequences are immediate.

Corollary 9.1. *Suppose that G is a nice and pure subgroup of the abelian p-group A such that A/G has a nice basis. If G is Σ-cyclic, then A has a nice basis.*

Proof. Since G is Σ-cyclic and is pure in A, it easily follows that it is strongly Σ-cyclic in A. □

The condition on G to be pure can be removed when we impose additional restrictions on A/G.

Proposition 9.2. *Suppose that A is an abelian p-group with a subgroup G such that A/G is separable. If G is Σ-cyclic, then A has a nice basis.*

Proof. Since A/G is separable, we can write $A/G = \bigcup_{n<\omega} A_n/G$ where $A_n \subseteq A_{n+1} \leq A$ and, for each $n \geq 1$, A_n/G is nice in A/G and bounded. Since G is nice in A, it follows from [13], vol. II, Lemma 79.3, that A_n is nice in A. By [13], vol. I, Proposition 18.3, G being Σ-cyclic implies the same property for A_n, and hence we are done. □

Observe that since $p^\omega A \subseteq G$ we have that $p^\omega A$ is also Σ-cyclic. Thus Proposition 9.2 follows from [7], Proposition 3, as well.

Problem 9.1. Assume that A is an abelian p-group with a nice subgroup G such that A/G has a nice basis. Does it follow that A has a nice basis if and only if G has a nice basis in A, or equivalently $G = \bigcup_{i<\omega} G_i$, $G_i \subseteq G_{i+1} \leq G$ and all G_i are nice in A Σ-cyclic subgroups?

We conjecture that the answer is negative. This is motivated by the observation that if N is a nice subgroup of A, then $N \cap G$ need not be nice neither in G nor in A even when G is nice and isotype in A. Nevertheless, under some additional assumptions on G this is so; for instance let $(G+N) \cap p^\alpha A = (G \cap p^\alpha A) + (N \cap p^\alpha A)$. Then, for each limit ordinal α, we have $\bigcap_{\tau<\alpha}(N \cap G + p^\tau A) \subseteq \bigcap_{\tau<\alpha}(G + p^\tau A) = G + p^\alpha A$ and $\bigcap_{\tau<\alpha}(N \cap G + p^\tau A) \subseteq \bigcap_{\tau<\alpha}(N + p^\tau A) = N + p^\alpha A$. Furthermore, by modularity we deduce that $\bigcap_{\tau<\alpha}(N \cap G + p^\tau A) \subseteq (N + p^\alpha A) \cap (G + p^\alpha A) = p^\alpha A + N \cap (G + p^\alpha A) = p^\alpha A + G \cap (N + p^\alpha A)$. Since $(G+N) \cap p^\alpha A = (G \cap p^\alpha A) + (N \cap p^\alpha A)$ we obtain that $G \cap (N + p^\alpha A) \subseteq G \cap N + p^\alpha A$, which implies that $\bigcap_{\tau<\alpha}(N \cap G + p^\tau A) = N \cap G + p^\alpha A$.

So we conclude that some extra requirements on G are necessary for answering the above question in the affirmative.

Problem 9.2. Decide whether or not there are p-groups with a nice basis which are precisely the separable p-groups.

We finish with a general question.

Problem 9.3. Find other classes of primary abelian groups for which the Dieudonné-type theorems hold. In particular, find out whether such theorems hold for the classes of quasi-complete p-groups, pure-complete p-groups, and direct sums of torsion-complete p-groups.

We recall that an abelian p-group A is known to be *quasi-complete* if $p^{\omega}(A/H)$ is divisible whenever H is a pure subgroup of A. We note the simple fact that a balanced subgroup of a quasi-complete p-group is again quasi-complete and the factor group of a quasi-complete p-group modulo its pure subgroup is quasi-complete too.

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