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An exact multiplicity result for a class of symmetric problems

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Abstract. We consider positive solutions of a class of semilinear problems

$$u'' + \lambda a(x) f(u) = 0, \qquad -1 < x < 1, \qquad u(-1) = u(1) = 0,$$

with even and positive $a(x)$, depending on a positive parameter λ. In case $f(u)$ is convex, an exact multiplicity result was given in P. Korman, Y. Li and T. Ouyang [6]; see also P. Korman [4] for the details. It was observed by P. Korman and J. Shi [7] that convexity requirement can be relaxed for large u (see also [5]). We show that convexity requirement can also be relaxed for small u.

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We consider positive solutions of a class of non-autonomous problems

$$u'' + \lambda a(x) f(u) = 0, \qquad -1 < x < 1, \qquad u(-1) = u(1) = 0, \tag{1}$$

with even and positive $a(x)$, depending on a positive parameter λ. We assume that the function $a(x) \in C^1(-1,1) \cap C[-1,1]$ satisfies

$$a(x) > 0, \qquad a(-x) = a(x), \qquad a'(x) < 0 \qquad \text{for } x \in (0,1), \tag{2}$$

while $f(u) \in C^2(\bar{R}_+)$ satisfies

$$f(u) > 0 \qquad \text{for } u \geq 0. \tag{3}$$

We study *exactly* how many solutions the problem (1) has, and how these solutions are connected when λ is varied.

It follows from B. Gidas, W.-M. Ni and L. Nirenberg [3] that under these conditions any positive solution of (1) is an even function, with $u'(x) < 0$ for $x > 0$

(i.e., $u(0)$ gives the maximum value of the solution). It is also known that in this case the problem (1) has properties similar to those of autonomous problems; see a recent review paper [4]. In particular, any non-trivial solution of the corresponding linearized problem

$$w'' + \lambda a(x) f'(u) w = 0, \qquad -1 < x < 1, \qquad w(-1) = w(1) = 0, \tag{4}$$

is an even function of one sign, so that we may assume that $w(x) > 0$ on $(-1, 1)$. It is also known that as one varies λ, solutions of (1) lie on smooth solution curves, which admit only simple turns at singular solutions (i.e., solutions $\big(\lambda, u(x)\big)$ of (1), at which the problem (4) admits non-trivial solutions).

We now make a further assumption on $f(u)$. We assume there is an $\gamma > 0$, so that

$$f'(u) < 0 \quad \text{for } 0 < u < \gamma, \qquad f'(u) > 0 \quad \text{and} \quad f''(u) > 0 \quad \text{for } u > \gamma. \tag{5}$$

(Observe that we do not restrict convexity on the interval $(0, \gamma)$.)

Lemma 1. *Under the assumption* (5), *for any singular solution $u(x)$ and the corresponding non-trivial solution of* (4) *one has*

$$p(x) \equiv 3u'(x)w'(x) - u''(x)w(x) > 0 \qquad \textit{for all } x \in [0, 1].$$

Proof. It follows by maximum principle that (4) cannot have a non-trivial solution in the region where $f'(u) < 0$, i.e. $u(0) > \gamma$ at any singular solution. Since $w(x) > 0$, we see from the equation (4) that $w(x)$ changes concavity exactly once on $(0, 1)$, being concave for small x and convex near $x = 1$. It follows that $w(x)$ is a non-increasing function, i.e., $w'(x) \le 0$ for all $x \in (0, 1)$. But then $p(x) = 3u'(x)w'(x) + \lambda a(x) f\big(u(x)\big) w(x) > 0$. □

Theorem 1. *For the problem* (1) *assume that the conditions* (2), (3) *and* (5) *hold. Then the problem* (1) *has at most two positive solutions, and moreover all solutions lie on a unique smooth solution curve, starting at* $(\lambda = 0, u = 0)$. *This curve makes at most one turn, and it tends to infinity at some $\bar{\lambda} \ge 0$. If, in addition, we assume that*

$$\lim_{u \to \infty} \frac{f(u)}{u} = \infty, \tag{6}$$

then this curve of solutions makes exactly one turn at some $\lambda = \lambda_0$, and it tends to infinity as $\lambda \downarrow 0$, so that for $0 < \lambda < \lambda_0$ the problem (1) *has exactly two strictly ordered positive solutions, it has exactly one positive solution at $\lambda = \lambda_0$, and none for $\lambda > \lambda_0$.*

Proof. By the implicit function theorem there is a curve of positive solutions, starting at $(\lambda = 0, u = 0)$. We can continue this curve for increasing λ by using either the implicit function theorem or the Crandall–Rabinowitz bifurcation theorem [2]; see [4] for the details. Since $f(u)$ is increasing and convex for large u, it follows that it grows at least linearly, and then the Sturm comparison theorem implies that this curve cannot extend for all $\lambda > 0$, so that it will either go to infinity at a finite $\bar{\lambda}$, or it will turn left eventually. If condition (6) holds, then the second possibility must be true, again by the Sturm comparison theorem.

The key is to show that a turn to the left in the $(\lambda, u(0))$ plane occurs at any turning point. This will imply that there is only one turning point, and the theorem will follow.

Let $u(x)$ be a singular solution of (1). We denote by $\alpha = u(0)$ its maximal value. By maximum principle $\alpha > \gamma$. It then follows by our assumption (5) that

$$f'(u(x)) - f'(\alpha) < 0 \qquad \text{for all } x \in (0,1). \tag{7}$$

The direction of turn at any critical solution of (1) is governed by the sign of the integral $I \equiv \int_0^1 a(x) f''(u) w^3 \, dx$. We show next that this integral is positive, which implies that a turn to the left must occur at any singular solution; see, e.g., [4]. Indeed

$$\begin{aligned} I &= \int_0^1 \frac{d}{dx} [f'(u) - f'(\alpha)] \frac{a(x) w^3(x)}{u'(x)} \, dx \\ &= [f'(u) - f'(\alpha)] \frac{a w^3}{u'} \bigg|_0^1 - \int_0^1 [f'(u) - f'(\alpha)] \frac{a' u' w^3 + a w^2 p}{u'^2} \, dx. \end{aligned} \tag{8}$$

The first term on the right vanishes, since

$$\lim_{x \to 0} \frac{f'(u) - f'(\alpha)}{u'} = \lim_{x \to 0} f''(u) u' u'' = 0, \tag{9}$$

while the second term is positive by (7) and Lemma 1. (Observe that the integrand in the second term is bounded, as seen by a computation similar to (9).) □

We remark that our result might be new even for constant $a(x)$, although in that case it can probably be obtained by the time-map method; see, e.g., I. Addou and S.-H. Wang [1].

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