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Weak theorems on differential inequalities for two-dimensional functional differential systems

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Abstract. New theorems on differential inequalities for two-dimensional systems of linear functional differential equations are established. Differential systems with argument deviations are considered in more detail, in which case further results are obtained.

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1. Introduction

On the interval $[a,b]$ we consider the two-dimensional differential system

$$\begin{aligned} u'(t) &= p(v)(t) + q_1(t), \\ v'(t) &= g(u)(t) + q_2(t), \end{aligned} \tag{1.1}$$

where $p, g : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ are bounded linear operators and $q_1, q_2 \in L([a,b];\mathbb{R})$. By a solution of the system (1.1) we understand a pair (u,v) of absolutely continuous on $[a,b]$ functions satisfying (1.1) almost everywhere on $[a,b]$.

We shall study the system (1.1) in the case where either p or g is a monotone operator. Thus, we shall assume in the sequel that

$$p \in \mathscr{P}_{ab}, \tag{1.2}$$

where $\mathscr{P}_{ab}$ denotes the set of nondecreasing operators (see Definition 2.1).

It is well known that the theorems on differential inequalities play very important role in the theory of differential equations. Therefore, the question on the

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validity of the theorems on differential inequalities is studied by many authors (see, e.g., [24], [4], [25], [17], [9], [27], [13], [18], [19], [11], [20], [22], [1], [5], [2], [26], [6], [7], [10]). Although for ordinary differential equations and their systems the question indicated is studied in detail (see, e.g., [9], [3], [27], [13], [1], [2], [10], [26] and references therein), for functional differential systems, and even for the rather simple system (1.1), there is still a broad field for further investigations.

We have investigated the n-dimensional systems of functional differential inequalities in [24]. In the present paper, new results in this line, namely, the so-called *weak theorems on differential inequalities*, are established for the system (1.1). In other words, we obtain efficient conditions for the operators p and g which guarantee that a certain maximum principle holds for the system (1.1). All results are finally applied in the case where (1.1) is the differential system with argument deviations

$$u'(t) = f(t)v\big(\mu(t)\big) + q_1(t), \qquad v'(t) = h(t)u\big(\tau(t)\big) + q_2(t), \tag{1.3}$$

in which $f, h, q_1, q_2 \in L([a,b];\mathbb{R})$ and $\mu, \tau : [a,b] \to [a,b]$ are measurable functions. It should be noted that the second order functional differential equation

$$u''(t) = \ell(u)(t) + q(t), \tag{1.4}$$

where $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is a linear bounded operator and $q \in L([a,b];\mathbb{R})$, can also be regarded as a particular case of (1.1). Some of the results stated below correspond to those obtained in [25], [17] for the equation (1.4).

2. Notation and definitions

We use the following notation throughout the paper.

$\mathbb{R}$ is the set of all the real numbers, $\mathbb{R}_+ = [0, +\infty[$;

$C([a,b];\mathbb{R})$ is the Banach space of continuous functions $u : [a,b] \to \mathbb{R}$ equipped with the norm

$$\|u\|_C = \max\{|u(t)| \mid t \in [a,b]\};$$

$C([a,b];\mathbb{R}_+) = \{u \in C([a,b];\mathbb{R}) \mid u(t) \geq 0 \text{ for } t \in [a,b]\}$;

$C_{\mathrm{loc}}([a,b[;\mathbb{R})$ is the set of continuous functions $u : [a,b[\to \mathbb{R}$.

$\widetilde{C}([a,b];\mathbb{R})$ is the set of absolutely continuous functions $u : [a,b] \to \mathbb{R}$;

$\widetilde{C}_{\mathrm{loc}}([a,b[;\mathbb{R})$ is the set of functions $u : [a,b[\to \mathbb{R}$ such that $u \in \widetilde{C}([a,\beta];\mathbb{R})$ for every $\beta \in \,]a,b[$;

$L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $h:[a,b]\to\mathbb{R}$ equipped with the norm

$$\|h\|_L = \int_a^b |h(s)|\,ds;$$

$L([a,b];\mathbb{R}_+) = \{h \in L([a,b];\mathbb{R}) \mid h(t) \ge 0 \text{ for a.a. } t \in [a,b]\}$;
$\mathscr{L}_{ab}$ is the set of linear bounded operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$.

Definition 2.1. An operator $\ell \in \mathscr{L}_{ab}$ is said to be *nondecreasing* if it maps the set $C([a,b];\mathbb{R}_+)$ to the set $L([a,b];\mathbb{R}_+)$. The class of nondecreasing operators is denoted by $\mathscr{P}_{ab}$. We say that an operator $\ell \in \mathscr{L}_{ab}$ is *nonincreasing* if $-\ell \in \mathscr{P}_{ab}$.

Example 2.1. Let $\ell \in \mathscr{L}_{ab}$ be the operator defined by the formula

$$\ell(z)(t) \stackrel{\text{def}}{=} h(t)z\big(\tau(t)\big) \qquad \text{for } t \in [a,b],\ z \in C([a,b];\mathbb{R}), \tag{2.1}$$

where $h \in L([a,b];\mathbb{R})$ and $\tau : [a,b] \to [a,b]$ is a measurable function. Then $\ell \in \mathscr{P}_{ab}$ if and only if

$$h(t) \ge 0 \qquad \text{for a.e. } t \in [a,b].$$

Definition 2.2. We say that $\ell \in \mathscr{L}_{ab}$ is an a-Volterra operator if, for arbitrary $b_0 \in\,]a,b]$ and $z \in C([a,b];\mathbb{R})$ with the property

$$z(t) = 0 \qquad \text{for } t \in [a,b_0],$$

we have

$$\ell(z)(t) = 0 \qquad \text{for a.e. } t \in [a,b_0].$$

Example 2.2. The operator $\ell \in \mathscr{L}_{ab}$ given by (2.1) is an a-Volterra operator if and only if

$$|h(t)|\big(\tau(t) - t\big) \le 0 \qquad \text{for a.e. } t \in [a,b].$$

Definition 2.3. Let $\ell \in \mathscr{L}_{ab}$ and $b_0 \in\,]a,b[$. The operator $\ell_{ab_0} : C([a,b_0];\mathbb{R}) \to L([a,b_0];\mathbb{R})$ defined by the equality

$$\ell_{ab_0}(z)(t) \stackrel{\text{def}}{=} \ell(\tilde{z})(t) \qquad \text{for a.e. } t \in [a,b_0],\ z \in C([a,b_0];\mathbb{R}),$$

where

$$\tilde{z}(t) = \begin{cases} z(t) & \text{for } t \in [a, b_0[, \\ z(b_0) & \text{for } t \in [b_0, b], \end{cases}$$

is called the restriction of the operator ℓ to the space $C([a, b_0]; \mathbb{R})$.

If $b_0 < b_1 \le b$ and $z \in C([a, b_1]; \mathbb{R})$, then we write $\ell_{ab_0}(z)$ instead of $\ell_{ab_0}(z|_{[a,b_0]})$.

Remark 2.1. If ℓ is an a-Volterra operator then it is clear that, for every $b_0 \in\,]a, b[$ and $z \in C([a, b]; \mathbb{R})$, the condition

$$\ell_{ab_0}(z)(t) = \ell(z)(t) \qquad \text{for a.e. } t \in [a, b_0]$$

is satisfied.

Along with the system (1.1), we consider the corresponding homogeneous system

$$u'(t) = p(v)(t), \qquad v'(t) = g(u)(t). \tag{1.1$_0$}$$

The following statement is well known from the general theory of functional differential equations (see, e.g., [23], [15], [16], [8]).

Proposition 2.1. *The Cauchy problem*

$$u(a) = c_1, \qquad v(a) = c_2 \tag{2.2}$$

for the system (1.1) *is uniquely solvable for arbitrary* $q_1, q_2 \in L([a, b]; \mathbb{R})$ *and* $c_1, c_2 \in \mathbb{R}$ *if and only if the corresponding homogeneous problem*

$$u(a) = 0, \qquad v(a) = 0 \tag{2.2$_0$}$$

for the system (1.1$_0$) *has only the trivial solution.*

3. Main Results

We give some definitions and remarks before we formulate the main results.

Definition 3.1 ([24], Definition 3.1). A pair $(p, g) \in \mathscr{L}_{ab} \times \mathscr{L}_{ab}$ is said to belong to the set $\mathscr{S}^2_{ab}(a)$ if, for any $u, v \in \tilde{C}([a, b]; \mathbb{R})$ such that

$$u'(t) \ge p(v)(t), \qquad v'(t) \ge g(u)(t) \qquad \text{for a.e. } t \in [a, b] \tag{3.1}$$

and

$$u(a) \geq 0, \qquad v(a) \geq 0, \tag{3.2}$$

the relations

$$u(t) \geq 0, \qquad v(t) \geq 0 \qquad \text{for } t \in [a,b] \tag{3.3}$$

are satisfied.

If $(p,g) \in \mathscr{S}_{ab}^2(a)$ then we say that the *theorem on differential inequalities* holds for the system (1.1).

In [24], efficient conditions are found for the validity of the inclusion $(p,g) \in \mathscr{S}_{ab}^2(a)$, provided that $p, g \in \mathscr{P}_{ab}$. The question of obtaining such conditions is still open for the cases where at least one of the operators p and g is not nondecreasing.

On the other hand it is well known that for the ordinary differential system

$$u' = f(t)v + q_1(t), \qquad v' = h(t)u + q_2(t),$$

where $f, h, q_1, q_2 \in L([a,b];\mathbb{R})$, the theorem on differential inequalities holds if

$$f(t) \geq 0, \qquad h(t) \geq 0 \qquad \text{for a.e. } t \in [a,b]. \tag{3.4}$$

In other words, the condition (3.4) is sufficient for the validity of the inclusion $(p,g) \in \mathscr{S}_{ab}^2(a)$, where

$$p(z)(t) \overset{\text{def}}{=} f(t)z(t), \qquad g(z)(t) \overset{\text{def}}{=} h(t)z(t) \qquad \text{for a.e. } t \in [a,b],\ z \in C([a,b];\mathbb{R}).$$

If $f, h \in C([a,b];\mathbb{R})$ then the condition (3.4) is not only sufficient but also necessary (see, e.g., [14], §1.7).

Therefore, the requirement of the validity of the condition (3.3) in Definition 3.1 seems to be too restrictive in the case where the operators p and g are not both nondecreasing. We shall weaken the condition (3.3) in the following way.

Definition 3.2. A pair $(p,g) \in \mathscr{L}_{ab} \times \mathscr{L}_{ab}$ is said to belong to the set $\hat{\mathscr{S}}_{ab}^2(a)$ if, for any $u, v \in \tilde{C}([a,b];\mathbb{R})$ satisfying (3.1) and (3.2), the relation

$$u(t) \geq 0 \qquad \text{for } t \in [a,b] \tag{3.5}$$

is fulfilled.

If $(p,g) \in \hat{\mathscr{S}}_{ab}^2(a)$ then we say that the *weak theorem on differential inequalities* holds for the system (1.1).

Remark 3.1. It follows immediately from Definition 3.2 that

(a) $(0, g) \in \hat{\mathscr{S}}^2_{ab}(a)$ for every $g \in \mathscr{L}_{ab}$;

(b) $\mathscr{S}^2_{ab}(a)$ is a proper subset of $\hat{\mathscr{S}}^2_{ab}(a)$;

(c) $\hat{\mathscr{S}}^2_{ab}(a) \cap (\mathscr{L}_{ab} \times \mathscr{P}_{ab}) = \mathscr{S}^2_{ab}(a) \cap (\mathscr{L}_{ab} \times \mathscr{P}_{ab})$, i.e.,

$$(p, g) \in \hat{\mathscr{S}}^2_{ab}(a) \iff (p, g) \in \mathscr{S}^2_{ab}(a) \quad \text{for every } p \in \mathscr{L}_{ab},\ g \in \mathscr{P}_{ab};$$

(d) $(p, g) \in \hat{\mathscr{S}}^2_{ab}(a) \iff (g, p) \in \hat{\mathscr{S}}^2_{ab}(a)$ for every $p, g \in \mathscr{P}_{ab}$.

Remark 3.2. It is clear that the homogeneous problem (1.1_0), (2.2_0) has only the trivial solution under the assumption $(p, g) \in \hat{\mathscr{S}}^2_{ab}(a)$. Therefore, according to Proposition 2.1, the Cauchy problem (1.1), (2.2) has a unique solution for all $q_1, q_2 \in L([a, b]; \mathbb{R})$ and $c_1, c_2 \in \mathbb{R}$. However, the inclusion $(p, g) \in \hat{\mathscr{S}}^2_{ab}(a)$ (resp. $(p, g) \in \mathscr{S}^2_{ab}(a)$) guarantees that, in addition, the solution (u, v) of this problem satisfies (3.5) (resp. (3.3)) whenever q_1, q_2 and c_1, c_2 are such that

$$q_k(t) \geq 0 \quad \text{for a.e. } t \in [a, b], \quad c_k \geq 0\ (k = 1, 2).$$

As has been indicated above, we investigate the system (1.1) in the case where the condition (1.2) is satisfied. Let us now formulate the main results, namely, efficient conditions for the operators p and g guaranteeing the validity of the inclusion $(p, g) \in \hat{\mathscr{S}}^2_{ab}(a)$. The proofs are given later in Section 4.

The following statement describes a characteristic property of the set $\hat{\mathscr{S}}^2_{ab}(a)$.

Theorem 3.1. *Let $p \in \mathscr{P}_{ab}$ and $g = g_0 - g_1$ with $g_0, g_1 \in \mathscr{P}_{ab}$. If*

$$(p, g_0) \in \hat{\mathscr{S}}^2_{ab}(a) \quad \text{and} \quad (p, -g_1) \in \hat{\mathscr{S}}^2_{ab}(a) \tag{3.6}$$

then $(p, g) \in \hat{\mathscr{S}}^2_{ab}(a)$.

It is proved in [12], Ch.VII, §1.2, that $g \in \mathscr{L}_{ab}$ admits the representation $g = g_0 - g_1$ with $g_0, g_1 \in \mathscr{P}_{ab}$ if and only if the operator g is strongly bounded, i.e., if there exists $\eta \in L([a, b]; \mathbb{R}_+)$ such that

$$|g(z)(t)| \leq \eta(t)\|z\|_C \quad \text{for a.e. } t \in [a, b] \text{ and every } z \in C([a, b]; \mathbb{R}).$$

Consequently, due to the results given in Sections 3.1 and 3.2, Theorem 3.1 allows one to obtain several efficient conditions for the validity of the inclusion $(p, g) \in \hat{\mathscr{S}}^2_{ab}(a)$ for every nondecreasing p and strongly bounded g.

3.1. The case $g \in \mathscr{P}_{ab}$. We first consider the case where both operators p and g are nondecreasing. In this case, we have

$$(p,g) \in \hat{\mathscr{S}}_{ab}^2(a) \iff (p,g) \in \mathscr{S}_{ab}^2(a)$$

(see Remark 3.1(c)). As mentioned above, properties of the set $\mathscr{S}_{ab}^2(a)$ are studied in [24]. For the sake of completeness, we formulate here a general result (see Theorem 3.2) and one of its corollaries. Then we derive two new corollaries of this general theorem, which are not contained in the paper mentioned.

Theorem 3.2 ([24], Theorem 3.2). *Let $p, g \in \mathscr{P}_{ab}$. Then $(p,g) \in \hat{\mathscr{S}}_{ab}^2(a)$ if and only if there exist functions $\gamma_1, \gamma_2 \in \tilde{C}([a,b];\mathbb{R})$ such that*

$$\gamma_1(t) > 0, \qquad \gamma_2(t) > 0 \qquad \text{for } t \in [a,b], \tag{3.7}$$

and

$$\gamma_1'(t) \geq p(\gamma_2)(t), \qquad \gamma_2'(t) \geq g(\gamma_1)(t) \qquad \text{for a.e. } t \in [a,b]. \tag{3.8}$$

Corollary 3.1 ([24], Corollary 3.5). *Let $p, g \in \mathscr{P}_{ab}$ and let there exist operators $\tilde{p}, \tilde{g} \in \mathscr{P}_{ab}$ such that the inequalities*

$$\begin{aligned} p\big(\varphi\big(g(w)\big)\big)(t) - p(1)(t)\varphi\big(g(w)\big)(t) &\leq \tilde{p}(w)(t) \qquad \text{for a.e. } t \in [a,b], \\ g\big(\varphi\big(p(w)\big)\big)(t) - g(1)(t)\varphi\big(p(w)\big)(t) &\leq \tilde{g}(w)(t) \qquad \text{for a.e. } t \in [a,b] \end{aligned}$$

hold on the set $C([a,b];\mathbb{R}_+)$, where

$$\varphi(h)(t) \stackrel{\text{def}}{=} \int_a^t h(s)\,ds \qquad \text{for } t \in [a,b],\ h \in L([a,b];\mathbb{R}). \tag{3.9}$$

Let, moreover,

$$\max\{\lambda_1, \lambda_2\} < 1, \tag{3.10}$$

where

$$\lambda_1 = \int_a^b \cosh\Big(\int_s^b \omega(\xi)\,d\xi\Big)\tilde{p}(1)(s)\,ds + \int_a^b \sinh\Big(\int_s^b \omega(\xi)\,d\xi\Big)\tilde{g}(1)(s)\,ds, \tag{3.11}$$

$$\lambda_2 = \int_a^b \cosh\Big(\int_s^b \omega(\xi)\,d\xi\Big)\tilde{g}(1)(s)\,ds + \int_a^b \sinh\Big(\int_s^b \omega(\xi)\,d\xi\Big)\tilde{p}(1)(s)\,ds, \tag{3.12}$$

and

$$\omega(t) \stackrel{\text{def}}{=} \max\{p(1)(t), g(1)(t)\} \qquad \text{for a.e. } t \in [a,b]. \tag{3.13}$$

Then $(p,g) \in \hat{\mathscr{S}}_{ab}^2(a)$.

Remark 3.3. The strict inequality (3.10) in the previous corollary cannot be replaced by the nonstrict one (see [24], Example 5.3).

We introduce a simple notation.

Notation 3.1. For any $\ell \in \mathscr{L}_{ab}$, we put

$$b_\ell^* = \inf \mathscr{A}(\ell),$$

where $\mathscr{A}(\ell)$ is the set of all $t \in [a,b]$ for which the implication

$$z \in C([a,b];\mathbb{R}), \qquad z(\xi) = 0 \text{ for } \xi \in [a,t] \implies \ell(v)(\xi) = 0 \text{ for a.a. } \xi \in [a,b]$$

is true.

Remark 3.4. It is easy to verify that $b_\ell^* \in \mathscr{A}(\ell)$, i.e.,

$$z \in C([a,b];\mathbb{R}), \qquad z(\xi) = 0 \text{ for } \xi \in [a,b_\ell^*] \implies \ell(z)(\xi) = 0 \text{ for a.a. } \xi \in [a,b].$$

The following statements can also be derived from Theorem 3.2.

Corollary 3.2. *Let $p, g \in \mathscr{P}_{ab}$ be such that*

$$\int_a^{b_g^*} p\big(\varphi\big(g(1)\big)\big)(s)\,ds < 1, \tag{3.14}$$

where the operator φ is given by (3.9) *and the number b_g^* is defined in Notation* 3.1. *Then $(p,g) \in \hat{\mathscr{S}}_{ab}^2(a)$.*

The next proposition can be regarded as a complement of the previous corollary.

Proposition 3.1. *Let $p, g \in \mathscr{P}_{ab}$ be such that*

$$\int_a^{b_g^*} p\big(\varphi\big(g(1)\big)\big)(s)\,ds = 1, \tag{3.15}$$

where the operator φ is given by (3.9) *and the number b_g^* is defined in Notation* 3.1. *Then $(p,g) \in \hat{\mathscr{S}}_{ab}^2(a)$ if and only if the homogeneous problem* (1.1_0), (2.2_0) *has only the trivial solution.*

In view of Remark 3.1(d), Corollary 3.2 and Proposition 3.1 immediately yield

Corollary 3.2′. *Let $p, g \in \mathscr{P}_{ab}$ be such that*

$$\int_a^{b_p^*} g\big(\varphi\big(p(1)\big)\big)(s)\,ds < 1,$$

where the operator φ *is given by* (3.9) *and the number* b_p^* *is defined in Notation* 3.1. *Then* $(p,g) \in \hat{\mathscr{S}}_{ab}^2(a)$.

Proposition 3.1′. *Let* $p, g \in \mathscr{P}_{ab}$ *be such that*

$$\int_a^{b_p^*} g\big(\varphi\big(p(1)\big)\big)(s)\,ds = 1,$$

where the operator φ *is given by* (3.9) *and the number* b_p^* *is defined in Notation* 3.1. *Then* $(p,g) \in \hat{\mathscr{S}}_{ab}^2(a)$ *if and only if the homogeneous problem* (1.1_0), (2.2_0) *has only the trivial solution.*

Example 3.1. On the interval $[0, \pi/4]$, we consider the differential system

$$\begin{aligned} u'(t) &= d_1 \sin t \int_0^{t/2} s v(s/2)\,ds + q_1(t), \\ v'(t) &= d_2 \cos(2t) \int_0^t \cos(2s) u\big(\tau(s)\big)\,ds + q_2(t), \end{aligned} \tag{3.16}$$

where $\tau : [0, \pi/4] \to [0, \pi/4]$ is a measurable function, $q_1, q_2 \in L([0, \pi/4]; \mathbb{R})$, and $d_1, d_2 \in \mathbb{R}_+$ are such that

$$d_1 d_2 < \frac{2^{12}}{4\pi(1 + 2\sqrt{2}) - \pi^2(1 + \sqrt{2}) - 24}.$$

It is clear that (3.16) is a particular case of (1.1) in which $a = 0$, $b = \pi/4$, and p, g are given by formulae

$$\begin{aligned} p(z)(t) &= d_1 \sin t \int_0^{t/2} s z(s/2)\,ds, \\ g(z)(t) &= d_2 \cos(2t) \int_0^t \cos(2s) z\big(\tau(s)\big)\,ds \end{aligned} \tag{3.17}$$

for a.e. $t \in [0, \pi/4]$ and all $z \in C([0, \pi/4]; \mathbb{R})$. It is not difficult to verify that $b_g^* = \operatorname{ess\,sup}\{\tau(t) \mid t \in [0, \pi/4]\}$ (see Notation 3.1) and

$$\varphi\big(g(1)\big)(t) = \int_0^t g(1)(s)\,ds = \int_0^t d_2 \cos(2s) \int_0^s \cos(2\xi)\,d\xi\,ds = \frac{d_2}{16}\big(1 - \cos(4t)\big)$$

for $t \in [0, \pi/4]$. Consequently, we have

$$\int_0^{b_g^*} p\big(\varphi\big(g(1)\big)\big)(s)\,ds \le \int_0^{\pi/4} p\big(\varphi\big(g(1)\big)\big)(s)\,ds$$
$$= \int_0^{\pi/4} d_1 \sin s \int_0^{s/2} \xi \frac{d_2}{16}\big(1-\cos(2\xi)\big)\,d\xi\,ds$$
$$= \frac{d_1 d_2}{2^{12}}\big(4\pi(1+2\sqrt{2}) - \pi^2(1+\sqrt{2}) - 24\big) < 1.$$

Therefore, according to Corollary 3.2, Remark 3.1(c), and Remark 3.2, the Cauchy problem

$$u(0) = c_1, \qquad v(0) = c_2 \tag{3.18}$$

for the system (3.16) has a unique solution for arbitrary $q_1, q_2 \in L([0,\pi/4];\mathbb{R})$ and $c_1, c_2 \in \mathbb{R}$. Moreover, if q_1, q_2 and c_1, c_2 fulfil the additional condition

$$q_k(t) \ge 0 \quad \text{ for a.e. } t \in [0,\pi/4], \qquad c_k \ge 0 \ (k=1,2) \tag{3.19}$$

then the unique solution (u,v) of this problem satisfies the relation

$$u(t) \ge 0, \qquad v(t) \ge 0 \qquad \text{for } t \in [0,\pi/4].$$

Example 3.2. On the interval $[0,1]$, we consider the Cauchy problem

$$u''(t) = \frac{d}{(1-t)^\lambda}\int_0^t \frac{u\big(\tau(s)\big)}{(1-s)^\lambda}\,ds + q(t), \qquad u(0)=c_1,\ u'(0)=c_2, \tag{3.20}$$

where $\lambda < 1$, $0 \le d < (3-2\lambda)(2-\lambda)$, $\tau : [0,1] \to [0,1]$ is a measurable function, $q \in L([0,1];\mathbb{R})$ and $c_1, c_2 \in \mathbb{R}$.

It is clear that (3.20) is a particular case of (1.1), (2.2) in which $a=0$, $b=1$, $q_1 \equiv 0$, $q_2 \equiv q$, and p, g are given by formulae

$$p(z)(t) = z(t), \qquad g(z)(t) = \frac{d}{(1-t)^\lambda}\int_0^t \frac{z\big(\tau(s)\big)}{(1-s)^\lambda}\,ds \tag{3.21}$$

for a.e. $t \in [0,1]$ and all $z \in C([0,1];\mathbb{R})$. It is not difficult to verify that $b_g^* = \operatorname{ess\,sup}\{\tau(t) \mid t \in [0,1]\}$ (see Notation 3.1) and

$$\int_0^{b_g^*} p\big(\varphi\big(g(1)\big)\big)(s)\,ds = \int_0^{b_g^*}(1-s)g(1)(s)\,ds \le \int_0^1 (1-s)g(1)(s)\,ds$$
$$= d\int_0^1 (1-s)^{1-\lambda}\int_0^s \frac{d\xi}{(1-\xi)^\lambda}\,ds = \frac{d}{(3-2\lambda)(2-\lambda)}.$$

Therefore, according to Corollary 3.2, Remark 3.1(c), and Remark 3.2, the problem (3.20) has a unique solution for arbitrary $q \in L([0,1];\mathbb{R})$ and $c_1, c_2 \in \mathbb{R}$. Moreover, if q and c_1, c_2 fulfil the additional condition

$$q(t) \geq 0 \quad \text{for a.e. } t \in [0,1], \quad c_1 \geq 0,\ c_2 \geq 0, \tag{3.22}$$

then the unique solution u of this problem satisfies the relation

$$u(t) \geq 0, \quad u'(t) \geq 0 \quad \text{for } t \in [0,1].$$

3.2. The case $-g \in \mathcal{P}_{ab}$. Now we consider the case where the operators p and g are nondecreasing and nonincreasing, respectively. Here we have a sufficient and necessary condition for the validity of the inclusion $(p,g) \in \hat{\mathcal{S}}^2_{ab}(a)$, provided that p, g are a-Volterra operators.

Theorem 3.3. *Let $-g, p \in \mathcal{P}_{ab}$ and let p, g be a-Volterra operators. Then $(p,g) \in \hat{\mathcal{S}}^2_{ab}(a)$ if and only if there exist functions $\gamma_1, \gamma_2 \in \tilde{C}_{\mathrm{loc}}([a,b[;\mathbb{R})$ such that $\gamma_1 \in C([a,b];\mathbb{R})$,*

$$\gamma_1'(t) \leq p(\gamma_2)(t) \quad \textit{for a.e. } t \in [a,b],^{1} \tag{3.23}$$

$$\gamma_2'(t) \leq g(\gamma_1)(t) \quad \textit{for a.e. } t \in [a,b], \tag{3.24}$$

$$\gamma_1(t) \geq 0 \quad \textit{for } t \in [a,b], \tag{3.25}$$

$$\gamma_1(a) > 0, \quad \gamma_2(a) \leq 0, \tag{3.26}$$

and

$$|\gamma_1(t)| + |\gamma_2(t)| \neq 0 \quad \textit{for } t \in\,]a,b[. \tag{3.27}$$

Remark 3.5. The condition (3.23) of the previous theorem is understood in the sense that, for any $b_0 \in\,]a,b[$, the relation

$$\gamma_1'(t) \leq p_{ab_0}(\gamma_2)(t) \quad \text{for a.e. } t \in [a,b_0] \tag{3.28}$$

holds, where p_{ab_0} is the restriction of the operator p to the space $C([a,b_0];\mathbb{R})$.

Remark 3.6. Observe that the function γ_2 in Theorem 3.3 necessarily satisfies

$$\gamma_2(t) \leq 0 \quad \text{for } t \in [a,b[. \tag{3.29}$$

Theorem 3.3 yields the following statement.

---

[1] See Remark 3.5.

Corollary 3.3. *Let $-g, p \in \mathcal{P}_{ab}$ and let p, g be a-Volterra operators. If, moreover,*

$$\int_a^b \left| p\big(\varphi(g(1))\big)(s) \right| ds \le 1, \tag{3.30}$$

where the operator φ is defined by (3.9)*, then $(p, g) \in \hat{\mathcal{S}}^2_{ab}(a)$.*

Remark 3.7. The inequality (3.30) of the previous corollary cannot be replaced by the inequality

$$\int_a^b \left| p\big(\varphi(g(1))\big)(s) \right| ds \le 1 + \varepsilon, \tag{3.31}$$

no matter how small $\varepsilon > 0$ would be (see Example 6.1).

Example 3.3. On the interval $[0, \pi/4]$, we consider the differential system (3.16), where $\tau : [0, \pi/4] \to [0, \pi/4]$ is a measurable function, $\tau(t) \le t$ for a.e. $t \in [0, \pi/4]$, $q_1, q_2 \in L([0, \pi/4]; \mathbb{R})$, and $d_1 \ge 0$, $d_2 \le 0$ are such that

$$d_1|d_2| \le \frac{2^{12}}{4\pi(1 + 2\sqrt{2}) - \pi^2(1 + \sqrt{2}) - 24}.$$

It is clear that (3.16) is a particular case of (1.1) in which $a = 0$, $b = \pi/4$, and p, g are given by formulae (3.17). Analogously to Example 3.1, we get the relation

$$\int_0^{\pi/4} \left| p\big(\varphi(g(1))\big)(s)\, ds \right| = \frac{d_1|d_2|}{2^{12}} \big(4\pi(1 + 2\sqrt{2}) - \pi^2(1 + \sqrt{2}) - 24\big) \le 1.$$

Therefore, according to Corollary 3.3 and Remark 3.2, the problem (3.16), (2.2) has a unique solution for arbitrary $q_1, q_2 \in L([0, \pi/4]; \mathbb{R})$ and $c_1, c_2 \in \mathbb{R}$. Moreover, if q_1, q_2 and c_1, c_2 fulfil the additional condition (3.19), then the unique solution (u, v) of this problem satisfies the relation $u(t) \ge 0$ for $t \in [0, \pi/4]$.

Example 3.4. On the interval $[0, 1]$ we consider the problem (3.20), where $\lambda < 1$, $d \le 0$, $|d| \le (3 - 2\lambda)(2 - \lambda)$, $\tau : [0, 1] \to [0, 1]$ is a measurable function, $\tau(t) \le t$ for a.e. $t \in [0, 1]$, $q \in L([0, 1]; \mathbb{R})$, and $c_1, c_2 \in \mathbb{R}$.

It is clear that (3.20) is a particular case of (1.1), (2.2) in which $a = 0$, $b = 1$, $q_1 \equiv 0$, $q_2 \equiv q$, and p, g are given by formulae (3.21). Analogously to Example 3.2, we get the relation

$$\int_0^1 \left| p\big(\varphi(g(1))\big)(s) \right| ds = \frac{|d|}{(3 - 2\lambda)(2 - \lambda)} \le 1.$$

Therefore, according to Corollary 3.3 and Remark 3.2, the problem (3.20) has a unique solution for arbitrary $q \in L([0,1];\mathbb{R})$ and $c_1, c_2 \in \mathbb{R}$. Moreover, if q and c_1, c_2 fulfil the additional condition (3.22) then the unique solution u of the this problem satisfies the relation $u(t) \geq 0$ for $t \in [0,1]$.

4. Proofs of the main results

Proof of Theorem 3.1. Let the functions $u, v \in \tilde{C}([a,b];\mathbb{R})$ satisfy (3.1) and (3.2). We will show that the function u is nonnegative. Put

$$[u(t)]_- = \frac{1}{2}\big(|u(t)| - u(t)\big) \qquad \text{for } t \in [a,b].$$

According to the inclusion $(p, -g_1) \in \hat{\mathscr{S}}^2_{ab}(a)$ and Remark 3.2, the problem

$$\alpha'(t) = p(\beta)(t), \qquad \beta'(t) = -g_1(\alpha)(t) + g_0([u]_-)(t), \tag{4.1}$$

$$\alpha(a) = 0, \qquad \beta(a) = 0 \tag{4.2}$$

has a unique solution (α, β) and

$$\alpha(t) \geq 0 \qquad \text{for } t \in [a,b]. \tag{4.3}$$

In view (3.1), (3.2), (4.1), (4.2) and the assumption $g_0 \in \mathscr{P}_{ab}$, we get

$$\begin{aligned} \alpha'(t) + u'(t) &\geq p(\beta + v)(t) \qquad \text{for a.e. } t \in [a,b], \\ \beta'(t) + v'(t) &\geq -g_1(\alpha + u)(t) + g_0(u + [u]_-)(t) \\ &\geq -g_1(\alpha + u)(t) \qquad \text{for a.e. } t \in [a,b], \end{aligned}$$

and

$$\alpha(a) + u(a) \geq 0, \qquad \beta(a) + v(a) \geq 0.$$

Consequently, the inclusion $(p, -g_1) \in \hat{\mathscr{S}}^2_{ab}(a)$ yields

$$\alpha(t) + u(t) \geq 0 \qquad \text{for } t \in [a,b]. \tag{4.4}$$

Now (4.3) and (4.4) imply

$$[u(t)]_- \leq \alpha(t) \qquad \text{for } t \in [a,b]. \tag{4.5}$$

On the other hand, by virtue of (4.1), (4.3), (4.5), and the assumptions $g_0, g_1 \in \mathscr{P}_{ab}$, we obtain that

$$\alpha'(t) = p(\beta)(t), \qquad \beta'(t) \leq g_0(\alpha)(t) \qquad \text{for a.e. } t \in [a,b].$$

Hence the inclusion $(p, g_0) \in \hat{\mathscr{S}}^2_{ab}(a)$, in view of (4.2), implies that

$$\alpha(t) \le 0 \qquad \text{for } t \in [a, b],$$

which, together with (4.4), guarantees (3.5). □

Proof of Corollary 3.2. According to (3.14) and the assumption $p \in \mathscr{P}_{ab}$, there exists $\varepsilon > 0$ such that

$$\varepsilon\Big(1 + \int_a^{b_g^*} p(1)(s)\,ds\Big) + \int_a^{b_g^*} p\big(\varphi\big(g(1)\big)\big)(s)\,ds \le 1. \tag{4.6}$$

Put

$$\gamma_2(t) = \varepsilon + \int_a^t g(1)(s)\,ds \qquad \text{for } t \in [a, b], \tag{4.7}$$

$$\gamma_1(t) = \varepsilon + \int_a^t p(\gamma_2)(s)\,ds \qquad \text{for } t \in [a, b]. \tag{4.8}$$

It is clear that $\gamma_1, \gamma_2 \in \tilde{C}([a, b]; \mathbb{R})$ satisfy (3.7) because the operators p and g are supposed to be nondecreasing. Put

$$\tilde{\gamma}_1(t) = \begin{cases} \gamma_1(t) & \text{for } t \in [a, b_g^*[. \\ \gamma_1(b_g^*) & \text{for } t \in [b_g^*, b]. \end{cases} \tag{4.9}$$

Then (4.6)–(4.8) yield

$$\begin{aligned} \tilde{\gamma}_1(t) \le \gamma_1(b_g^*) &= \varepsilon + \int_a^{b_g^*} p\big(\varepsilon + \varphi\big(g(1)\big)\big)(s)\,ds \\ &= \varepsilon\Big(1 + \int_a^{b_g^*} p(1)(s)\,ds\Big) + \int_a^{b_g^*} p\big(\varphi\big(g(1)\big)\big)(s)\,ds \le 1 \end{aligned} \tag{4.10}$$

for $t \in [a, b]$. On the other hand, in view of relations (4.9), (4.10), the assumption $g \in \mathscr{P}_{ab}$, and Remark 3.4, it follows from (4.7) and (4.8) that

$$\gamma_1'(t) = p(\gamma_2)(t), \qquad \gamma_2'(t) = g(1)(t) \ge g(\tilde{\gamma}_1)(t) = g(\gamma_1)(t) \qquad \text{for a.e. } t \in [a, b],$$

i.e., the inequalities (3.8) are fulfilled. Consequently, using Theorem 3.2, we get $(p, g) \in \hat{\mathscr{S}}^2_{ab}(a)$. □

Proof of Proposition 3.1. Suppose that (3.15) holds and the problem (1.1_0), (2.2_0) has only the trivial solution. We will show that $(p, g) \in \hat{\mathscr{S}}^2_{ab}(a)$. According to Proposition 2.1, the problem

$$\gamma_1'(t) = p(\gamma_2)(t), \qquad \gamma_2'(t) = g(\gamma_1)(t), \tag{4.11}$$

$$\gamma_1(a) = 1, \qquad \gamma_2(a) = 1 \tag{4.12}$$

has a unique solution (γ_1, γ_2). Put

$$m = \min\{\gamma_1(t) \mid t \in [a, b_g^*]\} \tag{4.13}$$

and choose $t_m \in [a, b_g^*]$ such that $\gamma_1(t_m) = m$.

Assume that

$$m \le 0. \tag{4.14}$$

By virtue of (4.13) and the assumption $g \in \mathscr{P}_{ab}$, the relations (4.11) and (4.12) yield

$$\gamma_2(t) = 1 + \int_a^t g(\gamma_1)(s)\,ds \ge m \int_a^t g(1)(s)\,ds = m\varphi\big(g(1)\big)(t) \qquad \text{for } t \in [a, b].$$

Consequently, in view of (4.14) and the assumption $p \in \mathscr{P}_{ab}$, the relations (4.11) and (4.12) imply

$$m = 1 + \int_a^{t_m} p(\gamma_2)(s)\,ds \ge 1 + m \int_a^{t_m} p\big(\varphi\big(g(1)\big)\big)(s)\,ds \ge 1 + m \int_a^{b_g^*} p\big(\varphi\big(g(1)\big)\big)(s)\,ds.$$

Using (3.15) in the last relation, we get the contradiction $m \ge m + 1$.

The contradiction obtained proves that $m > 0$, i.e.,

$$\gamma_1(t) > 0 \qquad \text{for } t \in [a, b_g^*]. \tag{4.15}$$

Now we define the function $\tilde{\gamma}_1$ by (4.9). Obviously, $\tilde{\gamma}_1(t) > 0$ for $t \in [a, b]$ and therefore, by virtue of the assumption $g \in \mathscr{P}_{ab}$ and Remark 3.4, (4.11) yields

$$\gamma_2'(t) = g(\gamma_1)(t) = g(\tilde{\gamma}_1)(t) \ge 0 \qquad \text{for a.e. } t \in [a, b].$$

Since $\gamma_2(a) > 0$, the last relation yields that $\gamma_2(t) > 0$ for $t \in [a, b]$. Now (4.11) implies that

$$\gamma_1'(t) = p(\gamma_2)(t) \ge 0 \qquad \text{for a.e. } t \in [a, b],$$

which, together with (4.15), gives that $\gamma_1(t) > 0$ for $t \in [a, b]$. Consequently, Theorem 3.2 guarantees $(p, g) \in \hat{\mathscr{S}}_{ab}^2(a)$.

Now suppose that $(p, g) \in \hat{\mathscr{S}}_{ab}^2(a)$. If (u, v) is a solution of the homogeneous problem (1.1_0), (2.2_0), then the inclusion $(p, g) \in \hat{\mathscr{S}}_{ab}^2(a)$ yields that $u \equiv 0$. Consequently, $v \equiv 0$ as well, and thus the problem (1.1_0), (2.2_0) has only the trivial solution. □

To prove Theorem 3.3 we need the following lemma.

Lemma 4.1. *Let* $-g, p \in \mathcal{P}_{ab}$ *and let* p, g *be* a*-Volterra operators. Assume that there exist functions* $\gamma_1, \gamma_2 \in \tilde{C}_{\mathrm{loc}}([a,b[;\mathbb{R})$ *satisfying* $\gamma_1 \in C([a,b];\mathbb{R})$ *and* (3.23)–(3.26). *Then, for any* $u, v \in \tilde{C}([a,b];\mathbb{R})$ *fulfilling* (3.1) *and* (3.2), *the condition*

$$u(t) \geq 0 \quad \text{for } t \in [a, b_0] \tag{4.16}$$

holds, where

$$b_0 = \sup\{x \in \,]a,b] \mid \gamma_1(t) > 0 \text{ for } t \in [a,x]\}. \tag{4.17}$$

Proof. Let the functions $u, v \in \tilde{C}([a,b];\mathbb{R})$ satisfy (3.1) and (3.2). Define the number b_0 by (4.17). It is clear that $b_0 > a$ and

$$\gamma_1(t) > 0 \quad \text{for } t \in [a, b_0[. \tag{4.18}$$

Assume that, on the contrary, the relation (4.16) is not true. Then there exists $t_0 \in \,]a, b_0[$ such that

$$u(t_0) < 0. \tag{4.19}$$

Put

$$\lambda = \max\left\{ \frac{u(t)}{\gamma_1(t)} \,\middle|\, t \in [a, t_0] \right\}. \tag{4.20}$$

It is clear that

$$0 \leq \lambda < +\infty. \tag{4.21}$$

Define the functions w_1 and w_2 by setting

$$w_1(t) = \lambda\gamma_1(t) - u(t), \qquad w_2(t) = \lambda\gamma_2(t) - v(t) \quad \text{for } t \in [a, t_0]. \tag{4.22}$$

Since p, g are a-Volterra operators, using (3.1), (3.23), (3.24), (4.21), and Remark 3.5, we get

$$w_1'(t) = \lambda\gamma_1'(t) - u'(t) \leq p_{at_0}(\lambda\gamma_2 - v)(t) = p_{at_0}(w_2)(t) \quad \text{for a.e. } t \in [a, t_0] \tag{4.23}$$

and

$$w_2'(t) = \lambda\gamma_2'(t) - v'(t) \leq g_{at_0}(\lambda\gamma_1 - u)(t) = g_{at_0}(w_1)(t) \quad \text{for a.e. } t \in [a, t_0], \tag{4.24}$$

where p_{at_0}, g_{at_0} are the restrictions of the operators p, g to the space $C([a, t_0];\mathbb{R})$.

On the other hand, by virtue of (4.20), it is clear that

$$w_1(t) \geq 0 \qquad \text{for } t \in [a, t_0] \tag{4.25}$$

and there exists $t_1 \in [a, t_0[$ such that

$$w_1(t_1) = 0. \tag{4.26}$$

Since we suppose that $-g \in \mathscr{P}_{ab}$, we get from (3.2), (3.26), (4.21), (4.22), (4.24), and (4.25) that

$$w_2'(t) \leq g_{at_0}(w_1)(t) \leq 0 \qquad \text{for a.e. } t \in [a, t_0], \qquad w_2(a) = \lambda\gamma_2(a) - v(a) \leq 0.$$

Hence we obtain

$$w_2(t) \leq 0 \qquad \text{for } t \in [a, t_0]. \tag{4.27}$$

However, we suppose that $p \in \mathscr{P}_{ab}$ and thus we get from (4.23) and (4.27) that

$$w_1'(t) \leq p_{at_0}(w_2)(t) \leq 0 \qquad \text{for a.e. } t \in [a, t_0]. \tag{4.28}$$

Finally, by virtue of (3.25), (4.21), and (4.26), the relation (4.28) yields that

$$0 = w_1(t_1) \geq w_1(t_0) = \lambda\gamma_1(t_0) - u(t_0) \geq -u(t_0),$$

which contradicts (4.19).

The contradiction obtained proves the relation (4.16). □

Proof of Theorem 3.3. First suppose that $(p, g) \in \hat{\mathscr{S}}_{ab}^2(a)$. According to Remark 3.2, the system (4.11) has a unique solution (γ_1, γ_2) satisfying the initial conditions

$$\gamma_1(a) = 1, \qquad \gamma_2(a) = 0, \tag{4.29}$$

and, moreover, the relation

$$\gamma_1(t) \geq 0 \qquad \text{for } t \in [a, b] \tag{4.30}$$

holds. It is clear that $\gamma_1, \gamma_2 \in \tilde{C}([a, b]; \mathbb{R})$ satisfy (3.23)–(3.26). We will show that (3.27) holds. Assume that, on the contrary, the relation (3.27) is not satisfied. Then there exists $t_0 \in]a, b[$ such that $\gamma_2(t_0) = 0$ and

$$\gamma_1(t_0) = 0. \tag{4.31}$$

By virtue of (4.30) and the assumption $-g \in \mathscr{P}_{ab}$, the conditions (3.24), (4.29), and $\gamma_2(t_0) = 0$ imply $\gamma_2(t) = 0$ for $t \in [a, t_0]$. Since we suppose that p is an a-Volterra operator, (4.11) implies that

$$\gamma_1'(t) = 0 \qquad \text{for a.e. } t \in [a, t_0].$$

This relation, together with the condition $\gamma_1(a) = 1$, yields that $\gamma_1(t_0) = 1$, which contradicts (4.31). This proves the relation (3.27).

Now suppose that there exist functions $\gamma_1, \gamma_2 \in \tilde{C}_{\text{loc}}([a,b[;\mathbb{R})$ satisfying $\gamma_1 \in C([a,b];\mathbb{R})$ and (3.23)–(3.27). We will show that $(p,g) \in \hat{\mathscr{S}}^2_{ab}(a)$. Let the functions $u, v \in \tilde{C}([a,b];\mathbb{R})$ satisfy (3.1) and (3.2). By virtue of Lemma 4.1, the relation (4.16) is true, where the number b_0 is defined by (4.17).

If $b_0 = b$ then the proof is complete. Assume that $b_0 < b$ and let $b_1 \in \,]b_0, b[$ be arbitrary but fixed. We will show that

$$u(t) \geq 0 \qquad \text{for } t \in [a, b_1]. \tag{4.32}$$

It follows from (3.25) and (4.17) that (4.18) holds and

$$\gamma_1(t) = 0 \qquad \text{for } t \in [b_0, b]. \tag{4.33}$$

Consequently, by virtue of the assumptions (3.27) and (3.29), there exist $a_0 \in \,]a, b_0[$ and $\lambda_1 \in \mathbb{R}_+$ such that

$$v(t) \geq \lambda_1 \gamma_2(t) \qquad \text{for } t \in [a_0, b_1]. \tag{4.34}$$

On the other hand, in view of (4.18), there exist $\lambda_2 \in \mathbb{R}_+$ such that

$$u(t) \leq \lambda_2 \gamma_1(t) \qquad \text{for } t \in [a, a_0]. \tag{4.35}$$

Since the nonincreasing operator g is an a-Volterra one, using (3.1), (3.24), and (4.35), we get

$$v'(t) - \lambda_2 \gamma_2'(t) \geq g_{aa_0}(u - \lambda_2 \gamma_1)(t) \geq 0 \qquad \text{for a.e. } t \in [a, a_0]. \tag{4.36}$$

However, the functions v and γ_2 satisfy (3.2) and (3.26), and thus (4.36) yields that

$$v(t) \geq \lambda_2 \gamma_2(t) \qquad \text{for } t \in [a, a_0].$$

Therefore, if we put $\lambda = \max\{\lambda_1, \lambda_2\}$ then, in view of (3.29), we get

$$v(t) \geq \lambda \gamma_2(t) \qquad \text{for } t \in [a, b_1]. \tag{4.37}$$

Since we suppose that p is a nondecreasing a-Volterra operator, the inequalities (3.23) and (3.29) imply that

$$\gamma_1'(t) \le p_{ab_1}(\gamma_2)(t) \le 0 \qquad \text{for a.e. } t \in [a, b_1],$$

where p_{ab_1} is the restriction of the operator p to the space $C([a, b_1]; \mathbb{R})$. The function γ_1 vanishes on the interval $[b_0, b]$, and thus we have

$$p_{ab_1}(\gamma_2)(t) = 0 \qquad \text{for a.e. } t \in [b_0, b_1]. \tag{4.38}$$

Now (4.37) and (4.38) imply that

$$p_{ab_1}(v)(t) \ge \lambda p_{ab_1}(\gamma_2)(t) = 0 \qquad \text{for a.e. } t \in [b_0, b_1],$$

which, together with (3.1) and (4.16), results in (4.32). Since the point b_1 was chosen arbitrarily, we have proved that the function u is nonnegative on $[a, b]$. Consequently, $(p, g) \in \hat{\mathscr{S}}^2_{ab}(a)$. □

Proof of Corollary 3.3. Put

$$\gamma_1(t) = 1 - \int_a^t \big| p\big(\varphi\big(g(1)\big)\big)(s)\big| \, ds \qquad \text{for } t \in [a, b], \tag{4.39}$$

$$\gamma_2(t) = \int_a^t g(1)(s) \, ds \qquad \text{for } t \in [a, b]. \tag{4.40}$$

In view of (3.30), it is clear that the conditions (3.25) and (3.26) are satisfied and

$$\gamma_1(t) \le 1 \qquad \text{for } t \in [a, b]. \tag{4.41}$$

Since we suppose $-g, p \in \mathscr{P}_{ab}$, we get from (4.39)–(4.41) that

$$\gamma_1'(t) = p\big(\varphi\big(g(1)\big)\big)(t) = p(\gamma_2)(t) \qquad \text{for a.e. } t \in [a, b]$$

and

$$\gamma_2'(t) = g(1)(t) \le g(\gamma_1)(t) \qquad \text{for a.e. } t \in [a, b],$$

i.e., the functions γ_1, γ_2 satisfy (3.23) and (3.24).

We will show that the condition (3.27) holds. Assume that, on the contrary, the relation (3.27) is not true. Then there exists $t_0 \in \,]a, b[$ such that $\gamma_2(t_0) = 0$ and

$$\gamma_1(t_0) = 0. \tag{4.42}$$

Therefore, (4.40) yields that

$$g(1)(t) = 0 \qquad \text{for a.e. } t \in [a, t_0].$$

Supposing that p is an a-Volterra operator, the last relation gives

$$p\big(\varphi\big(g(1)\big)\big)(t) = 0 \qquad \text{for a.e. } t \in [a, t_0].$$

Hence (4.39) implies $\gamma_1(t_0) = 1$, which contradicts (4.42). The contradiction obtained proves the condition (3.27).

Consequently, using Theorem 3.3, we get $(p, g) \in \hat{\mathscr{S}}^2_{ab}(a)$. □

5. Corollaries for operators with argument deviations

In this section we give efficient conditions under which the weak theorem on differential inequalities holds for the system with argument deviations (1.3). More precisely, some corollaries of the main results are established for the case where the operators $p, g \in \mathscr{L}_{ab}$ are defined by the formulae

$$p(z)(t) \stackrel{\text{def}}{=} f(t)z\big(\mu(t)\big) \qquad \text{for a.e. } t \in [a, b],\ z \in C([a, b]; \mathbb{R}), \tag{5.1}$$

$$g(z)(t) \stackrel{\text{def}}{=} h_0(t)z\big(\tau_0(t)\big) \qquad \text{for a.e. } t \in [a, b],\ z \in C([a, b]; \mathbb{R}), \tag{5.2}$$

$$g(z)(t) \stackrel{\text{def}}{=} -h_1(t)z\big(\tau_1(t)\big) \qquad \text{for a.e. } t \in [a, b],\ z \in C([a, b]; \mathbb{R}), \tag{5.3}$$

and

$$g(z)(t) \stackrel{\text{def}}{=} h_0(t)z\big(\tau_0(t)\big) - h_1(t)z\big(\tau_1(t)\big) \qquad \text{for a.e. } t \in [a, b],\ z \in C([a, b]; \mathbb{R}), \tag{5.4}$$

where $f, h_0, h_1 \in L([a, b]; \mathbb{R}_+)$ and $\mu, \tau_0, \tau_1 : [a, b] \to [a, b]$ are measurable functions.

Throughout this section, the following notation is used:

$$\mu^* = \text{ess sup}\{\mu(t) \mid t \in [a, b]\}, \qquad \tau_0^* = \text{ess sup}\{\tau_0(t) \mid t \in [a, b]\}, \tag{5.5}$$

and

$$b^* = \max\{\mu^*, \tau_0^*\}.$$

All the statements are formulated in Section 5.1, the proofs are given later in Section 5.2.

5.1. Formulation of the results. The next statement can be derived from Theorem 3.2.

Theorem 5.1 ([24], Theorem 4.2). *Let* p *and* g *be the operators defined by* (5.1) *and* (5.2), *respectively. Put*

$$\omega(t) \stackrel{\text{def}}{=} \max\{f(t), h_0(t)\} \qquad \text{for a.e. } t \in [a,b], \tag{5.6}$$

and assume that $\omega \not\equiv 0$ *on* $[a, b^*]$,

$$\operatorname{ess\,sup}\Big\{\int_t^{\mu(t)} \omega(s)\,ds \,|\, t \in [a,b]\Big\} < \eta^*,$$

and

$$\operatorname{ess\,sup}\Big\{\int_t^{\tau_0(t)} \omega(s)\,ds \,|\, t \in [a,b]\Big\} < \eta^*,$$

where

$$\eta^* = \sup\left\{\frac{1}{x}\ln\left(x + \frac{x}{\exp\big(x\int_a^{b^*}\omega(s)\,ds\big) - 1}\right)\Bigg|\, x > 0\right\}.$$

Then the pair (p, g) *belongs to the set* $\hat{\mathscr{S}}^2_{ab}(a)$.

Corollary 3.1 yields

Theorem 5.2 ([24], Corollary 4.5). *Let* p *and* g *be the operators defined by* (5.1) *and* (5.2), *respectively. Assume that the condition* (3.10) *is satisfied, where*

$$\begin{aligned}\lambda_1 = &\int_a^b \cosh\Big(\int_s^b \omega(\xi)\,d\xi\Big) f(s)\sigma_1(s)\Big(\int_s^{\mu(s)} h_0(\xi)\,d\xi\Big)\,ds \\ &+ \int_a^b \sinh\Big(\int_s^b \omega(\xi)\,d\xi\Big) h_0(s)\sigma_2(s)\Big(\int_s^{\tau_0(s)} f(\xi)\,d\xi\Big)\,ds, \end{aligned}\tag{5.7}$$

$$\begin{aligned}\lambda_2 = &\int_a^b \cosh\Big(\int_s^b \omega(\xi)\,d\xi\Big) h_0(s)\sigma_2(s)\Big(\int_s^{\tau_0(s)} f(\xi)\,d\xi\Big)\,ds \\ &+ \int_a^b \sinh\Big(\int_s^b \omega(\xi)\,d\xi\Big) f(s)\sigma_1(s)\Big(\int_s^{\mu(s)} h_0(\xi)\,d\xi\Big)\,ds, \end{aligned}\tag{5.8}$$

the function ω *is defined by* (5.6), *and*

$$\sigma_1(t) = \frac{1}{2}\big(1 + \operatorname{sgn}\big(\mu(t) - t\big)\big) \qquad \textit{for a.e. } t \in [a,b], \tag{5.9}$$

$$\sigma_2(t) = \frac{1}{2}\big(1 + \operatorname{sgn}\big(\tau_0(t) - t\big)\big) \qquad \textit{for a.e. } t \in [a,b]. \tag{5.10}$$

Then the pair (p, g) *belongs to the set* $\hat{\mathcal{S}}^2_{ab}(a)$.

Remark 5.1. The strict inequality (3.10) in the previous corollary cannot be replaced by the nonstrict one (see [24], Example 5.3).

For the operator g given by (5.2), according to Notation 3.1, we have $b_g^* \le \tau_0^*$. It is however easy to see that the equality $b_g^* = \tau_0^*$ is not true in general. On the other hand, it is clear that the number τ_0^* is easier to compute than b_g^*. Therefore, the results obtained below by using Corollary 3.2 and Proposition 3.1 are formulated in terms of the number τ_0^* instead of b_g^*.

Theorem 5.3. *Let* p *and* g *be the operators defined by* (5.1) *and* (5.2), *respectively. If*

$$\int_a^{\tau_0^*} f(s)\Big(\int_a^{\mu(s)} h_0(\xi)\, d\xi\Big)\, ds < 1, \tag{5.11}$$

then the pair (p, g) *belongs to the set* $\hat{\mathcal{S}}^2_{ab}(a)$.

The next theorem can be regarded as a complement of the previous one.

Theorem 5.4. *Let* p *and* g *be the operators defined by* (5.1) *and* (5.2), *respectively, and let*

$$\int_a^{\tau_0^*} f(s)\Big(\int_a^{\mu(s)} h_0(\xi)\, d\xi\Big)\, ds = 1. \tag{5.12}$$

Then $(p, g) \in \hat{\mathcal{S}}^2_{ab}(a)$ *if and only if*

$$\int_a^{\tau_0^*} f(s)\Big(\int_a^{\mu(s)} h_0(\xi) x\big(\tau_0(\xi)\big)\, d\xi\Big)\, ds < 1, \tag{5.13}$$

where

$$x(t) = \int_a^{t} f(s)\Big(\int_a^{\mu(s)} h_0(\xi)\, d\xi\Big)\, ds \qquad \textit{for } t \in [a,b]. \tag{5.14}$$

In view of Remark 3.1(d), Theorems 5.3 and 5.4 immediately yield

Theorem 5.3′. *Let p and g be the operators defined by* (5.1) *and* (5.2), *respectively. If*

$$\int_a^{\mu^*} h_0(s)\Big(\int_a^{\tau_0(s)} f(\xi)\,d\xi\Big)\,ds < 1,$$

then the pair (p, g) belongs to the set $\hat{\mathscr{S}}^2_{ab}(a)$.

Theorem 5.4′. *Let p and g be the operators defined by* (5.1) *and* (5.2), *respectively, and let*

$$\int_a^{\mu^*} h_0(s)\Big(\int_a^{\tau_0(s)} f(\xi)\,d\xi\Big)\,ds = 1.$$

Then $(p, g) \in \hat{\mathscr{S}}^2_{ab}(a)$ if and only if

$$\int_a^{\mu^*} h_0(s)\Big(\int_a^{\tau_0(s)} f(\xi)y\big(\mu(\xi)\big)\,d\xi\Big)\,ds < 1,$$

where

$$y(t) = \int_a^t h_0(s)\Big(\int_a^{\tau_0(s)} f(\xi)\,d\xi\Big)\,ds \quad \text{for } t \in [a, b].$$

The next statement follows from Corollary 3.3.

Theorem 5.5. *Let p and g be the operators defined by* (5.1) *and* (5.3), *respectively. If*

$$f(t)\big(\mu(t) - t\big) \le 0, \quad h_1(t)\big(\tau_1(t) - t\big) \le 0 \quad \text{for a.e. } t \in [a, b], \tag{5.15}$$

and

$$\int_a^b f(s)\Big(\int_a^{\mu(s)} h_1(\xi)\,d\xi\Big)\,ds \le 1, \tag{5.16}$$

then the pair (p, g) belongs to the set $\hat{\mathscr{S}}^2_{ab}(a)$.

Remark 5.2. The inequality (5.16) in the previous theorem cannot be replaced by the inequality

$$\int_a^b f(s)\Big(\int_a^{\mu(s)} h_1(\xi)\,d\xi\Big)\,ds \le 1 + \varepsilon, \tag{5.17}$$

no matter how small $\varepsilon > 0$ would be (see Example 6.1).

The next statement contains the so-called Vallée-Poussin type conditions.

Theorem 5.6. *Let p and g be the operators defined by* (5.1) *and* (5.3), *respectively, and let the condition* (5.15) *hold. Assume that there exist numbers $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $\alpha_3 > 0$, $\lambda \in [0, 1[$, and $\nu \in [0, \lambda]$ such that*

$$\int_0^{+\infty} \frac{ds}{\alpha_1 + \alpha_2 s + \alpha_3 s^2} \geq \frac{(b-a)^{1-\lambda}}{1-\lambda}, \tag{5.18}$$

$$(b-t)^{\lambda-\nu} f(t) \leq \alpha_3 \left[1 + \sigma_3(t) \int_{\mu(t)}^{t} \left(\frac{\nu}{b-s} + \frac{\alpha_2}{(b-s)^{\lambda}} \right) ds \right] \quad \text{for a.e. } t \in [a,b], \tag{5.19}$$

$$(b-t)^{\lambda+\nu} h_1(t) \leq \alpha_1 \quad \text{for a.e. } t \in [a,b], \tag{5.20}$$

and

$$\alpha_3 (b-t)^{\nu} h_1(t) \big(t - \tau_1(t) \big) \leq \alpha_2 + \frac{\nu}{(b-t)^{1-\lambda}} \quad \text{for a.e. } t \in [a,b], \tag{5.21}$$

where

$$\sigma_3(t) = \frac{1}{2} \big(1 + \operatorname{sgn}\big(t - \mu(t) \big) \big) \quad \text{for a.e. } t \in [a,b]. \tag{5.22}$$

Then the pair (p, g) belongs to the set $\hat{\mathscr{S}}^2_{ab}(a)$.

Remark 5.3. The inequality (5.18) in the previous theorem cannot be replaced by the inequality

$$\int_0^{+\infty} \frac{ds}{\alpha_1 + \alpha_2 s + \alpha_3 s^2} \geq (1-\varepsilon) \frac{(b-a)^{1-\lambda}}{1-\lambda}, \tag{5.23}$$

no matter how small $\varepsilon > 0$ would be (see Example 6.2).

Theorem 5.7. *Let p and g be the operators defined by* (5.1) *and* (5.4), *respectively. Assume that the functions f, μ, h_0, τ_0 satisfy the assumptions of one of Theorems* 5.1–5.4, *whereas the functions f, μ, h_1, τ_1 satisfy the assumptions of Theorem* 5.5 *or* 5.6. *Then the pair (p, g) belongs to the set $\hat{\mathscr{S}}^2_{ab}(a)$.*

5.2. Proofs. Now we prove the statements formulated above.

Proof of Theorem 5.3. It is clear that $p, g \in \mathscr{P}_{ab}$. According to Notation 3.1, we get $b_g^* \leq \tau_0^*$. Therefore, the validity of the theorem follows immediately from Corollary 3.2. □

For the sake of clarity, we give the following lemma before we prove Theorem 5.4.

Lemma 5.1. *Let* $f, h_0 \in L([a,b];\mathbb{R}_+)$ *and* $\mu, \tau_0 : [a,b] \to [a,b]$ *be measurable functions such that the condition* (5.12) *holds. Then the homogeneous problem*

$$u'(t) = f(t)v\big(\mu(t)\big), \qquad v'(t) = h_0(t)u\big(\tau_0(t)\big), \tag{5.24}$$

$$u(a) = 0, \qquad v(a) = 0 \tag{5.25}$$

has only the trivial solution if and only if the inequality (5.13) *is satisfied, where the function* x *is defined by* (5.14).

Proof. Let (u, v) be a solution of the problem (5.24), (5.25). We first show that the function u does not change its sign on the interval $[a, \tau_0^*]$. Assume that, on the contrary, u changes its sign on $[a, \tau_0^*]$. Put

$$M = \max\{u(t) \mid t \in [a, \tau_0^*]\}, \qquad m = -\min\{u(t) \mid t \in [a, \tau_0^*]\}, \tag{5.26}$$

and choose $t_M, t_m \in [a, \tau_0^*]$ such that

$$u(t_M) = M, \qquad u(t_m) = -m. \tag{5.27}$$

Obviously,

$$M > 0, \qquad m > 0, \tag{5.28}$$

and we can assume without loss of generality that $t_m < t_M$. By virtue of (5.26), it follows from (5.25) and the second equation in (5.24) that

$$v(t) = \int_a^t h_0(s)u\big(\tau_0(s)\big)\, ds \le M \int_a^t h_0(s)\, ds \qquad \text{for } t \in [a,b]. \tag{5.29}$$

Therefore, the integration of the first equation in (5.24) from t_m to t_M, in view of (5.12), (5.27), and (5.29), yields that

$$M + m = \int_{t_m}^{t_M} f(s)v\big(\mu(s)\big)\, ds \le M \int_{t_m}^{t_M} f(s)\bigg(\int_a^{\mu(s)} h_0(\xi)\, d\xi\bigg)\, ds \le M,$$

which contradicts (5.28).

The contradiction obtained proves that the function u does not change its sign on the interval $[a, \tau_0^*]$. Therefore, we can assume without loss of generality that

$$u(t) \ge 0 \qquad \text{for } t \in [a, \tau_0^*]. \tag{5.30}$$

It follows from (5.24) and (5.25) that

$$u(t) = \int_a^t f(s)\Big(\int_a^{\mu(s)} h_0(\xi)u\big(\tau_0(\xi)\big)\, d\xi\Big)\, ds \qquad \text{for } t \in [a,b]. \tag{5.31}$$

Since $\tau_0(t) \leq \tau_0^*$ for a.a. $t \in [a,b]$ and the function u is nonnegative on $[a,\tau_0^*]$, the last relation implies that

$$u\big(\tau_0(t)\big) \leq u(\tau_0^*) \qquad \text{for a.e. } t \in [a,b]. \tag{5.32}$$

Therefore, in view of (5.14), it follows from the representation (5.31) that

$$u(t) \leq u(\tau_0^*)\int_a^t f(s)\Big(\int_a^{\mu(s)} h_0(\xi)\, d\xi\Big)\, ds = u(\tau_0^*)x(t) \qquad \text{for } t \in [a,b], \tag{5.33}$$

and

$$\begin{aligned} u(\tau_0^*) - u(t) &= \int_t^{\tau_0^*} f(s)\Big(\int_a^{\mu(s)} h_0(\xi)u\big(\tau_0(\xi)\big)\, d\xi\Big)\, ds \\ &\leq u(\tau_0^*)\int_t^{\tau_0^*} f(s)\Big(\int_a^{\mu(s)} h_0(\xi)\, d\xi\Big)\, ds \qquad \text{for } t \in [a,\tau_0^*]. \end{aligned} \tag{5.34}$$

Using (5.12) and (5.14) in the relation (5.34), we obtain

$$u(\tau_0^*)x(t) = u(\tau_0^*)\Big(1 - \int_t^{\tau_0^*} f(s)\Big(\int_a^{\mu(s)} h_0(\xi)\, d\xi\Big)\, ds\Big) \leq u(t) \qquad \text{for } t \in [a,\tau_0^*],$$

which, together with (5.33), gives

$$u(t) = u(\tau_0^*)x(t) \qquad \text{for } t \in [a,\tau_0^*]. \tag{5.35}$$

Finally, (5.31) and (5.35) result in

$$u(t) = u(\tau_0^*)\int_a^t f(s)\Big(\int_a^{\mu(s)} h_0(\xi)x\big(\tau_0(\xi)\big)\, d\xi\Big)\, ds \qquad \text{for } t \in [a,b], \tag{5.36}$$

whence we obtain

$$u(\tau_0^*)\Big[1 - \int_a^{\tau_0^*} f(s)\Big(\int_a^{\mu(s)} h_0(\xi)x\big(\tau_0(\xi)\big)\, d\xi\Big)\, ds\Big] = 0. \tag{5.37}$$

We have proved that every solution (u,v) of the problem (5.24), (5.25) satisfies (5.36), where $u(\tau_0^*)$ fulfils (5.37). Consequently, if (5.13) holds, then the homogeneous problem (5.24), (5.25) has only the trivial solution.

It remains to show that if (5.13) is not satisfied, i.e.,

$$\int_a^{\tau_0^*} f(s)\Big(\int_a^{\mu(s)} h_0(\xi)x\big(\tau_0(\xi)\big)\,d\xi\Big)\,ds = 1, \tag{5.38}$$

then the homogeneous problem (5.24), (5.25) has a nontrivial solution. Indeed, in view of (5.12), from (5.14) we obtain that

$$x\big(\tau_0(t)\big) \le x(\tau_0^*) = 1 \qquad \text{for a.e. } t \in [a,b].$$

Therefore, using (5.38), it is easy to verify that

$$\begin{aligned}
0 &\le \int_a^{t} f(s)\Big(\int_a^{\mu(s)} h_0(\xi)\big[1 - x\big(\tau_0(\xi)\big)\big]\,d\xi\Big)\,ds \\
&\le \int_a^{\tau_0^*} f(s)\Big(\int_a^{\mu(s)} h_0(\xi)\big[1 - x\big(\tau_0(\xi)\big)\big]\,d\xi\Big)\,ds \\
&= 1 - \int_a^{\tau_0^*} f(s)\Big(\int_a^{\mu(s)} h_0(\xi)x\big(\tau_0(\xi)\big)\,d\xi\Big)\,ds = 0 \qquad \text{for } t \in [a,\tau_0^*].
\end{aligned}$$

Hence we get

$$x(t) = \int_a^{t} f(s)\Big(\int_a^{\mu(s)} h_0(\xi)x\big(\tau_0(\xi)\big)\,d\xi\Big)\,ds \qquad \text{for } t \in [a,\tau_0^*]. \tag{5.39}$$

Put

$$v(t) = \int_a^t h_0(s)x\big(\tau_0(s)\big)\,ds, \qquad u(t) = \int_a^t f(s)v\big(\mu(s)\big)\,ds \qquad \text{for } t \in [a,b].$$

By virtue of (5.39), it is clear that $u(t) = x(t)$ for $t \in [a,\tau_0^*]$, and thus

$$v(t) = \int_a^t h_0(s)u\big(\tau_0(s)\big)\,ds \qquad \text{for } t \in [a,b].$$

Consequently, (u,v) is a nontrivial solution of the problem (5.24), (5.25). □

Now we can prove Theorem 5.4.

Proof of Theorem 5.4. It is clear that $p, g \in \mathcal{P}_{ab}$.

First suppose that $(p,g) \in \hat{\mathcal{S}}^2_{ab}(a)$. In view of Remark 3.2, the homogeneous problem (5.24), (5.25) has only the trivial solution. Thus Lemma 5.1 guarantees that the inequality (5.13) is satisfied.

Now suppose that the inequality (5.13) is fulfilled. According to Notation 3.1, we have $b_g^* \le \tau_0^*$. If

$$\int_a^{b_g^*} f(s)\Big(\int_a^{\mu(s)} h_0(\xi)\,d\xi\Big)\,ds < 1,$$

Corollary 3.2 implies that $(p,g) \in \hat{\mathscr{S}}_{ab}^2(a)$. If

$$\int_a^{b_g^*} f(s)\Big(\int_a^{\mu(s)} h_0(\xi)\,d\xi\Big)\,ds = 1$$

then, by virtue of Lemma 5.1, the homogeneous problem (5.24), (5.25) has only the trivial solution. Consequently, using Proposition 3.1, we get $(p,g) \in \hat{\mathscr{S}}_{ab}^2(a)$ as well. □

Proof of Theorem 5.5. It is clear that $p \in \mathscr{P}_{ab}$ and $-g \in \mathscr{P}_{ab}$. Moreover, (5.15) guarantees that p and g are a-Volterra operators. Therefore, the validity of the theorem follows immediately from Corollary 3.3. □

To prove Theorem 5.6 we need the following lemma.

Lemma 5.2. *Let the numbers* $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $\alpha_3 > 0$, *and* $\lambda \in [0,1[$ *be such that*

$$\int_0^{+\infty} \frac{ds}{\alpha_1 + \alpha_2 s + \alpha_3 s^2} = \frac{(b-a)^{1-\lambda}}{1-\lambda}. \tag{5.40}$$

Then, for any $\nu \in [0,\lambda]$, *there exist* $\gamma_1 \in C([a,b];\mathbb{R})$ *and* $\gamma_2 \in C_{\mathrm{loc}}([a,b[;\mathbb{R})$ *such that* $\gamma_1', \gamma_1'', \gamma_2' \in C_{\mathrm{loc}}([a,b[;\mathbb{R})$,

$$\gamma_1(t) > 0 \quad \text{for } t \in [a,b[, \tag{5.41}$$

$$\gamma_2(a) = 0, \quad \gamma_2(t) < 0 \quad \text{for } t \in]a,b[, \tag{5.42}$$

$$\gamma_1'(t) = \frac{\alpha_3}{(b-t)^{\lambda-\nu}}\gamma_2(t) \quad \text{for } t \in [a,b[, \tag{5.43}$$

$$\gamma_2'(t) = -\frac{\alpha_1}{(b-t)^{\lambda+\nu}}\gamma_1(t) + \left(\frac{\nu}{b-t} + \frac{\alpha_2}{(b-t)^{\lambda}}\right)\gamma_2(t) \quad \text{for } t \in [a,b[, \tag{5.44}$$

and

$$\gamma_1''(t) \le 0 \quad \text{for } t \in [a,b[. \tag{5.45}$$

Proof. Define the function $\varrho : [a,b[\to \mathbb{R}_+$ by setting

$$\int_{\varrho(t)}^{+\infty} \frac{ds}{\alpha_1 + \alpha_2 s + \alpha_3 s^2} = \frac{(b-t)^{1-\lambda}}{1-\lambda} \qquad \text{for } t \in [a,b[.$$

In view of (5.40), we get

$$\varrho(a) = 0, \qquad \varrho(t) > 0 \qquad \text{for } t \in \,]a,b[, \tag{5.46}$$

and

$$\varrho'(t) = \frac{\alpha_1 + \alpha_2 \varrho(t) + \alpha_3 \varrho^2(t)}{(b-t)^{\lambda}} \qquad \text{for } t \in [a,b[.$$

Put

$$\gamma_1(t) = \exp\Big(-\int_a^t \frac{\alpha_3 \varrho(s)}{(b-s)^{\lambda}}\, ds\Big), \qquad \gamma_2(t) = -\frac{\varrho(t)\gamma_1(t)}{(b-t)^{\nu}} \qquad \text{for } t \in [a,b[.$$

It is not difficult to verify that $\gamma_1, \gamma_2 \in C_{\text{loc}}([a,b[;\mathbb{R})$ and the conditions (5.43) and (5.44) are satisfied. Therefore, $\gamma_1', \gamma_2' \in C_{\text{loc}}([a,b[;\mathbb{R})$, as well. Moreover, in view of (5.46), it is clear that (5.41) and (5.42) are fulfilled. Consequently, by direct calculation we can check that $\gamma_1'' \in C_{\text{loc}}([a,b[;\mathbb{R})$ and (5.45) is satisfied. Since the function γ_1 is positive and nonincreasing on $[a,b[$, there exists a finite limit $\lim_{t\to b-} \gamma_1(t)$. Therefore, $\gamma_1 \in C([a,b];\mathbb{R})$ when we put $\gamma_1(b) = \lim_{t\to b-} \gamma_1(t)$. □

Now we are in position to prove Theorem 5.6.

Proof of Theorem 5.6. It is clear that $p \in \mathscr{P}_{ab}$ and $-g \in \mathscr{P}_{ab}$. Moreover, (5.15) guarantees that p and g are a-Volterra operators. According to (5.18) and (5.20), the number α_1 can be increased such that the equality (5.40) is satisfied instead of the inequality (5.18), and the condition (5.20) is still true. Then, by virtue of Lemma 5.2, there exist functions $\gamma_1 \in C([a,b];\mathbb{R})$ and $\gamma_2 \in C_{\text{loc}}([a,b[;\mathbb{R})$ such that $\gamma_1', \gamma_1'', \gamma_2' \in C_{\text{loc}}([a,b[;\mathbb{R})$, and the conditions (5.41)–(5.45) are satisfied. Obviously, $\gamma_1, \gamma_2 \in \tilde{C}_{\text{loc}}([a,b[;\mathbb{R})$. Using (5.41)–(5.44), we get

$$\gamma_1'(t) \le 0, \qquad \gamma_2'(t) \le 0 \qquad \text{for } t \in [a,b[. \tag{5.47}$$

Put

$$A = \{t \in [a,b] \mid f(t) > 0\}, \qquad B = \{t \in [a,b] \mid h_1(t) > 0\}.$$

If we take (5.15) into account, by direct calculation we obtain

$$
\begin{aligned}
\gamma_2\big(\mu(t)\big) &= \gamma_2(t) - \int_{\mu(t)}^{t} \gamma_2'(s)\, ds \\
&= \gamma_2(t) + \int_{\mu(t)}^{t} \frac{\alpha_1}{(b-s)^{\lambda+\nu}} \gamma_1(s)\, ds - \int_{\mu(t)}^{t} \left[\frac{\nu}{b-s} + \frac{\alpha_2}{(b-s)^{\lambda}}\right] \gamma_2(s)\, ds \\
&\geq \gamma_2(t) - \gamma_2\big(\mu(t)\big) \int_{\mu(t)}^{t} \left[\frac{\nu}{b-s} + \frac{\alpha_2}{(b-s)^{\lambda}}\right] ds \qquad \text{for a.e. } t \in A,
\end{aligned}
$$

and

$$
\begin{aligned}
-\gamma_1\big(\tau_1(t)\big) &= -\gamma_1(t) + \int_{\tau_1(t)}^{t} \gamma_1'(s)\, ds \geq -\gamma_1(t) + \gamma_1'(t)\big(t - \tau_1(t)\big) \\
&= -\gamma_1(t) + \frac{\alpha_3}{(b-t)^{\lambda-\nu}} \big(t - \tau_1(t)\big)\gamma_2(t) \qquad \text{for a.e. } t \in B.
\end{aligned}
$$

By virtue of (5.19), (5.20), (5.21), and (5.41)–(5.44), we get from the last relations

$$
f(t)\gamma_2\big(\mu(t)\big) \geq \frac{f(t)}{1 + \int_{\mu(t)}^{t} \left[\frac{\nu}{b-s} + \frac{\alpha_2}{(b-s)^{\lambda}}\right] ds} \gamma_2(t) \geq \frac{\alpha_3}{(b-t)^{\lambda-\nu}} \gamma_2(t) = \gamma_1'(t)
$$

for a.e. $t \in A$, and

$$
\begin{aligned}
-h_1(t)\gamma_1\big(\tau_1(t)\big) &\geq -h_1(t)\gamma_1(t) + \frac{\alpha_3}{(b-t)^{\lambda-\nu}} h_1(t)\big(t - \tau_1(t)\big)\gamma_2(t) \\
&\geq -\frac{\alpha_1}{(b-t)^{\lambda+\nu}} \gamma_1(t) + \left(\frac{\nu}{b-t} + \frac{\alpha_2}{(b-t)^{\lambda}}\right) \gamma_2(t) \\
&= \gamma_2'(t) \qquad \text{for a.e. } t \in B,
\end{aligned}
$$

which, together with (5.47), guarantees

$$
\gamma_1'(t) \leq f(t)\gamma_2\big(\mu(t)\big), \qquad \gamma_2'(t) \leq -h_1(t)\gamma_1\big(\tau_1(t)\big) \qquad \text{for a.e. } t \in [a,b],
$$

i.e., γ_1 and γ_2 satisfies (3.23) and (3.24). Consequently, using Theorem 3.3, we get $(p,g) \in \hat{\mathscr{S}}_{ab}^2(a)$. □

Proof of Theorem 5.7. The validity of the theorem follows immediately from Theorem 3.1 and Theorems 5.1–5.6. □

6. Counter-examples

Example 6.1. Let the operators p and g be defined by (5.1) and (5.3), respectively, where $\tau_1 \equiv a$. Then the condition (3.30) (i.e., (5.16)) is not only sufficient but also necessary for the validity of the inclusion $(p,g) \in \hat{\mathscr{S}}_{ab}^2(a)$.

Indeed, let $(p, g) \in \hat{\mathcal{S}}^2_{ab}(a)$. Then, according to Remark 3.2, the problem

$$u'(t) = f(t)v\big(\mu(t)\big), \qquad v'(t) = -h_1(t)u(a), \tag{6.1}$$

$$u(a) = 1, \qquad v(a) = 0 \tag{6.2}$$

has a unique solution (u, v) and, moreover, (3.5) is satisfied. It follows from (6.1) and (6.2) that

$$u(t) = 1 + \int_a^t f(s)v\big(\mu(s)\big)\,ds = 1 - \int_a^t f(s)\Big(\int_a^{\mu(s)} h_1(\xi)\,d\xi\Big)\,ds \qquad \text{for } t \in [a, b].$$

Hence we get

$$u(b) = 1 - \int_a^b f(s)\Big(\int_a^{\mu(s)} h_1(\xi)\,d\xi\Big)\,ds,$$

which, together with (3.5), guarantees (5.16).

This example shows that the inequalities (3.30) and (5.16) in Corollary 3.3 and Theorem 5.5 cannot be replaced by the inequalities (3.31) and (5.17), respectively, no matter how small $\varepsilon > 0$ would be.

Example 6.2. Let $\varepsilon > 0$, $\alpha = \frac{\pi}{2(1-\varepsilon)(b-a)}$, and let the operators p and g be defined by (5.1) and (5.3), respectively, where $f \equiv \alpha$, $h_1 \equiv \alpha$, and $\mu(t) = t$, $\tau_1(t) = t$ for $t \in [a, b]$. It is clear that the conditions (5.15), (5.19)–(5.21), and (5.23) are fulfilled with $\alpha_1 = \alpha_3 = \alpha$, $\alpha_2 = 0$, and $\lambda = \nu = 0$. On the other hand, the functions

$$u(t) = \cos\alpha(t-a), \qquad v(t) = -\sin\alpha(t-a) \qquad \text{for } t \in [a, b]$$

fulfils (3.1) and (3.2). However, the function u is not nonnegative on the entire interval $[a, b]$, and thus $(p, g) \notin \hat{\mathcal{S}}^2_{ab}(a)$.

This example shows that the inequality (5.18) cannot be replaced by the inequality (5.23), no matter how small $\varepsilon > 0$ would be.

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