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$\tau(p;q)$-summing mappings and the domination theorem

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Abstract. We extend the concept of τ-summing operators presented by Pietsch in his monograph on Operator Ideals to multilinear mappings and polynomials. We also present a domination theorem for $\tau(p)$-summing mappings and polynomials, showing their relation with p-semi-integral mappings and polynomials.

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1. Introduction

In the 1950s Alexandre Grothendieck developed a substantial body of work on p-summing operators. Later on, in 1967, Albrecht Pietsch isolated this class of operators and established many of their fundamental properties. In his book *Operator Ideals* several of those operators are studied, among which are τ-summing operators.

In Section 2 we establish our notation and recall results needed later. In Section 3 we extend to multilinear mappings the concept of τ-summing operators and establish a domination theorem for the multilinear case. In Section 4 we extend to polynomials the results of the previous section. In the last section we give a few examples of such mappings.

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2. Background

We denote by n a positive integer, $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$. $\mathbb{K}$ is a scalar field ($\mathbb{R}$ or $\mathbb{C}$), and D, D_i, E, E_i, F, F_i, G, G_i stand for Banach spaces over $\mathbb{K}$.

B_E is the closed unit ball in E. For $p \geq 1$ we denote by p' its conjugate, i.e., $1 = \frac{1}{p} + \frac{1}{p'}$. Recall that $(\ell_p)' = \ell_{p'}$ if $1 \leq p < \infty$. Let K be a compact set. Then $\mathscr{C}(K)$ stands for all continuous scalar-valued functions on K. $\mathscr{L}(E_1, \dots, E_n; F)$ denotes the continuous n-linear mappings from $E_1 \times \cdots \times E_n$ into F.

Lemma 2.1. *Let F be a Banach space, $p \geq 1$, and $(b_j)_{j=1}^{\infty} \in \ell_p^w(F')$. Then*

$$\sup_{\beta \in B_{F''}} \left(\sum_{j=1}^{\infty} |\beta(b_j)| \right)^{1/p} = \sup_{y \in B_F} \left(\sum_{j=1}^{\infty} |b_j(y)| \right)^{1/p}.$$

A proof can be found in [4], p. 1. A proof of the following lemma can be found in [5], p. 40.

Lemma 2.2 (Ky Fan). *Let K be a compact convex subset of a linear topological Hausdorff space, and let $\mathscr{F}$ be a concave collection of lower semi-continuous convex real-valued functions on $\mathscr{F}$. Suppose that for each $\Phi \in \mathscr{F}$ there exists $x \in K$ with $\Phi(x) \leq \varrho$. Then we can find $x_0 \in K$ such that $\Phi(x_0) \leq \varrho$ for all $\Phi \in \mathscr{F}$ simultaneously.*

3. $\tau(p; q)$-summing n-linear mappings

Let $\mathscr{L}_n$ be the class of all n-linear mappings between arbitrary Banach spaces, and for a subset $\mathscr{M}$ in $\mathscr{L}_n$ we write

$$\mathscr{M}(E_1, \dots, E_n; F) := \mathscr{M} \cap \mathscr{L}(E_1, \dots, E_n; F).$$

Definition 3.1. Let $\mathscr{M}$ be a subclass of $\mathscr{L}_n$ with an $\mathbb{R}^+$-valued function $\|\cdot\|_{\mathscr{M}}$ such that the following conditions are satisfied for some $0 < r \leq 1$:

(0) $I_n \in \mathscr{M}$ and $\|I_n\|_{\mathscr{M}} = 1$ for $I_n(\lambda_1, \dots, \lambda_n) := \lambda_1 \dots \lambda_n$.

(1) If $S_1, S_2, \dots \in \mathscr{M}(E_1, \dots, E_n; F)$ and $\sum_{k=1}^{\infty} \|S_k\|_{\mathscr{M}}^r < \infty$, then $S = \sum_{k=1}^{\infty} S_k \in \mathscr{M}(E_1, \dots, E_n; F)$ and $\|S\|_{\mathscr{M}}^r \leq \sum_{k=1}^{\infty} \|S_k\|_{\mathscr{M}}^r$.

(2) If $T_i \in \mathscr{L}(D_i; E_i)$, $S \in \mathscr{M}(E_1, \dots, E_n; F)$ and $R \in \mathscr{L}(F; G)$, then $RS(T_1, \dots, T_n) \in \mathscr{M}(D_1, \dots, D_n; G)$ and $\|RST\|_{\mathscr{M}} \leq \|R\| \, \|S\|_{\mathscr{M}} \|T_1\| \dots \|T_n\|$.

We say that $[\mathscr{M}, \|\cdot\|_{\mathscr{M}}]$ is an r-normed $\mathscr{L}_n$-module of n-linear mappings; if $r = 1$ then $[\mathscr{M}, \|\cdot\|_{\mathscr{M}}]$ is a Banach $\mathscr{L}_n$-module of n-linear mappings.

Some authors have called such class of mappings "ideals of n-linear mappings", but in order to keep an analogy with algebra we chose this new terminology.

Definition 3.2. Let $S \in \mathscr{L}(E_1, \dots, E_n; F)$ and $1 \le q \le p$. We say that S is $\tau(p;q)$-*summing* if there exists a constant $\sigma \ge 0$ such that, for all $m \in \mathbb{N}$, $x_{ij} \in E_i$, $b_j \in F'$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$,

$$\left(\sum_{j=1}^{m} \left|b_j\left(S(x_{1j}, \dots, x_{nj})\right)\right|^p\right)^{1/p} \le \sigma \sup_{\substack{\|a_i\| \le 1 \\ \|y\| \le 1}} \left(\sum_{j=1}^{m} |a_1(x_{1j}) \dots a_n(x_{nj}) b_j(y)|^q\right)^{1/q}$$

where $a_i \in E_i'$, $y \in F$ and $\beta \in F''$. We denote this class of mappings by $\mathscr{L}_{\tau(p;q)}(E_1, \dots, E_n; F)$ and on it we define the norm $\|S\|_{\tau(p;q)} := \inf \sigma$ for the constant that appears in the expression above.

$\mathscr{L}_{\tau(p;q)}$ with the norm $\|S\|_{\tau(p;q)}$ is a Banach $\mathscr{L}_n$-module of n-linear mappings. When $p = q$, we write $\mathscr{L}_{\tau(p)}$ and $\|S\|_{\tau(p)}$ instead of $\mathscr{L}_{\tau(p;p)}$ and $\|S\|_{\tau(p;p)}$, respectively. We say that S is $\tau(p)$-summing. If $p = q = 1$, we simply write $\mathscr{L}_\tau$ and $\|S\|_\tau$ and say that S is τ-summing. Notice that if $1 \le s \le r \le q \le p$, then $\mathscr{L}_{\tau(q;r)} \subseteq \mathscr{L}_{\tau(p;s)}$, because if $S \in \mathscr{L}_{\tau(q;r)}$, we have

$$\begin{aligned}
\left(\sum_{j=1}^{m} \left|b_j\left(S(x_{1j}, \dots, x_{nj})\right)\right|^p\right)^{1/p} &\le \left(\sum_{j=1}^{m} \left|b_j\left(S(x_{1j}, \dots, x_{nj})\right)\right|^q\right)^{1/q} \\
&\le \sigma \sup_{\substack{\|a_i\| \le 1 \\ \|y\| \le 1}} \left(\sum_{j=1}^{m} |a_1(x_{1j}) \dots a_n(x_{nj}) b_j(y)|^r\right)^{1/r} \\
&\le \sigma \sup_{\substack{\|a_i\| \le 1 \\ \|y\| \le 1}} \left(\sum_{j=1}^{m} |a_1(x_{1j}) \dots a_n(x_{nj}) b_j(y)|^s\right)^{1/s}.
\end{aligned}$$

Moreover, from the above inequalities it follows that

$$\|S\|_{\tau(p;r)} \le \|S\|_{\tau(q;r)} \qquad \text{for all } S \in \mathscr{L}_{\tau(q;r)} \subseteq \mathscr{L}_{\tau(p;r)}$$

and

$$\|S\|_{\tau(q;s)} \le \|S\|_{\tau(q;r)} \qquad \text{for all } S \in \mathscr{L}_{\tau(q;r)} \subseteq \mathscr{L}_{\tau(q;s)}.$$

Remark 3.3. Recall that if $\frac{1}{p} \le \frac{1}{q_1} + \dots + \frac{1}{q_n}$, $T \in \mathscr{L}(E_1, \dots, E_n; F)$ is *absolutely* $(p; q_1, \dots, q_n)$-*summing* if there exists a constant $\sigma \ge 0$ such that for all $m, n \in \mathbb{N}$, $x_{ij} \in E_i$, and $i = 1, \dots, n$, $j = 1, \dots, m$,

$$\left(\sum_{j=1}^{m} \|T(x_{1j}, \dots, x_{nj})\|^p\right)^{1/p} \le \sigma \prod_{i=1}^{n} \sup_{\|a_i\| \le 1} \left(\sum_{j=1}^{m} |a_i(x_{ij})|^{q_i}\right)^{1/q_i};$$

and we write $T \in \mathscr{L}_{\mathrm{as}(p;q_1,\dots,q_n)}(E_1,\dots,E_n;F)$. On the other hand, if $\frac{1}{q} \le \frac{1}{q_1} + \dots + \frac{1}{q_n}$, then by Hölder's generalized inequality

$$\Big(\sum_{j=1}^{m}(\|x_{1j}\|\dots\|x_{nj}\|)^{q}\Big)^{1/q} \le \Big(\sum_{j=1}^{m}\|x_{1j}\|^{q_1}\Big)^{1/q_1}\dots\Big(\sum_{j=1}^{m}\|x_{nj}\|^{q_n}\Big)^{1/q_n}.$$

Given $S \in \mathscr{L}_{\tau(p;q)}(E_1,\dots,E_n;F)$, we define $S_F \in \mathscr{L}(E_1,\dots,E_n,F';\mathbb{K})$ by $S_F(x_1,\dots,x_n,b) := bS(x_1,\dots,x_n)$. It follows that if $\frac{1}{q} = \frac{1}{q_1} + \dots + \frac{1}{q_n} + \frac{1}{q_{n+1}}$, then

$$\begin{aligned}
&\Big(\sum_{j=1}^{m}|S_F(x_{1_j},\dots,x_{n_j},b_j)|^{p}\Big)^{1/p}\\
&\quad = \Big(\sum_{j=1}^{m}|b_jS(x_{1_j},\dots,x_{n_j})|^{p}\Big)^{1/p}\\
&\quad \le \sigma \sup_{\substack{\|a_i\|\le 1\\ \|y\|\le 1}} \Big(\sum_{j=1}^{m}|a_1(x_{1j})\dots a_n(x_{nj})b_j(y)|^{q}\Big)^{1/q} \qquad (\text{since } S \in \mathscr{L}_{\tau(p;q)})\\
&\quad \le \sigma \sup_{\substack{\|a_i\|\le 1\\ \|y\|\le 1}} \Big(\sum_{j=1}^{m}|a_1(x_{1j})|^{q_1}\Big)^{1/q_1}\dots\Big(\sum_{j=1}^{m}|a_n(x_{nj})|^{q_n}\Big)^{1/q_n}\Big(\sum_{j=1}^{m}|b_j(y)|^{q_{n+1}}\Big)^{1/q_{n+1}}\\
&\qquad (\text{by Hölder})\\
&\quad = \sigma \sup_{\substack{\|a_i\|\le 1\\ \|\beta\|\le 1}} \Big(\sum_{j=1}^{m}|a_1(x_{1j})|^{q_1}\Big)^{1/q_1}\dots\Big(\sum_{j=1}^{m}|a_n(x_{nj})|^{q_n}\Big)^{1/q_n}\Big(\sum_{j=1}^{m}|\beta(b_j)|^{q_{n+1}}\Big)^{1/q_{n+1}}\\
&\qquad (\text{by Lemma 2.1}).
\end{aligned}$$

So $S \in \mathscr{L}_{\tau(p;q)}(E_1,\dots,E_n;F)$ implies $S_F \in \mathscr{L}_{\mathrm{as}(p;q_1,\dots,q_n,q_{n+1})}(E_1,\dots,E_n,F';\mathbb{K})$.

Remark 3.4. Given $\frac{1}{p} = \frac{1}{q_1} + \dots + \frac{1}{q_n}$, an n-linear mapping $S \in \mathscr{L}(E_1,\dots,E_n;F)$ is $(q_1,\dots,q_n)$-*dominated* if $S \in \mathscr{L}_{\mathrm{as}(p;q_1,\dots,q_n)}(E_1,\dots,E_n;F)$; and it is p-*dominated* if $S \in \mathscr{L}_{\mathrm{as}(p/n;p)}(E_1,\dots,E_n;F)$. It follows that if S is $\tau(p)$-summing, then S_F is p-dominated.

Theorem 3.5. *Let $1 \le p < \infty$. A mapping $S \in \mathscr{L}(E_1,\dots,E_n;F)$ is $\tau(p)$-summing if and only if there exist a constant $\sigma \ge 0$ and $\mu \in W(B_{E_1'} \times \dots \times B_{E_n'} \times B_{F''})$, the set of Borel probability measures in $B_{E_1} \times \dots \times B_{E_n} \times B_{F''}$, such that*

$$\begin{aligned}
&\big|b\big(S(x_1,\dots,x_n)\big)\big|\\
&\qquad \le \sigma\Big(\int_{B_{E_1'}\times\dots\times B_{E_n'}\times B_{F''}}|a_1(x_1)\dots a_n(x_n)\beta(b)|^{p}\,d\mu(a_1,\dots,a_n,\beta)\Big)^{1/p} \qquad (1)
\end{aligned}$$

for all $x_i \in E_i$ and all $b \in F'$. In this case, $\|S\|_{\tau(p)} = \inf \sigma$.

Proof. Here we exploit ideas already used by Matos and Alencar for absolutely p-summing mappings in [1], 4.3, p. 11.

If inequality (1) holds, then:

$$\begin{aligned}
&\Big(\sum_{j=1}^{m}\big|b_j\big(S(x_{1j},\dots,x_{nj})\big)\big|^p\Big)^{1/p}\\
&\le \Big\{\sum_{j=1}^{m}\Big[\sigma\Big(\int_{B_{E_1'}\times\cdots\times B_{E_n'}\times B_{F''}}|a_1(x_{1j})\cdots a_n(x_{nj})\beta(b_j)|^p\,d\mu(a_1,\dots,a_n,\beta)\Big)^{1/p}\Big]^p\Big\}^{1/p}\\
&\quad\text{(by (1))}\\
&=\sigma\Big\{\int_{B_{E_1'}\times\cdots\times B_{E_n'}\times B_{F''}}\sum_{j=1}^{m}|a_1(x_{1j})\cdots a_n(x_{nj})\beta(b_j)|^p\,d\mu(a_1,\dots,a_n,\beta)\Big\}^{1/p}\\
&\le\sigma\Big\{\int_{B_{E_1'}\times\cdots\times B_{E_n'}\times B_{F''}}\sup_{\substack{\|a_i\|\le 1\\ \|\beta\|\le 1}}\sum_{j=1}^{m}|a_1(x_{1j})\cdots a_n(x_{nj})\beta(b_j)|^p\,d\mu(a_1,\dots,a_n,\beta)\Big\}^{1/p}\\
&=\sigma\sup_{\substack{\|a_i\|\le 1\\ \|y\|\le 1}}\Big\{\sum_{j=1}^{m}|a_1(x_{1j})\cdots a_n(x_{nj})b_j(y)|^p\Big\}^{1/p}\quad\text{(by Lemma 2.1).}
\end{aligned}$$

So S is $\tau(p)$-summing and $\|S\|_{\tau(p)}\le\inf\sigma$.

On the other hand, if $S\in\mathscr{L}_{\tau(p)}(E_1,\dots,E_n;F)$ let $\sigma=\|S\|_{\tau(p)}$. Take $\mathscr{C}(B_{E_1'}\times\cdots\times B_{E_n'}\times B_{F''})'$ equipped with the weak-$\mathscr{C}(B_{E_1'}\times\cdots\times B_{E_n'}\times B_{F''})$ topology. Then $W(B_{E_1'}\times\cdots\times B_{E_n'}\times B_{F''})$ is a compact convex subset. For any finite family of elements $x_{i1},\dots,x_{im}\in E_i$ and functionals $b_1,\dots,b_m\in F'$ the equation

$$\begin{aligned}
\phi(\mu):=\sum_{j=1}^{m}\Big\{&\big|b_j\big(S(x_{1j},\dots,x_{nj})\big)\big|^p\\
&-\|S\|_{\tau(p)}^p\int_{B_{E_1'}\times\cdots\times B_{E_n'}\times B_{F''}}|a_1(x_{1j})\dots a_n(x_{nj})\beta(b_j)|^p\,d\mu(a_1,\dots,a_n,\beta)\Big\}
\end{aligned}$$

defines a real continuous convex function ϕ on $W(B_{E_1'}\times\cdots\times B_{E_n'}\times B_{F''})$. Choose $a_{10}\in B_{E_1'},\dots,a_{n0}\in B_{E_n'}$ and $\beta_0\in B_{F''}$ such that

$$\begin{aligned}
&\sup\Big\{\sum_{j=1}^{m}|a_1(x_{1j})\dots a_n(x_{nj})\beta(b_j)|^p : \|a_i\|,\|\beta\|\le 1\Big\}\\
&\quad=\sum_{j=1}^{m}|a_{10}(x_{1j})\dots a_{n0}(x_{nj})\beta_0(b_j)|^p.
\end{aligned}$$

If $\delta(a_{10}, \ldots, a_{n0}, \beta_0)$ denotes the Dirac measure at $(a_{10}, \ldots, a_{n0}, \beta_0)$, then we have

$$\phi\big(\delta(a_{10}, \ldots, a_{n0}, \beta_0)\big) = \sum_{j=1}^{m} \big|b_j\big(S(x_{1j}, \ldots, x_{nj})\big)\big|^p - \|S\|_{\tau(p)}^p |a_{10}(x_{1j}) \ldots a_{n0}(x_{nj})\beta_0(b_j)|^p \le 0.$$

Since the collection $\mathscr{F}$ of all functions obtained this way is concave, by the Ky Fan Lemma 2.2 there exists a measure $\mu_0 \in W(B_{E_1'} \times \cdots \times B_{E_n'} \times B_{F''})$ such that $\phi(\mu_0) \le 0$, for all $\phi \in \mathscr{F}$ simultaneously. In particular, if ϕ is generated by $x_1, \ldots, x_n$ and b, i.e.,

$$\phi(\mu) := \Big\{\big|b\big(S(x_1, \ldots, x_n)\big)\big|^p - \|S\|_{\tau(p)}^p \int_{B_{E_1'} \times \cdots \times B_{E_n'} \times B_{F''}} |a_1(x_1) \ldots a_n(x_n)\beta(b)|^p \, d\mu(a_1, \ldots, a_n, \beta)\Big\},$$

it follows that

$$\big|b\big(S(x_1, \ldots, x_n)\big)\big| \le \|S\|_{\tau(p)} \Big(\int_{B_{E_1'} \times \cdots \times B_{E_n'} \times B_{F''}} |a_1(x_1) \ldots a_n(x_n)\beta(b)|^p \, d\mu_0(a_1, \ldots, a_n, \beta)\Big)^{1/p}$$

and $\|S\|_{\tau(p)} = \inf \sigma$. □

We also have the following result.

Theorem 3.6. *Let $1 \le p < \infty$. A mapping $S \in \mathscr{L}(E_1, \ldots, E_n; F)$ is $\tau(p)$-summing if and only if there exist a constant $\sigma \ge 0$, $\mu_i \in W(B_{E_i'})$, $\mu_{n+1} \in W(B_{F''})$, the sets of Borel probability measures in $B_{E_i'}$ $(i = 1, \ldots, n)$ and $B_{F''}$, such that*

$$\big|b\big(S(x_1, \ldots, x_n)\big)\big| \le \sigma\Big(\int_{B_{E_1'}} \cdots \int_{B_{E_n'}} \int_{B_{F''}} |a_1(x_1) \cdots a_n(x_n)\beta(b)|^p \, d\mu_{n+1}(\beta)\, d\mu_n(a_n) \cdots d\mu_1(a_1)\Big)^{1/p} \tag{2}$$

for all $x_i \in E_i$ and $b \in F'$. In this case, $\|S\|_{\tau(p)} = \inf \sigma$.

Proof. Here we repeat the proof of the previous theorem with minor changes.

($\Leftarrow$): Using (2) one easily shows that S is $\tau(p)$-summing and $\|S\|_{\tau(p)} \le \inf \sigma$.

($\Rightarrow$): Let $S \in \mathscr{L}_{\tau(p)}(E_1, \ldots, E_n; F)$, and put $\sigma = \|S\|_{\tau(p)}$. Take each $\mathscr{C}(B_{E_1'})', \ldots, \mathscr{C}(B_{E_n'})', \mathscr{C}(B_{F''})'$ equipped with the weak-$\mathscr{C}(B_{E_1'})', \ldots,$ weak-$\mathscr{C}(B_{E_n'})'$, weak-$\mathscr{C}(B_{F''})'$ topology, respectively. Then $W(B_{E_1'}), \ldots, W(B_{E_n'})$, $W(B_{F''})$ as well as $W(B_{E_1'}) \times \cdots \times W(B_{E_n'}) \times W(B_{F''})$ are compact convex subsets. For any finite family of elements $x_{i1},\dots,x_{im} \in E_i$, $i=1,\dots,n$ and functionals $b_1,\dots,b_m \in F'$ the equation

$$\begin{aligned}\phi(\mu_1,\dots,\mu_n,\mu_{n+1}) \\ := \sum_{j=1}^{m} \Big\{ \big|b_j\big(S(x_{1j},\dots,x_{nj})\big)\big|^p \\ - \sigma^p \int_{B_{E_1'}} \dots \int_{B_{E_n'}} \int_{B_{F''}} |a_1(x_1)\dots a_n(x_n)\beta(b)|^p \, d\mu_{n+1}(\beta)\, d\mu_n(a_n)\dots d\mu_1(a_1)\Big\}\end{aligned}$$

defines a real continuous convex function ϕ on $W(B_{E_1'}) \times \cdots \times W(B_{E_n'}) \times W(B_{F''})$. Choose $a_{10} \in B_{E_1'},\dots,a_{n0} \in B_{E_n'}$ and $\beta_0 \in B_{F''}$ such that

$$\begin{aligned}\sup\Big\{\sum_{j=1}^{m} |a_1(x_{1j})\dots a_n(x_{nj})\beta(b_j)|^p : \|a_i\|, \|\beta\| \le 1\Big\} \\ = \sum_{j=1}^{m} |a_{10}(x_{1j})\dots a_{n0}(x_{nj})\beta_0(b_j)|^p.\end{aligned}$$

If $\delta_1(a_{10}),\dots,\delta_n(a_{n0}),\delta_{n+1}(\beta_0)$ denote the Dirac measures at $a_{10},\dots,a_{n0}$, β_0 respectively, then we have

$$\begin{aligned}\phi\big(\delta_1(a_{10}) \times \cdots \times \delta_n(a_{n0}) \times \delta_{n+1}(\beta_0)\big) \\ = \sum_{j=1}^{m} \big|b_j\big(S(x_{1j},\dots,x_{nj})\big)\big|^p - \sigma^p |a_{10}(x_{1j})\dots a_{n0}(x_{nj})\beta_0(b_j)|^p \le 0.\end{aligned}$$

Since the collection $\mathscr{F}$ of all functions obtained this way is concave, by the Ky Fan Lemma 2.2 there exists a measure $\bar{\mu}_1 \times \cdots \times \bar{\mu}_n \times \bar{\mu}_{n+1} \in W(B_{E_1'}) \times \cdots \times W(B_{E_n'}) \times W(B_{F''})$ such that $\phi(\bar{\mu}_1 \times \cdots \times \bar{\mu}_n \times \bar{\mu}_{n+1}) \le 0$, for all $\phi \in \mathscr{F}$ simultaneously. In particular, if ϕ is generated by $x_1,\dots,x_n$ and b, we have

$$\begin{aligned}\phi(\bar{\mu}_1 \times \cdots \times \bar{\mu}_n \times \bar{\mu}_{n+1}) \\ = \Big\{ \big|b\big(S(x_1,\dots,x_n)\big)\big|^p \\ - \sigma^p \int_{B_{E_1'}} \dots \int_{B_{E_n'}} \int_{B_{F''}} |a_1(x_1)\dots a_n(x_n)\beta(b)|^p \, d\bar{\mu}_{n+1}(\beta)\, d\bar{\mu}_n(a_n)\dots d\bar{\mu}_1(a_1)\Big\}.\end{aligned}$$

It follows that

$$\begin{aligned}\big|b\big(S(x_1,\dots,x_n)\big)\big| \\ \le \sigma\Big(\int_{B_{E_1'}} \dots \int_{B_{E_n'}} \int_{B_{F''}} |a_1(x_1)\dots a_n(x_n)\beta(b)|^p \, d\bar{\mu}_{n+1}(\beta)\, d\bar{\mu}_n(a_n)\dots d\bar{\mu}_1(a_1)\Big)^{1/p}. \quad \square\end{aligned}$$

4. *n*-homogeneous polynomials

Let $\mathscr{L}({}^nE;F) := \mathscr{L}(E,\overset{n\,\text{times}}{\ldots},E;F)$ and Σ_n the set of all permutations over $\{1,\ldots,n\}$. We say $T \in \mathscr{L}({}^nE;F)$ is a *symmetric n-linear mapping* if

$$T(x_1,\ldots,x_n) = T(x_{\sigma(1)},\ldots,x_{\sigma(n)}) \qquad \text{for all } \sigma \in \Sigma_n,\ x_1,\ldots,x_n \in E.$$

We write $\mathscr{L}_s({}^nE;F)$ for the class of all symmetric n-linear mappings. The set $[\mathscr{L}_s({}^nE;F),\|\cdot\|]$ is closed in $[\mathscr{L}({}^nE;F),\|\cdot\|]$ (see [2], Proposition 1, p. 2). The *symmetrization n-linear mapping*

$$\begin{gathered} s : \mathscr{L}({}^nE;F) \to \mathscr{L}_s({}^nE;F), \\ T \mapsto T_s, \end{gathered}$$

where

$$T(x_1,\ldots,x_n) = \frac{1}{n!}\sum_{\sigma\in\Sigma_n} T(x_{\sigma(1)},\ldots,x_{\sigma(n)})$$

is a continuous projection (see [2], Proposition 2, p. 3).

We write $x^n := (x,\overset{n\,\text{times}}{\ldots},x)$. A *continuous n-homogeneous polynomial* $P : E \to F$ is a mapping for which there is some $T \in \mathscr{L}({}^nE;F)$ such that $P(x) = T(x^n)$ for every $x \in E$. To denote that T corresponds to P, we will write $\hat{T} = P$. We shall denote by $\mathscr{P}({}^nE;F)$ the Banach space of continuous n-homogeneous polynomials from E into F with respect to pointwise operations and the norm defined by

$$\|P\| := \sup\{\|P(x)\| : x \in B_E\}.$$

Note that $\|P(x)\| \le \|P\|\,\|x\|^n$. The mapping

$$\begin{gathered} \varphi : \mathscr{L}_s({}^nE;F) \to \mathscr{P}({}^nE;F), \\ T \mapsto \hat{T}, \end{gathered}$$

is a vector space isomorphism and a homeomorphism of the Banach space $\mathscr{L}_s({}^nE;F)$ onto $\mathscr{P}({}^nE;F)$ (see [2], Proposition 3, p. 4). A *continuous polynomial* P from E into F is a mapping $P : E \to F$ for which there are $n \in \mathbb{N}_0$ and $P_k \in \mathscr{P}({}^kE;F)$, $k = 0,1,\ldots,n$, such that $P = P_0 + P_1 + \cdots + P_n$. We shall denote by $\mathscr{P}(E;F)$ the Banach space of continuous polynomials from E into F, and by $\mathscr{P}$ the class of all continuous polynomials between arbitrary Banach spaces. If $\mathscr{I}$ is a subset of $\mathscr{P}$ we write

$$\mathscr{I}({}^nE;F) := \mathscr{I} \cap \mathscr{P}({}^nE;F).$$

All definitions and results valid for n-linear mapping $\mathscr{L}_n$-modules can be transferred to polynomials via the φ mapping just described.

Definition 4.1. Let $\mathscr{I}$ be a subclass of $\mathscr{P}$ with a $\mathbb{R}^+$-valued function $\|\cdot\|_{\mathscr{I}}$ such that the following conditions are satisfied for some $0 < r \leq 1$:

(0) $I_n \in \mathscr{I}$ and $\|I_n\|_{\mathscr{I}} = 1$ for $I_n(\lambda) := \lambda^n$.

(1) If $P_1, P_2, \ldots \in \mathscr{I}(^nE;F)$ and $\sum_{k=1}^{\infty} \|P_k\|_{\mathscr{I}}^r < \infty$, then $P = \sum_{k=1}^{\infty} P_k \in \mathscr{I}(^nE;F)$ and $\|P\|_{\mathscr{I}}^r \leq \sum_{k=1}^{\infty} \|P_k\|_{\mathscr{I}}^r$.

(2) If $Q \in \mathscr{P}(^1D;E)$, $P \in \mathscr{I}(^nE;F)$ and $O \in \mathscr{P}(^1F;G)$, then $OPQ \in \mathscr{I}(^nD;G)$ and $\|OPQ\|_{\mathscr{I}} \leq \|O\|\,\|P\|_{\mathscr{I}}\|Q\|$.

We say that $[\mathscr{I}, \|\cdot\|_{\mathscr{I}}]$ is an r-normed ideal of n-homogeneous polynomials, and if $r = 1$, then $[\mathscr{I}, \|\cdot\|_{\mathscr{I}}]$ is a Banach ideal of n-homogeneous continuous polynomials.

Definition 4.2. Given a n-homogeneous polynomial $P \in \mathscr{P}(^nE;F)$, we shall say that it is $\tau(p;q)$-*summing* if there exists a constant $\sigma \geq 0$ such that for all $m \in \mathbb{N}$, $x_j \in E$, $b_j \in F'$, $j = 1, 2, \ldots, m$, we have

$$\left(\sum_{j=1}^{m} \left|b_j\big(P(x_j)\big)\right|^p\right)^{1/p} \leq \sigma \sup_{\substack{\|a\|\leq 1\\ \|y\|\leq 1}} \left(\sum_{j=1}^{m} |a(x_j)^n b_j(y)|^q\right)^{1/q}.$$

By defining the norm $\|P\|_{\tau(p;q)} := \inf \sigma$ for the constant that appears in the expression above, it can be shown that this class of polynomials is an ideal, which we shall denote by $\mathscr{P}_{\tau(p;q)}(^nE;F)$. When $p = q$ we write $\mathscr{P}_{\tau(p)}(^nE;F)$ and $\|P\|_{\tau(p)}$ respectively and say that it is the ideal of n-homogeneous $\tau(p)$-summing polynomials from E into F. When $p = q = 1$ we simply write $\mathscr{P}_{\tau}(^nE;F)$, $\|P\|_{\tau}$ and speak of the ideal of n-homogeneous τ-summing polynomials from E into F.

Theorem 4.3. *A n-homogeneous polynomial $P \in \mathscr{P}(^nE;F)$ is $\tau(p)$-summing if and only if there exist a constant $\sigma \geq 0$ and $\mu \in W(B_{E'} \times B_{F''})$, the Borel probability measures set in $B_{E'} \times B_{F''}$, such that*

$$\left|b\big(P(x)\big)\right| \leq \sigma\left(\int_{B_{E'}\times B_{F''}} |a(x)^n \beta(b)|^p \, d\mu(a,\beta)\right)^{1/p} \qquad \text{for all } x \in E,\ b \in F'.$$

In this case, $\|P\|_{\tau(p)} = \inf \sigma$.

Proof. ($\Leftarrow$): If the inequality holds, we have

$$\Big(\sum_{j=1}^{m}|b_jP(x_j)|^p\Big)^{1/p} \le \Big\{\sum_{j=1}^{m}\Big[\sigma\Big(\int_{B_{E'}\times B_{F''}}|a(x_j)^n\beta(b_j)|^p\,d\mu(a,\beta)\Big)^{1/p}\Big]^p\Big\}^{1/p}$$

$$(\text{finite sum } \Rightarrow) = \sigma\Big\{\int_{B_{E'}\times B_{F''}}\sum_{j=1}^{m}|a(x_j)^n\beta(b_j)|^p\,d\mu(a,\beta)\Big\}^{1/p}$$

$$\le \sigma\Big\{\int_{B_{E'}\times B_{F''}}\sup_{\substack{\|a\|\le 1\\ \|\beta\|\le 1}}\sum_{j=1}^{m}|a(x_j)^n\beta(b_j)|^p\,d\mu(a,\beta)\Big\}^{1/p}$$

$$= \sigma\sup_{\substack{\|a\|\le 1\\ \|\beta\|\le 1}}\Big\{\sum_{j=1}^{m}|a(x_j)^n\beta(b_j)|^p\Big\}^{1/p}.$$

($\Rightarrow$): Conversely, let $P \in \mathscr{P}_{\tau(p)}({}^nE;F)$ and $\sigma = \|P\|_{\tau(p)}$. Consider $\mathscr{C}(B_{E'}\times B_{F''})'$ with the weak-$\mathscr{C}(B_{E'}\times B_{F''})$ topology. Then $W(B_{E'}\times B_{F''})$ is a convex compact subset. For any finite family of elements $x_1,\dots,x_m \in E$ and functionals $b_1,\dots,b_m \in F'$ the equation

$$\phi(\mu) := \sum_{j=1}^{m}\Big\{|b_jP(x_j)|^p - \sigma^p\int_{B_{E'}\times B_{F''}}|a(x_j)^n\beta(b_j)|^p\,d\mu(a,\beta)\Big\}$$

defines a real convex continuous function ϕ over $W(B_{E'}\times B_{F''})$. Choose $a_0 \in B_{E'}$ and $\beta_0 \in B_{F''}$ such that

$$\sup\Big\{\sum_{j=1}^{m}|a(x_j)^n\beta(b_j)|^p : \|a\|,\|\beta\| \le 1\Big\} = \sum_{j=1}^{m}|a_0(x_j)^n\beta_0(b_j)|^p.$$

If $\delta(a_0,\beta_0)$ denotes the Dirac measure at (a_0,β_0), then

$$\phi\big(\delta(a_0,\beta_0)\big) = \sum_{j=1}^{m}|b_jP(x_j)|^p - \sigma^p|a_0(x_j)^n\beta_0(b_j)|^p \le 0.$$

Since the collection $\mathscr{F}$ of functions thus obtained is concave, by the Ky Fan Lemma 2.2, there is a measure $\mu_0 \in W(B_{E'}\times B_{F''})$ such that $\phi(\mu_0) \le 0$, for all $\phi \in \mathscr{F}$ simultaneously. In particular, if ϕ is generated by x and b, it follows that

$$\big|b\big(P(x)\big)\big|^p - \sigma^p\int_{B_{E'}\times B_{F''}}|a(x)^n\beta(b)|^p\,d\mu_0(a,\beta) \le 0$$

and so

$$\left|b\big(P(x)\big)\right| \le \sigma\Big(\int_{B_{E'}\times B_{F''}} |a(x)^n\beta(b)|^p\, d\mu_0(a,\beta)\Big)^{1/p}.$$

From the above implications we obtain $\|P\|_{\tau(p)} = \inf\sigma$. □

With a similar proof we also have the following:

An n-homogeneous polynomial $P \in \mathscr{P}(^nE;F)$ is $\tau(p)$-summing if and only if there exist a constant $\sigma \ge 0$ and $\mu \in W(B_{E'})$, $\nu \in W(B_{F'p})$ sets of Borel probability measures in $B_{E'}$ and $B_{E'p}$ respectively, such that

$$b(P(x))| \le \sigma\Big(\int_{B_{E'}}\int_{B_{F'p}} |a(x)^n\beta(b)|^p d\nu(\beta)d\mu(a)\Big)^{1/p}$$

for all $x \in E_i$ and $b \in F'$. In this case, $\|P\|_{\tau(p)} = \inf\sigma$.

Remark 4.4. Recall the following definition (see [1], p. 10): A n-homogeneous polynomial $P \in \mathscr{P}(^nE;F)$ is said to be *p-semi-integral*; we write $P \in \mathscr{P}_{\mathrm{si}(p)}(^nE;F)$ if there exist a constant $\sigma \ge 0$ and a regular probability measure $\mu \in W(B_{E'})$ such that

$$\|P(x)\| \le \sigma\Big(\int_{B_{E'}} |a(x)^n|^p\, d\mu(a)\Big)^{1/p} \qquad \text{for all } x \in E.$$

Let ν be a regular probability measure given by $\nu(C) = \mu(C \times B_{F''})$ for each Borel subset C of $B_{E'}$. If $P \in \mathscr{P}_{\tau(p)}(^nE;F)$, then, for every $x \in E$,

$$\begin{aligned}|bP(x)| &= \left|b\big(P(x)\big)\right|\\ &\le \sigma\Big(\int_{B_{E'}\times B_{F''}} |a(x)^n\beta(b)|^p\, d\mu(a,\beta)\Big)^{1/p} \le \|b\|\sigma\Big(\int_{B_{E'}} |a(x)^n|^p\, d\nu(a)\Big)^{1/p}.\end{aligned}$$

It follows that

$$\begin{aligned}\|P(x)\| &= \sup_{\|b\|\le 1} |bP(x)|\\ &\le \sup_{\|b\|\le 1} \|b\|\sigma\Big(\int_{B_{E'}} |a(x)^n|^p\, d\nu(a)\Big)^{1/p} \le \sigma\Big(\int_{B_{E'}} |a(x)^n|^p\, d\nu(a)\Big)^{1/p}.\end{aligned}$$

In other words, P is p-semi-integral, ie $\mathscr{P}_{\tau(p)}(^nE;F) \subseteq \mathscr{P}_{\mathrm{si}(p)}(^nE;F)$.

5. Examples

Example 5.1. Consider E_1, E_2, F Banach spaces and fix $\bar{a}_1^k \in E_1$, $\bar{a}_2^k \in E_2$, $\bar{y}^k \in F$, $\bar{a}_1^k, \bar{a}_2^k, \bar{y}^k \ne 0$ for $k = 1,\dots,M$. Then a finite type mapping $S \in \mathscr{L}_f(E_1,E_2;F)$,

$$S : E_1 \times E_2 \to F,$$
$$(x_1, x_2) \mapsto \sum_{k=1}^{M} \bar{a}_1^k(x_1)\bar{a}_2^k(x_2)\,\bar{y}^k,$$

is a $\tau(p,q)$-summing 2-linear mapping for all $1 \le q \le p$ with $\|S\|_{\tau(p;q)} \le \sum_{k=1}^{M} \|\bar{a}_1^k\|\,\|\bar{a}_2^k\|\,\|\bar{y}^k\|$. We have

$$\begin{aligned}
&\Big(\sum_{j=1}^{m} |b_j S(x_{1j}, x_{2j})|^p\Big)^{1/p} \\
&\quad = \Big(\sum_{j=1}^{m} \Big|\sum_{k=1}^{M} \bar{a}_1^k(x_{1j})\bar{a}_2^k(x_{2j})b_j(\bar{y}^k)\Big|^p\Big)^{1/p} \\
&\quad \le \sum_{k=1}^{M} \Big(\sum_{j=1}^{m} \big|\bar{a}_1^k(x_{1j})\bar{a}_2^k(x_{2j})b_j(\bar{y}^k)\big|^p\Big)^{1/p} \\
&\quad = \sum_{k=1}^{M} \left(\|\bar{a}_1^k\|\,\|\bar{a}_2^k\|\,\|\bar{y}^k\| \left(\sum_{j=1}^{m} \left|\frac{\bar{a}_1^k}{\|\bar{a}_1^k\|}(x_{1j})\frac{\bar{a}_2^k}{\|\bar{a}_2^k\|}(x_{2j})b_j\left(\frac{\bar{y}^k}{\|\bar{y}^k\|}\right)\right|^p\right)^{1/p}\right) \\
&\quad \le \sum_{k=1}^{M} \Big(\|\bar{a}_1^k\|\,\|\bar{a}_2^k\|\,\|\bar{y}^k\| \sup_{\substack{\|a_i\| \le 1 \\ \|y\| \le 1}} \Big(\sum_{j=1}^{m} \big|a_1(x_{1j})a_2(x_{2j})b_j(y)\big|^p\Big)^{1/p}\Big) \\
&\quad \le \Big(\sum_{k=1}^{M} \|\bar{a}_1^k\|\,\|\bar{a}_2^k\|\,\|\bar{y}^k\|\Big) \sup_{\substack{\|a_i\| \le 1 \\ \|y\| \le 1}} \Big(\sum_{j=1}^{m} \big|a_1(x_{1j})a_2(x_{2j})b_j(y)\big|^q\Big)^{1/q} \\
&\qquad (q \le p \Rightarrow \|\cdot\|_p \le \|\cdot\|_q).
\end{aligned}$$

Example 5.2. A mapping $S \in \mathscr{L}(E_1, E_2; F)$ is said to be nuclear if it admits a representation $S = \sum_{k=1}^{\infty} a_{1k} \otimes a_{2k} \otimes y_k$ with $\sigma = \sum_{k=1}^{\infty} \|a_{1k}\|\,\|a_{2k}\|\,\|y_k\| < \infty$ and norm $\|S\|_N = \inf \sigma$, with the infimum taken over all possible representations. Suppose $a_{1k}, a_{2k}, y_k \ne 0$ for all $k \in \mathbb{N}$. So we have

$$\begin{aligned}
&\Big(\sum_{j=1}^{m} |b_j S(x_{1j}, x_{2j})|^p\Big)^{1/p} \\
&\quad = \Big(\sum_{j=1}^{m} \Big|\sum_{k=1}^{\infty} a_{1k}(x_{1j})a_{2k}(x_{2j})b_j(y_k)\Big|^p\Big)^{1/p} \\
&\quad = \left(\sum_{j=1}^{m} \left|\sum_{k=1}^{\infty} \|a_{1k}\| \frac{a_{1k}}{\|a_{1k}\|}(x_{1j})\|a_{2k}\| \frac{a_{2k}}{\|a_{2k}\|}(x_{2j})\|y_k\| b_j\left(\frac{y_k}{\|y_k\|}\right)\right|^p\right)^{1/p}
\end{aligned}$$

$$\leq \sum_{k=1}^{\infty}\left(\sum_{j=1}^{m}\left|\|a_{1k}\|\frac{a_{1k}}{\|a_{1k}\|}(x_{1j})\|a_{2k}\|\frac{a_{2k}}{\|a_{2k}\|}(x_{2j})\|y_k\|b_j\left(\frac{y_k}{\|y_k\|}\right)\right|^p\right)^{1/p}$$

$$\leq \sum_{k=1}^{\infty}\left(\|a_{1k}\|\,\|a_{2k}\|\,\|y_k\|\left(\sum_{j=1}^{m}\left|\frac{a_{1k}}{\|a_{1k}\|}(x_{1j})\frac{a_{2k}}{\|a_{2k}\|}(x_{2j})b_j\left(\frac{y_k}{\|y_k\|}\right)\right|^p\right)^{1/p}\right)$$

$$\leq \left(\sum_{k=1}^{\infty}\|a_{1k}\|\,\|a_{2k}\|\,\|y_k\|\right)\sup_{\substack{\|a_i\|\leq 1\\ \|y\|\leq 1}}\left(\sum_{j=1}^{m}|a_1(x_{1j})a_2(x_{2j})b_j(y)|^p\right)^{1/p}.$$

Thus S is $\tau(p)$-summing and $\|S\|_{\tau(p)} \leq \|S\|_N$.

Example 5.3. Recall that an n-linear mapping $S \in \mathscr{L}(E_1,\dots,E_n;F)$ is *p-semi-integral* (see [1], p. 10) if there exist a constant $\sigma \geq 0$ and a regular probability measure $\mu \in W(B_{E_1'} \times \cdots \times B_{E_n'})$ such that

$$\|S(x_1,\dots,x_n)\| \leq \sigma\left(\int_{B_{E_1'}\times\cdots\times B_{E_n'}} |a_1(x_1)\dots a_n(x_n)|^p\, d\mu(a_1,\dots,a_n)\right)^{1/p}$$

for all $x_i \in E_i$. We write $S \in \mathscr{L}_{\mathrm{si}(p)}(E_1,\dots,E_n;F)$ and its norm is given by $\|S\|_{\mathrm{si}(p)} = \inf\sigma$, where the infimum is taken on the above inequality. Let v be a regular probability measure over $B_{E_1'} \times \cdots \times B_{E_n'}$ such that $v(C) = \mu(C \times B_{F''})$ for each C Borel subset of $B_{E_1'} \times \cdots \times B_{E_n'}$. If $S \in \mathscr{L}_{\tau(p)}(E_1,\dots,E_n;F)$, then by 3.5,

$$\begin{aligned}
&|bS(x_1,\dots,x_n)|\\
&\quad\leq \|S\|_{\tau(p)}\left(\int_{B_{E_1'}\times\cdots\times B_{E_n'}\times B_{F''}} |a_1(x_1)\dots a_n(x_n)\beta(b)|^p\, d\mu(a_1,\dots,a_n,\beta)\right)^{1/p}\\
&\quad\leq \|S\|_{\tau(p)}\left(\int_{B_{E_1'}\times\cdots\times B_{E_n'}\times B_{F''}} |a_1(x_1)\dots a_n(x_n)|^p\|\beta\|^p\|b\|^p\, d\mu(a_1,\dots,a_n,\beta)\right)^{1/p}\\
&\quad\leq \|b\|\,\|S\|_{\tau(p)}\left(\int_{B_{E_1'}\times\cdots\times B_{E_n'}} |a_1(x_1)\dots a_n(x_n)|^p\, dv(a_1,\dots,a_n)\right)^{1/p},
\end{aligned}$$

for all $x_i \in E_i$. So

$$\begin{aligned}
\|S(x_1,\dots,x_n)\| &= \sup_{\|b\|\leq 1}|bS(x_1,\dots,x_n)|\\
&\leq \|S\|_{\tau(p)}\left(\int_{B_{E_1'}\times\cdots\times B_{E_n'}} |a_1(x_1)\dots a_n(x_n)|^p\, dv(a_1,\dots,a_n)\right)^{1/p}
\end{aligned}$$

for all $x_i \in E_i$, i.e., S is p-semi-integral and $\|S\|_{\mathrm{si}(p)} \leq \|S\|_{\tau(p)}$. Observe that if $F = \mathbb{K}$, we also have that $\mathscr{L}_{\mathrm{si}(p)}(E_1,\dots,E_n;\mathbb{K}) \subseteq \mathscr{L}_{\tau(p)}(E_1,\dots,E_n;\mathbb{K})$: If $\delta(1) \in W(B_{\mathbb{K}})$ stands for the Dirac measure at $1 \in B_{\mathbb{K}''}$, then

$$
\begin{aligned}
&|bS(x_1,\dots,x_n)|\\
&\le \|b\|\,\|S\|_{\mathrm{si}(p)}\Big(\int_{B_{E_1'}\times\cdots\times B_{E_n'}} |a_1(x_1)\dots a_n(x_n)|^p\,dv(a_1,\dots,a_n)\Big)^{1/p}\\
&\le \Big(\int_{B_{\mathbb{K}''}} |\beta(b)|^p\,d\delta(1)(\beta)\Big)^{1/p}\|S\|_{\mathrm{si}(p)}\\
&\quad\cdot\Big(\int_{B_{E_1'}\times\cdots E_n'} |a_1(x_1)\dots a_n(x_n)|^p\,dv(a_1,\dots,a_n)\Big)^{1/p}\\
&\le \|S\|_{\mathrm{si}(p)}\Big(\int_{B_{E_1'}\times\cdots\times B_{E_n'}\times B_{\mathbb{K}''}} |a_1(x_1)\dots a_n(x_n)|^p|\beta(b)|^p\,d(v\times\delta(1))(a_1,\dots,a_n,\beta)\Big)^{1/p}.
\end{aligned}
$$

Since $(v\times\delta(1))\in W(B_{E_1'}\times\cdots\times B_{E_n'}\times B_{\mathbb{K}''})$ it follows that S is $\tau(p)$-summing and $\|S\|_{\mathrm{si}(p)}=\|S\|_{\tau(p)}$.

Example 5.4. A n-linear mapping S is p-dominated if and only if there exist a constant $\sigma\ge 0$ and probability measures $\mu_i\in W(B_{E_i'})$ $(i=1,\dots,n)$ such that

$$
\|S(x_1,\dots,x_n)\|\le\sigma\Big(\int_{B_{E_1'}}\cdots\int_{B_{E_n'}} |a_1(x_1)\dots a_n(x_n)|^p\,d\mu_n(a_n)\dots d\mu_1(a_1)\Big)^{1/p}
$$

for all $x_1\in E_1,\dots,x_n\in E_n$, in which case $\|S\|_{d(p)}=\inf\sigma$ (see 3.2 in [3], p. 12). Let $\mu_i\in W(B_{E_i'})$ and $\mu_{n+1}\in W(B_{F''})$ be regular probability measures over $B_{E_i'}$ $(i=1,\dots,n)$ and $B_{F''}$, respectively. If $S\in\mathscr{L}_{\tau(p)}(E_1,\dots,E_n;F)$, then by 2

$$
\begin{aligned}
&|bS(x_1,\dots,x_n)|\\
&\le \|S\|_{\tau(p)}\Big(\int_{B_{E_1'}}\cdots\int_{B_{E_n'}}\int_{B_{F''}} |a_1(x_1)\dots a_n(x_n)\beta(b)|^p\,d\mu_{n+1}(\beta)\,d\mu_n(a_n)\dots d\mu_1(a_1)\Big)^{1/p}\\
&\le \|S\|_{\tau(p)}\Big(\int_{B_{E_1'}}\cdots\int_{B_{E_n'}}\int_{B_{F''}} |a_1(x_1)\dots a_n(x_n)\|\beta\|\,\|b\||^p\,d\mu_{n+1}(\beta)\\
&\qquad\cdot d\mu_n(a_n)\dots d\mu_1(a_1)\Big)^{1/p}\\
&\le \|b\|\,\|S\|_{\tau(p)}\Big(\int_{B_{E_1'}}\cdots\int_{B_{E_n'}} |a_1(x_1)\dots a_n(x_n)|^p\,d\mu_n(a_n)\cdots d\mu_1(a_1)\Big)^{1/p}
\end{aligned}
$$

for all $x_i\in E_i$. So

$$
\begin{aligned}
\|S(x_1,\dots,x_n)\| &= \sup_{\|b\|\le 1}|bS(x_1,\dots,x_n)|\\
&\le \|S\|_{\tau(p)}\Big(\int_{B_{E_1'}}\cdots\int_{B_{E_n'}} |a_1(x_1)\dots a_n(x_n)|^p\,d\mu_n(a_n)\dots d\mu_1(a_1)\Big)^{1/p}
\end{aligned}
$$

for all $x_i \in E_i$, i.e., S is p-dominated with $\|S\|_{d(p)} \le \|S\|_{\tau(p)}$. Observe that if $F = \mathbb{K}$, we also have that $\mathscr{L}_{d(p)}(E_1, \dots, E_n; \mathbb{K}) \subseteq \mathscr{L}_{\tau(p)}(E_1, \dots, E_n; \mathbb{K})$: If $\delta(1) \in W(B_{\mathbb{K}''})$ stands for the Dirac measure at $1 \in B_{\mathbb{K}''}$, then

$$\begin{aligned}
&|bS(x_1, \dots, x_n)| \\
&\le \|b\| \, \|S\|_{d(p)} \Big(\int_{B_{E_1'}} \dots \int_{B_{E_n'}} |a_1(x_1) \dots a_n(x_n)|^p \, d\mu_n(a_n) \dots d\mu_1(a_1) \Big)^{1/p} \\
&\le \Big(\int_{B_{\mathbb{K}''}} |\beta(b)|^p \, d\delta(1)(\beta) \Big)^{1/p} \|S\|_{d(p)} \\
&\quad \cdot \Big(\int_{B_{E_1'}} \dots \int_{B_{E_n'}} |a_1(x_1) \dots a_n(x_n)|^p \, d\mu_n(a_n) \dots d\mu_1(a_1) \Big)^{1/p} \\
&\le \|S\|_{d(p)} \Big(\int_{B_{E_1'}} \dots \int_{B_{E_n'}} \int_{B_{\mathbb{K}''}} |a_1(x_1) \dots a_n(x_n) \beta(b)|^p \, d\delta(1)(\beta) \, d\mu_n(a_n) \dots d\mu_1(a_1) \Big)^{1/p}.
\end{aligned}$$

By 3.6, S is $\tau(p)$-summing and $\|S\|_{d(p)} = \|S\|_{\tau(p)}$. In other words, $S \in \mathscr{L}(E_1, \dots, E_n; \mathbb{K})$ is p-dominated if and only if it is $\tau(p)$-summing.

Example 5.5. Recall that a mapping $S \in \mathscr{L}(E_1, E_2; F)$ is said to be approximable if there exists a sequence $S_k \in \mathscr{L}_f(E_1, E_2; F)$ such that for all $\varepsilon > 0$ there is K_ε such that $\|S - S_k\| \le \varepsilon$ whenever $k \ge K_\varepsilon$. For $1 \le q \le p$, by 5.1, for each $k = 1, 2, \dots$ there exist constants σ_k such that

$$\Big(\sum_{j=1}^m |b_j S_k(x_{1j}, x_{2j})|^p \Big)^{1/p} \le \sigma_k \sup_{\substack{\|a_i\| \le 1 \\ \|y\| \le 1}} \Big(\sum_{j=1}^m |a_1(x_{1j}) a_2(x_{2j}) b_j(y)|^q \Big)^{1/q}$$

(for all $k = 1, 2, \dots$) holds for all $m \in \mathbb{N}$, $x_{ij} \in E_i$, $b_j \in F'$ and $j = 1, 2, \dots, m$. Given ε, choose $k \ge K_\varepsilon$ to obtain that

$$\begin{aligned}
&\Big(\sum_{j=1}^m |b_j S(x_{1j}, x_{2j})|^p \Big)^{1/p} \\
&\qquad \le \Big(\sum_{j=1}^m |b_j (S - S_k)(x_{1j}, x_{2j})|^p \Big)^{1/p} + \Big(\sum_{j=1}^m |b_j S_k(x_{1j}, x_{2j})|^p \Big)^{1/p} \\
&\qquad \le \Big(\sum_{j=1}^m (\|b_j\| \, \|S - S_k\| \, \|x_{1j}\| \, \|x_{2j}\|)^p \Big)^{1/p} + \Big(\sum_{j=1}^m |b_j S_k(x_{1j}, x_{2j})|^p \Big)^{1/p} \\
&\qquad \le \varepsilon \Big(\sum_{j=1}^m (\|b_j\| \, \|x_{1j}\| \, \|x_{2j}\|)^p \Big)^{1/p} + \sigma_k \sup_{\substack{\|a_i\| \le 1 \\ \|y\| \le 1}} \Big(\sum_{j=1}^m |a_1(x_{1j}) a_2(x_{2j}) b_j(y)|^q \Big)^{1/q}.
\end{aligned}$$

If the limit $\lim_{k\to\infty} \sigma_k$ exists and is finite, it follows that S is $\tau(p;q)$-summing, and $\|S\|_{\tau(p;q)} \leq \lim_{k\to\infty} \sigma_k$.

Analogous examples can be given for polynomials.

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