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Bivariate classical and q-series transformations

Wenchang Chu and Cangzhi Jia

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Abstract. By means of bivariate inverse series relations, we review several bivariate classical hypergeometric series transformation formulae and establish their q-analogues.

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1. Introduction and motivation

For a complex number x and a natural number n, denote the shifted factorial by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\dots(x+n-1) \quad \text{with } n = 1, 2, \dots.$$

Following Bailey [1], the hypergeometric series in variable z is defined by

$$ {}_{1+\ell}F_\ell\left[\begin{matrix} a_0, a_1, \dots, a_\ell \\ b_1, \dots, b_\ell \end{matrix}\middle| z\right] = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \dots (a_\ell)_n}{n! (b_1)_n \dots (b_\ell)_n} z^n,$$

where $\{a_i\}$ and $\{b_j\}$ are complex parameters such that no zero factors appear in the denominators of the summands on the right-hand side. If one of the numerator parameters $\{a_k\}$ is a negative integer, then the series terminates, which reduces to a polynomial in z.

Similarly, we have the basic hypergeometric series, called q-series briefly. According to Bailey [1] and Slater [9], it reads as

$$ {}_{1+\ell}\Phi_\ell\left[\begin{matrix} a_0, a_1, \dots, a_\ell \\ b_1, \dots, b_\ell \end{matrix}\middle| q; z\right] = \sum_{n=0}^{\infty} \frac{(a_0;q)_n (a_1;q)_n \dots (a_\ell;q)_n}{(q;q)_n (b_1;q)_n \dots (b_\ell;q)_n} z^n,$$

where the q-shifted factorial is defined by

$$(x;q)_0 = 1 \qquad \text{and} \qquad (x;q)_n = \prod_{k=0}^{n-1}(1-q^k x) \qquad \text{with } n = 1, 2, \ldots .$$

For the sake of brevity, we write

$$[a, b, \ldots, c; q]_n := (a;q)_n (b;q)_n \ldots (c;q)_n.$$

The ${}_{1+\ell}\Phi_\ell$-series is well defined, provided that no zero factors appear in the denominator on the right-hand side, i.e., none of the denominator parameters $\{b_k\}_{k=1}^{\ell}$ has the form q^{-m} with $m \in \mathbb{N}_0$. When a_i and b_j are replaced respectively by their q-exponential functions q^{a_i} and q^{b_j}, then the ${}_{1+\ell}\Phi_\ell$-series will become the classical ${}_{1+\ell}F_\ell$-series as $q \to 1$ under term by term limit.

From their work on integrals involving products of Laguerre polynomials, Lee et al. [6], Eq. 39, found the following interesting bivariate hypergeometric series transformation associated with the Kampé de Fériet function:

$$\sum_{i,j=0}^{\infty} \frac{(a)_{i+j}(b)_{i+j}}{(a+b)_{i+j}} \frac{(c+c'-1)_{i+j}}{(c)_i (c')_j} \frac{\{x(1-y)\}^i}{i!} \frac{\{y(1-x)\}^j}{j!}$$
$$= \sum_{i,j=0}^{\infty} \frac{(c+c'-1)_{i+j}}{(a+b)_{i+j}} \frac{(a)_i (b)_i}{(c)_i} \frac{(a)_j (b)_j}{(c')_j} \frac{x^i}{i!} \frac{y^j}{j!}.$$

A detailed proof can be found in Karlsson et al. [5], §2, where several transformation and reduction formulae are derived by means of integral representations for the Kampé de Fériet function. Most of these results have subsequently been reviewed through a combination of the formal power series method and a series rearrangement by Chu and Srivastava [3], who succeeded also in establishing q-analogues. In the same paper, Chu and Srivastava [3], Theorem 2, generalized a fundamental result due to Karlsson et al. [5], Eq. 1.12, with two extra parameters, but failed to figure out its q-analogue with the same approach.

Motivated by this intriguing fact, we find that the bivariate inverse series relations can be employed not only to establish the desired q-analogue, but also to derive several other transformations for bivariate hypergeometric series. The main body of the paper will be divided into two parts. In the next section, we shall present new proofs of four main theorems due to Chu–Srivastava [3], Theorem 2, Lievens–Jeugt [7], Eqs. 11a and 11b, and Singh [8], Eq. 1.2. Then their q-analogues will be presented in the last section.

2. Bivariate hypergeometric series transformations

We first consider transformation formulae for bivariate hypergeometric series with the help of the double inverse series relations [2]

$$f(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{m}{i}\binom{n}{j}g(i,j), \tag{1a}$$

$$g(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{m}{i}\binom{n}{j}f(i,j), \tag{1b}$$

which can be easily verified with the help of the binomial theorem.

2.1. Generalizing the transformation formula of Kampé de Fériet functions due to Karlsson et al. [5], Eq. 1.12, Chu and Srivastava found the following result.

Theorem 1 (Chu–Srivastava [3], Theorem 2).

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(a)_{i+j}(b)_{i+j}(c+c'-1)_{i+j}}{(a+b)_{i+j}(e)_{i+j}(e')_{i+j}}\frac{(-m)_i(e'+n)_i}{i!(c)_i}\frac{(-n)_j(e+m)_j}{j!(c')_j} \tag{2a}$$

$$= \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(c+c'-1)_{i+j}}{(a+b)_{i+j}}\frac{(-m)_i(a)_i(b)_i}{i!(c)_i(e)_i}\frac{(-n)_j(a)_j(b)_j}{j!(c')_j(e')_j}. \tag{2b}$$

This theorem can be shown alternatively by applying the double inverse series relations to the following closed double sum. The interested reader may write down the details as an exercise following the procedure exhibited in Sections 2.2–2.4.

Lemma 2 (Closed double sum).

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m-i-j-1}{m-i}\binom{n-i-j-1}{n-j}\frac{(a)_{i+j}(b)_{i+j}}{(a+b)_{i+j}}\frac{(c+c'-1)_{i+j}}{i!(c)_i j!(c')_j}$$
$$= \frac{(a)_m(b)_m}{m!(c)_m}\frac{(a)_n(b)_n}{n!(c')_n}\frac{(c+c'-1)_{m+n}}{(a+b)_{m+n}}.$$

Sketch of the proof. For the double sum displayed in the lemma, we can first invert the summation order by the replacements $i \to m-i$ and $j \to n-j$, then reformulate it by letting $k := i+j$, and finally reduce it to the closed expression by appealing successively to the Chu–Vandermonde summation theorem (cf. Bailey [1], §1.3) and the Pfaff–Saalschütz summation formula (cf. Bailey [1], §2.2). □

2.2. By using twice the Chu–Vandermonde summation theorem, it is not difficult to evaluate the following double sum

$$\sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}\binom{m}{r}\binom{n}{s}\frac{(d-b)_r(b)_s}{(d)_{r+s}}=\frac{(b)_m(d-b)_n}{(d)_{m+n}}.$$

Multiplying both sides by $\frac{(a)_m(a')_n}{(c)_m(c')_n}$, we may reformulate it as follows:

$$\frac{(a)_m(b)_m(a')_n(d-b)_n}{(d)_{m+n}(c)_m(c')_n}=\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r(a)_r(d-b)_r}{r!(c)_r(d)_{r+s}}\frac{(-n)_s(a')_s(b)_s}{s!(c')_s}\cdot\frac{(a+r)_{m-r}(a'+s)_{n-s}}{(c+r)_{m-r}(c'+s)_{n-s}}.$$

Replacing the last factorial fraction by the Chu–Vandermonde sums

$$\frac{(a+r)_{m-r}(a'+s)_{n-s}}{(c+r)_{m-r}(c'+s)_{n-s}}={}_2F_1\left[\begin{matrix}-m+r,c-a\\c+r\end{matrix}\middle|1\right]{}_2F_1\left[\begin{matrix}-n+s,c'-a'\\c'+s\end{matrix}\middle|1\right]$$
$$=\sum_{i=0}^{m-r}\sum_{j=0}^{n-s}\frac{(-m+r)_i(c-a)_i}{i!(c+r)_i}\frac{(-n+s)_j(c'-a')_j}{j!(c'+s)_j}$$

and then interchanging the summation order, we get

$$\frac{(a)_m(b)_m(a')_n(d-b)_n}{(d)_{m+n}(c)_m(c')_n}$$
$$=\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{m}{i}\binom{n}{j}\frac{(c-a)_i(c'-a')_j}{(c)_i(c')_j}$$
$$\cdot\sum_{r=0}^{i}\sum_{s=0}^{j}(-1)^{r+s}\binom{i}{r}\binom{j}{s}\frac{(a)_r(d-b)_r(a')_s(b)_s}{(d)_{r+s}(1+a-c-i)_r(1+a'-c'-j)_s}.$$

In view of (1a)–(1b), its dual relation recovers the following result.

Theorem 3 (Transformation formula, [8], Eq. 1.2).

$$\frac{(c)_m(c')_n}{(c-a)_m(c'-a')_n}\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(-m)_i(a)_i(b)_i}{(d)_{i+j}i!(c)_i}\frac{(-n)_j(a')_j(d-b)_j}{j!(c')_j}$$
$$=\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(-m)_i(a)_i(d-b)_i}{(d)_{i+j}i!(1+a-c-m)_i}\frac{(-n)_j(a')_j(b)_j}{j!(1+a'-c'-n)_j}.$$

2.3. Similarly, we have the following almost trivial double summation formula:

$$\sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}\binom{m}{r}\binom{n}{s}\frac{(c-d-n)_r(b'-a')_s}{(c)_r(b')_s}=\frac{(d+n)_m(a')_n}{(c)_m(b')_n}.$$

Then multiplying across by $\frac{(a)_m(d)_n}{(b)_m(1+d-c)_n}$, we may restate it as follows:

$$\frac{(a)_m(a')_n(d)_{m+n}}{(b)_m(c)_m(b')_n(1+d-c)_n}=\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r(a)_r}{r!(b)_r(c)_r}\frac{(-n)_s(d)_s(b'-a')_s}{s!(b')_s}\cdot\frac{(a+r)_{m-r}(d+s)_{n-s}(c-d-n)_r}{(b+r)_{m-r}(1+d-c)_n}.$$

Replacing the last fraction by the following double sum

$$\begin{aligned}&\frac{(a+r)_{m-r}(d+s)_{n-s}(c-d-n)_r}{(b+r)_{m-r}(1+d-c)_n}\\&=\frac{(c-d-s)_r}{(1+d-c)_s}\,{}_2F_1\left[\begin{matrix}-m+r,b-a\\b+r\end{matrix}\middle|1\right]{}_2F_1\left[\begin{matrix}-n+s,1-c-r\\1+d-c+s-r\end{matrix}\middle|1\right]\\&=\frac{(c-d-s)_r}{(1+d-c)_s}\sum_{i=0}^{m-r}\sum_{j=0}^{n-s}\frac{(-m+r)_i(b-a)_i}{i!(b+r)_i}\frac{(-n+s)_j(1-c-r)_j}{j!(1+d-c+s-r)_j}\end{aligned}$$

and then interchanging the summation order, we get

$$\frac{(a)_m(a')_n(d)_{m+n}}{(b)_m(c)_m(b')_n(1+d-c)_n}=\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{m}{i}\binom{n}{j}\frac{(b-a)_i(1-c)_j}{(b)_i(1+d-c)_j}\cdot\sum_{r=0}^{i}\sum_{s=0}^{j}\frac{(-i)_r(a)_r(c-d-j)_r(-j)_s(b'-a')_s(d)_s}{(c-j)_{r+s}r!(1+a-b-i)_r s!(b')_s}.$$

According to (1a)–(1b), its dual relation results in the following theorem.

Theorem 4 (Transformation formula, Lievens–Jeugt [7], Eq. 11a).

$$\begin{aligned}&\frac{(b)_m(1+d-c)_n}{(b-a)_m(1-c)_n}\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(d)_{i+j}(-m)_i(a)_i}{i!(b)_i(c)_i}\frac{(-n)_j(a')_j}{j!(b')_j(1+d-c)_j}\\&=\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(-m)_i(a)_i(c-d-n)_i}{(c-n)_{i+j}(1+a-b-m)_i i!}\frac{(-n)_j(b'-a')_j(d)_j}{j!(b')_j}.\end{aligned}$$

We remark that the reversal of this double series gives another transformation due to Lievens–Jeugt [7], Eq. 12a.

2.4. For an arbitrary sequence $\{\Omega_k\}$, there holds the almost trivial relation

$$\frac{(d)_m}{(c)_m}\Omega(m)=\sum_{r=0}^{m}\binom{m}{r}\frac{(d)_r}{(c)_r}\Omega(r)\delta_{r,m}=\sum_{r=0}^{m}\binom{m}{r}\frac{(d)_r}{(c)_r}\Omega(r)\sum_{i=0}^{m-r}(-1)^i\binom{m-r}{i},$$

where $\delta_{i,j}$ stands for the usual Kronecker symbol. Applying twice the Chu–Vander-monde summation formula, it is not hard to check the double sum identity

$$\frac{(d+m)_n(a')_n}{(1+d-c)_n(b')_n}=\sum_{s=0}^{n}\sum_{j=0}^{n-s}\frac{(-n)_{s+j}(b'-a')_s(d+m)_s(1-m-c)_j}{s!j!(b')_s(1+d-c)_{s+j}}.$$

Multiplying both equations, we get the expression

$$\begin{aligned}\frac{(a')_n(d)_{m+n}}{(c)_m(1+d-c)_n(b')_n}\Omega(m)&=\sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^r\frac{(d)_{r+s}(-m)_r}{r!(c)_r}\frac{(-n)_s(b'-a')_s}{s!(b')_s(1+d-c)_s}\Omega(r)\\&\quad\cdot\sum_{i=0}^{m-r}\sum_{j=0}^{n-s}\frac{(-m+r)_i}{i!}\frac{(-n+s)_j(1-r-c)_j}{j!(1+s+d-c)_j}.\end{aligned}$$

Interchanging the summation order, we can rewrite the last equality as

$$\begin{aligned}\frac{(a')_n(d)_{m+n}}{(c)_m(1+d-c)_n(b')_n}\Omega(m)&=\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{m}{i}\binom{n}{j}\frac{(1-c)_j}{(1+d-c)_j}\\&\quad\cdot\sum_{r=0}^{i}\sum_{s=0}^{j}(-1)^{r+s}\binom{i}{r}\binom{j}{s}\frac{(d)_{r+s}(b'-a')_s}{(c-j)_{r+s}(b')_s}\Omega(r).\end{aligned}$$

Its dual relation through (1a)–(1b) reads as the following general theorem.

Theorem 5 (New transformation formula).

$$\begin{aligned}&\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(d)_{i+j}(-m)_i}{i!(c)_i}\frac{(-n)_j(a')_j}{j!(b')_j(1+d-c)_j}\Omega(i)\\&\quad=\frac{(1-c)_n}{(1+d-c)_n}\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(d)_{i+j}}{(c-n)_{i+j}}\frac{(-m)_i}{i!}\frac{(-n)_j(b'-a')_j}{j!(b')_j}\Omega(i).\end{aligned}$$

The transformation due to Lievens–Jeugt [7], Eq. 11b, results in the very special case $\Omega(m)=(a)_m/(b)_m$ of this theorem.

3. Transformation formulae for double q-series

By means of Euler's q-binomial theorem (cf. [4], II-4), we can prove the following q-analogue for the inverse series relations displayed in (1a)–(1b):

$$F(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\begin{bmatrix} m \\ i \end{bmatrix}\begin{bmatrix} n \\ j \end{bmatrix} q^{\binom{m-i}{2}} G(i,j), \tag{3a}$$

$$G(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\begin{bmatrix} m \\ i \end{bmatrix}\begin{bmatrix} n \\ j \end{bmatrix} q^{\binom{n-j}{2}} F(i,j). \tag{3b}$$

For the subsequent applications, we need also the slightly varied form with the alternating factors and the q-exponents being migrated:

$$F(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{m+n-i-j}\begin{bmatrix} m \\ i \end{bmatrix}\begin{bmatrix} n \\ j \end{bmatrix} G(i,j), \tag{4a}$$

$$G(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n} q^{\binom{m-i}{2}+\binom{n-j}{2}}\begin{bmatrix} m \\ i \end{bmatrix}\begin{bmatrix} n \\ j \end{bmatrix} F(i,j). \tag{4b}$$

Now we are ready to derive transformation formulae for the terminating bivariate basic hypergeometric series.

3.1. In order to establish the q-analogue of (2a)–(2b), we first prove the following bivariate summation formula.

Lemma 6 (Closed q-double sum).

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\begin{bmatrix} m-i-j-1 \\ m-i \end{bmatrix}\begin{bmatrix} n-i-j-1 \\ n-j \end{bmatrix}\frac{q^{i(n-2j)}(q/c)^j}{(q;q)_i(q;q)_j}\frac{[a,b;q]_{i+j}}{(ab;q)_{i+j}}\frac{(cc'/q;q)_{i+j}}{(c;q)_i(c';q)_j} \tag{5a}$$

$$= \left(\frac{q}{c}\right)^n \frac{[a,b;q]_m}{[q,c;q]_m}\frac{[a,b;q]_n}{[q,c';q]_n}\frac{(cc'/q;q)_{m+n}}{(ab;q)_{m+n}}. \tag{5b}$$

Proof. With i and j being replaced by $m-i$ and $n-j$ respectively, the reversal of (5a) may be restated as follows:

$$\text{Eq. (5a)} = \frac{q^{-mn}(q/c)^n}{(q;q)_m(q;q)_n}\frac{[a,b;q]_{m+n}}{(ab;q)_{m+n}}\frac{(cc'/q;q)_{m+n}}{(c;q)_m(c';q)_n}$$

$$\cdot\sum_{i,j\geq 0}(q^{2-n}/c')^i\frac{(q^{1-m}/c;q)_i(q^{1-n}/c';q)_j}{(q;q)_i(q;q)_j}q^j$$

$$\cdot\frac{[q^{-m},q^{-n},q^{1-m-n}/ab;q]_{i+j}}{[q^{1-m-n}/a,q^{1-m-n}/b,q^{2-m-n}/cc';q]_{i+j}}.$$

Letting $k := i + j$, the last double sum may be simplified successively by means of the q-Gauss and the q-Saalschütz summation formulae (cf. [4], II-6 and II-12) as follows:

$$\sum_{k\geq 0} q^k \frac{(q^{1-n}/c';q)_k}{(q^{2-m-n}/cc';q)_k} {}_2\Phi_1\left[\begin{matrix} q^{-k}, q^{1-m}/c \\ c'q^{n-k} \end{matrix}\middle| q;q\right] \frac{[q^{-m},q^{-n},q^{1-m-n}/ab;q]_k}{[q,q^{1-m-n}/a,q^{1-m-n}/b;q]_k}$$
$$= {}_3\Phi_2\left[\begin{matrix} q^{-m}, & q^{-n}, & q^{1-m-n}/ab \\ & q^{1-m-n}/a, & q^{1-m-n}/b \end{matrix}\middle| q;q\right] = \frac{(a;q)_m(b;q)_m}{(q^n a;q)_m(q^n b;q)_m} q^{mn}.$$

Therefore we have the following expression

$$\text{Eq. (5a)} = \left(\frac{q}{c}\right)^n \frac{[a,b;q]_m}{[q,c;q]_m} \frac{[a,b;q]_n}{[q,c';q]_n} \frac{(cc'/q;q)_{m+n}}{(ab;q)_{m+n}} = \text{Eq. (5b)},$$

which completes the proof of the lemma. □

Now rewrite the formula in the lemma as

$$\text{Eq. (5b)} \cdot \frac{(q;q)_m(q;q)_n}{(e;q)_m(e';q)_n}$$
$$= \left(\frac{q}{c}\right)^n \frac{[a,b;q]_m}{[c,e;q]_m} \frac{[a,b;q]_n}{[c',e';q]_n} \frac{(cc'/q;q)_{m+n}}{(ab;q)_{m+n}} = \frac{(q;q)_m(q;q)_n}{(e;q)_m(e';q)_n}$$
$$\cdot \sum_{\imath=0}^{m}\sum_{\jmath=0}^{n} \begin{bmatrix} m-\imath-\jmath-1 \\ m-\imath \end{bmatrix} \begin{bmatrix} n-\imath-\jmath-1 \\ n-\jmath \end{bmatrix}$$
$$\times \frac{q^{\imath(n-2\jmath)}(q/c)^{\jmath}}{(q;q)_\imath(q;q)_\jmath} \frac{[a,b;q]_{\imath+\jmath}}{(ab;q)_{\imath+\jmath}} \frac{(cc'/q;q)_{\imath+\jmath}}{(c;q)_\imath(c';q)_\jmath}$$
$$= \sum_{\imath=0}^{m}\sum_{\jmath=0}^{n} \begin{bmatrix} m \\ \imath \end{bmatrix} \begin{bmatrix} n \\ \jmath \end{bmatrix} \frac{[a,b;q]_{\imath+\jmath}}{(ab;q)_{\imath+\jmath}} \frac{(cc'/q;q)_{\imath+\jmath}}{(c;q)_\imath(c';q)_\jmath}$$
$$\times \frac{q^{\imath(n-2\jmath)}(q/c)^{\jmath}}{(e;q)_\imath(e';q)_\jmath} \frac{(q^{-\jmath};q)_{m-\imath}(q^{-\imath};q)_{n-\jmath}}{(eq^{\imath};q)_{m-\imath}(e'q^{\jmath};q)_{n-\jmath}}.$$

In view of the q-Gauss summation theorem (cf. [4], II-6 and II-7), performing the replacements

$$\frac{(q^{-\jmath};q)_{m-\imath}}{(eq^{\imath};q)_{m-\imath}} = {}_2\Phi_1\left[\begin{matrix} q^{\imath-m}, eq^{\imath+\jmath} \\ eq^{\imath} \end{matrix}\middle| q;q^{m-\imath-\jmath}\right]$$
$$= \sum_{i=\imath}^{m} (-1)^{i-\imath} \begin{bmatrix} m-\imath \\ i-\imath \end{bmatrix} \frac{(eq^{\imath+\jmath};q)_{i-\imath}}{(eq^{\imath};q)_{i-\imath}} q^{\binom{i-\imath}{2}-\jmath(i-\imath)},$$

$$\frac{(q^{-\imath};q)_{n-\jmath}}{(e'q^{\jmath};q)_{n-\jmath}} = {}_2\Phi_1\left[\begin{matrix} q^{\jmath-n}, e'q^{\imath+\jmath} \\ e'q^{\jmath}\end{matrix}\middle| q;q\right](e'q^{\imath+\jmath})^{\jmath-n}$$

$$= \sum_{j=\jmath}^{n}(-1)^{j-\jmath}\begin{bmatrix} n-\jmath \\ j-\jmath\end{bmatrix}\frac{(e'q^{\imath+\jmath};q)_{j-\jmath}}{(e'q^{\jmath};q)_{j-\jmath}}\, q^{\binom{1+j-\jmath}{2}}(e'q^{\imath+j})^{\jmath-n}$$

and then interchanging the summation order, we find, after some routine modification, that

$$q^{\binom{n}{2}}\frac{[a,b;q]_m}{[c,e;q]_m}\frac{[a,b;q]_n}{[c',e';q]_n}\frac{(cc'/q;q)_{m+n}}{(ab;q)_{m+n}}\left(\frac{qe'}{c}\right)^n$$

$$= \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\begin{bmatrix} m \\ i\end{bmatrix}\begin{bmatrix} n \\ j\end{bmatrix}q^{\binom{n-j}{2}}$$

$$\cdot \sum_{\imath=0}^{i}\sum_{\jmath=0}^{j}(-1)^{\imath+\jmath}\begin{bmatrix} i \\ \imath\end{bmatrix}\begin{bmatrix} j \\ \jmath\end{bmatrix}q^{\binom{i-\imath-\jmath}{2}}\frac{[a,b,cc'/q;q]_{\imath+\jmath}}{[ab,e,e';q]_{\imath+\jmath}}\frac{(e'q^{j};q)_{\imath}(eq^{i};q)_{\jmath}}{(c;q)_{\imath}(c';q)_{\jmath}}\left(\frac{e'}{c}q^{-\imath}\right)^{\jmath}.$$

This relation matches with (3b) under the following specification:

$$F(m,n) = \sum_{\imath=0}^{m}\sum_{\jmath=0}^{n}(-1)^{\imath+\jmath}\begin{bmatrix} m \\ \imath\end{bmatrix}\begin{bmatrix} n \\ \jmath\end{bmatrix}q^{\binom{m-\imath-\jmath}{2}}$$

$$\times \frac{[a,b,cc'/q;q]_{\imath+\jmath}}{[ab,e,e';q]_{\imath+\jmath}}\frac{(e'q^{n};q)_{\imath}(eq^{m};q)_{\jmath}}{(c;q)_{\imath}(c';q)_{\jmath}}\left(q^{-\imath}\frac{e'}{c}\right)^{\jmath},$$

$$G(m,n) = q^{\binom{n}{2}}\frac{[a,b;q]_m}{[c,e;q]_m}\frac{[a,b;q]_n}{[c',e';q]_n}\frac{(cc'/q;q)_{m+n}}{(ab;q)_{m+n}}\left(\frac{qe'}{c}\right)^n.$$

Then the dual relation (3a) leads us to the transformation formula

$$\sum_{\imath=0}^{m}\sum_{\jmath=0}^{n}(-1)^{\imath+\jmath}\begin{bmatrix} m \\ \imath\end{bmatrix}\begin{bmatrix} n \\ \jmath\end{bmatrix}q^{\binom{m-\imath-\jmath}{2}}\frac{[a,b,cc'/q;q]_{\imath+\jmath}}{[ab,e,e';q]_{\imath+\jmath}}\frac{(e'q^{n};q)_{\imath}(eq^{m};q)_{\jmath}}{(c;q)_{\imath}(c';q)_{\jmath}}\left(\frac{e'}{c}q^{-\imath}\right)^{\jmath}$$

$$= \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\begin{bmatrix} m \\ i\end{bmatrix}\begin{bmatrix} n \\ j\end{bmatrix}q^{\binom{m-i}{2}+\binom{j}{2}}\frac{[a,b;q]_i}{[c,e;q]_i}\frac{[a,b;q]_j}{[c',e';q]_j}\frac{(cc'/q;q)_{i+j}}{(ab;q)_{i+j}}\left(\frac{qe'}{c}\right)^{j},$$

which can be highlighted as the following theorem.

Theorem 7 (New transformation formula).

$$\sum_{i=0}^{m}\sum_{j=0}^{n} q^{i}\frac{[a,b,cc'/q;q]_{i+j}}{[ab,e,e';q]_{i+j}}\frac{[q^{-m},e'q^{n};q]_{i}}{[q,c;q]_{i}}\frac{[q^{-n},eq^{m};q]_{j}}{[q,c';q]_{j}}\left(q^{1+n-m}\frac{e'}{c}\right)^{j}$$
$$=\sum_{i=0}^{m}\sum_{j=0}^{n} q^{i}\frac{(cc'/q;q)_{i+j}}{(ab;q)_{i+j}}\frac{[q^{-m},a,b;q]_{i}}{[q,c,e;q]_{i}}\frac{[q^{-n},a,b;q]_{j}}{[q,c',e';q]_{j}}\left(q^{1+n}\frac{e'}{c}\right)^{j}.$$

This is the full q-analogue of the transformation (2a)–(2b). The weaker q-analogue established by Chu–Srivastava [3], Theorem 6, results in the special case of this theorem with $e' = a$ and $e = b$.

3.2. Applying twice the q-Gauss theorem, we can evaluate the double sum:

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{[q^{-m},d/b;q]_{r}}{(d;q)_{r+s}(q;q)_{r}}\frac{[q^{-n},b;q]_{s}}{(q;q)_{s}}q^{r+s}=\frac{(b;q)_{m}(d/b;q)_{n}}{(d;q)_{m+n}}\left(\frac{q^{n}d}{b}\right)^{m}b^{n}.$$

Multiplying both sides by $\frac{(a;q)_m(a';q)_n}{(c;q)_m(c';q)_n}\left(\frac{c}{a}\right)^m\left(\frac{c'}{a'}\right)^n$, we may restate it as follows:

$$\frac{(a;q)_{m}(b;q)_{m}(a';q)_{n}(d/b;q)_{n}}{(d;q)_{m+n}(c;q)_{m}(c';q)_{n}}\left(\frac{q^{n}cd}{ab}\right)^{m}\left(\frac{bc'}{a'}\right)^{n}$$
$$=\frac{(a;q)_{m}(a';q)_{n}}{(c;q)_{m}(c';q)_{n}}\left(\frac{c}{a}\right)^{m}\left(\frac{c'}{a'}\right)^{n}\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{[q^{-m},d/b;q]_{r}}{(d;q)_{r+s}(q;q)_{r}}\frac{[q^{-n},b;q]_{s}}{(q;q)_{s}}q^{r+s}$$
$$=\sum_{r=0}^{m}\sum_{s=0}^{n}q^{r+s}\frac{[q^{-m},a,d/b;q]_{r}}{(d;q)_{r+s}[q,c;q]_{r}}\frac{[q^{-n},a',b;q]_{s}}{[q,c';q]_{s}}$$
$$\times\frac{(q^{r}a;q)_{m-r}(q^{s}a';q)_{n-s}}{(q^{r}c;q)_{m-r}(q^{s}c';q)_{n-s}}\left(\frac{c}{a}\right)^{m}\left(\frac{c'}{a'}\right)^{n}.$$

Again by means of the q-Gauss summation formula, making the substitution

$$\frac{(q^{r}a;q)_{m-r}}{(q^{r}c;q)_{m-r}}\frac{(q^{s}a';q)_{n-s}}{(q^{s}c';q)_{n-s}}\left(\frac{c}{a}\right)^{m-r}\left(\frac{c'}{a'}\right)^{n-s}$$
$$={}_{2}\Phi_{1}\left[\begin{matrix}q^{-m+r},c/a\\ q^{r}c\end{matrix}\middle|q;q\right]\times{}_{2}\Phi_{1}\left[\begin{matrix}q^{-n+s},c'/a'\\ q^{s}c'\end{matrix}\middle|q;q\right]$$
$$=\sum_{i=0}^{m-r}\sum_{j=0}^{n-s}\frac{(q^{-m+r};q)_{i}(c/a;q)_{i}}{(q;q)_{i}(q^{r}c;q)_{i}}\frac{(q^{-n+s};q)_{j}(c'/a';q)_{j}}{(q;q)_{j}(q^{s}c';q)_{j}}q^{i+j}$$

and then interchanging the summation order, we derive the equality

$$\frac{(a;q)_m(b;q)_m(a';q)_n(d/b;q)_n}{(d;q)_{m+n}(c;q)_m(c';q)_n}\left(\frac{q^n cd}{ab}\right)^m\left(\frac{bc'}{a'}\right)^n$$
$$=\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{[q^{-m},c/a;q]_i}{[q,c;q]_i}\frac{[q^{-n},c'/a';q]_j}{[q,c';q]_j}q^{i+j}$$
$$\cdot\sum_{r=0}^{i}\sum_{s=0}^{j}\frac{[q^{-i},a,d/b;q]_r}{(d;q)_{r+s}[q,q^{1-i}a/c;q]_r}\frac{[q^{-j},a',b;q]_s}{[q,q^{1-j}a'/c';q]_s}q^{r+s}.$$

Multiplying across by $q^{\binom{m}{2}+\binom{n}{2}}$, we may reformulate the last equation as

$$\frac{(a;q)_m(b;q)_m(a';q)_n(d/b;q)_n}{(d;q)_{m+n}(c;q)_m(c';q)_n}\left(\frac{q^n cd}{ab}\right)^m\left(\frac{bc'}{a'}\right)^n q^{\binom{m}{2}+\binom{n}{2}}$$
$$=\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}q^{\binom{m-i}{2}+\binom{n-j}{2}}\begin{bmatrix}m\\i\end{bmatrix}\begin{bmatrix}n\\j\end{bmatrix}\frac{(c/a;q)_i(c'/a';q)_j}{(c;q)_i(c';q)_j}$$
$$\cdot\sum_{r=0}^{i}\sum_{s=0}^{j}\frac{[q^{-i},a,d/b;q]_r}{(d;q)_{r+s}[q,q^{1-i}a/c;q]_r}\frac{[q^{-j},a',b;q]_s}{[q,q^{1-j}a'/c';q]_s}q^{r+s}.$$

In view of the double series inversions (4a)–(4b), we obtain the following dual formula.

Theorem 8 (Transformation formula, Singh [8], Eq. 3.2).

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(q^{-m};q)_i(a;q)_i(b;q)_i}{(d;q)_{i+j}(q;q)_i(c;q)_i}\frac{(q^{-n};q)_j(a';q)_j(d/b;q)_j}{(q;q)_j(c';q)_j}\left(\frac{q^m cd}{ab}\right)^i\left(\frac{q^{n+i}bc'}{a'}\right)^j$$
$$=\frac{(c/a;q)_m(c'/a';q)_n}{(c;q)_m(c';q)_n}\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{q^{i+j}}{(d;q)_{i+j}}\frac{[q^{-m},a,d/b;q]_i}{(q;q)_i(q^{1-m}a/c;q)_i}\frac{[q^{-n},a',b;q]_j}{(q;q)_j(q^{1-n}a'/c';q)_j}.$$

3.3. Similar to Section 3.2, we have another double sum formula:

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{[q^{-m},q^{-n}c/d;q]_r}{(q;q)_r(c;q)_r}\frac{[q^{-n},b'/a';q]_s}{(q;q)_s(b';q)_s}q^{r+s}=\frac{(q^n d;q)_m(a';q)_n}{(c;q)_m(b';q)_n}\left(\frac{q^{-n}c}{d}\right)^m\left(\frac{b'}{a'}\right)^n,$$

which, by multiplying both sides by $\frac{(a;q)_m(d;q)_n}{(b;q)_m(qd/c;q)_n}\left(\frac{b}{a}\right)^m\left(\frac{q}{c}\right)^n$, can be rewritten in the form

$$\frac{(d;q)_{m+n}(a;q)_m(a';q)_n}{(b;q)_m(c;q)_m(qd/c;q)_n(b';q)_n}\left(\frac{q^{-n}bc}{ad}\right)^m\left(\frac{qb'}{a'c}\right)^n$$
$$=\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{[q^{-m},a;q]_r}{[q,b,c;q]_r}\,\frac{[q^{-n},d,b'/a';q]_s}{[q,b';q]_s}\left(\frac{qb}{a}\right)^r\left(\frac{q^2}{c}\right)^s$$
$$\cdot\frac{(q^ra;q)_{m-r}(q^{-n}c/d;q)_r(q^sd;q)_{n-s}}{(q^rb;q)_{m-r}(qd/c;q)_n}\left(\frac{b}{a}\right)^{m-r}\left(\frac{q}{c}\right)^{n-s}.$$

Replacing the last expression by the double sum

$$\frac{(q^ra;q)_{m-r}(q^{-n}c/d;q)_r(q^sd;q)_{n-s}}{(q^rb;q)_{m-r}(qd/c;q)_n}\left(\frac{b}{a}\right)^{m-r}\left(\frac{q}{c}\right)^{n-s}$$
$$=\frac{(q^{-s}c/d;q)_r}{(qd/c;q)_s}\cdot{}_2\Phi_1\left[\begin{matrix}q^{-m+r},\,b/a\\ q^rb\end{matrix}\middle|q;q\right]{}_2\Phi_1\left[\begin{matrix}q^{-n+s},\quad q^{1-r}/c\\ q^{1+s-r}d/c\end{matrix}\middle|q;q\right]$$
$$=\frac{(q^{-s}c/d;q)_r}{(qd/c;q)_s}\sum_{i=0}^{m-r}\sum_{j=0}^{n-s}\frac{(q^{-m+r};q)_i(b/a;q)_i}{(q;q)_i(q^rb;q)_i}\,\frac{(q^{-n+s};q)_j(q^{1-r}/c;q)_j}{(q;q)_j(q^{1+s-r}d/c;q)_j}q^{i+j}$$

and then interchanging the summation order, we have the following equality:

$$\frac{(d;q)_{m+n}(a;q)_m(a';q)_n}{(b;q)_m(c;q)_m(qd/c;q)_n(b';q)_n}q^{\binom{m}{2}+\binom{n}{2}}\left(\frac{q^{-n}bc}{ad}\right)^m\left(\frac{qb'}{a'c}\right)^n$$
$$=\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\begin{bmatrix}m\\ i\end{bmatrix}\begin{bmatrix}n\\ j\end{bmatrix}q^{\binom{m-i}{2}+\binom{n-j}{2}}\frac{(b/a;q)_i(q/c;q)_j}{(b;q)_i(qd/c;q)_j}$$
$$\cdot\sum_{r=0}^{i}\sum_{s=0}^{j}\frac{q^{r+s}}{(q^{-j}c;q)_{r+s}}\,\frac{[q^{-i},a,q^{-j}c/d;q]_r}{[q,q^{1-i}a/b;q]_r}\,\frac{[q^{-j},b'/a',d;q]_s}{[q,b';q]_s}.$$

Applying again (4a)–(4b) to the last relation, we derive the following dual formula.

Theorem 9 (q-analogue of Lievens–Jeugt [7], Eq. 11a).

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(d;q)_{i+j}(q^{-m};q)_i(a;q)_i}{(q;q)_i(b;q)_i(c;q)_i}\,\frac{(q^{-n};q)_j(a';q)_j}{(q;q)_j(b';q)_j(qd/c;q)_j}\left(\frac{q^mbc}{ad}\right)^i\left(\frac{q^{1+n-i}b'}{a'c}\right)^j$$
$$=\frac{(b/a;q)_m(q/c;q)_n}{(b;q)_m(qd/c;q)_n}\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{q^{i+j}}{(q^{-n}c;q)_{i+j}}\,\frac{[q^{-m},a,q^{-n}c/d;q]_i}{(q;q)_i(q^{1-m}a/b;q)_i}\,\frac{[q^{-n},b'/a',d;q]_j}{(q;q)_j(b';q)_j}.$$

3.4. For an arbitrary sequence $\{\Omega(k)\}$, we can check, by means of the q-binomial theorem, the almost trivial relation

$$\frac{(d;q)_m}{(c;q)_m}\Omega(m) = \sum_{r=0}^{m}\frac{(d;q)_r}{(c;q)_r}\begin{bmatrix} m \\ r \end{bmatrix}\Omega(r)\sum_{i=0}^{m-r}(-1)^i\begin{bmatrix} m-r \\ i \end{bmatrix}q^{\binom{i}{2}}.$$

Applying twice the q-Gauss theorem, we have also the following identity

$$\frac{[a', q^m d; q]_n}{[b', qd/c; q]_n} = \sum_{s=0}^{n}\frac{[q^{-n}, b'/a', q^m d; q]_s}{[q, b', qd/c; q]_s}(q^n a')^s\sum_{j=0}^{n-s}\frac{[q^{-n+s}, q^{1-m}/c; q]_j}{[q, q^{1+s}d/c; q]_j}(q^{n+m}d)^j.$$

Multiplying both equations

$$\begin{aligned}&\frac{\Omega(m)(a';q)_n(d;q)_{m+n}}{(c;q)_m(b';q)_n(qd/c;q)_n}\\ &\quad= \sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^r\frac{(d;q)_{r+s}(q^{-m};q)_r[q^{-n}, b'/a'; q]_s}{[q,c;q]_r[q,b',qd/c;q]_s}q^{mr-\binom{r}{2}}(q^n a')^s\\ &\qquad\cdot\Omega(r)\sum_{i=0}^{m-r}\sum_{j=0}^{n-s}\frac{(q^{-m+r};q)_i}{(q;q)_i}\frac{[q^{-n+s}, q^{1-r}/c; q]_j}{[q, q^{1+s}d/c; q]_j}q^{(m-r)i+(n+r)j}d^j\end{aligned}$$

and then exchanging the summation order, we arrive at the following relation:

$$\begin{aligned}&\frac{(a';q)_n(d;q)_{m+n}}{(c;q)_m(b';q)_n(qd/c;q)_n}\Omega(m)\\ &\quad= \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\begin{bmatrix} m \\ i \end{bmatrix}\begin{bmatrix} n \\ j \end{bmatrix}\frac{(q/c;q)_j}{(qd/c;q)_j}q^{\binom{i}{2}+\binom{j}{2}}d^j\\ &\qquad\cdot\sum_{r=0}^{i}\sum_{s=0}^{j}\frac{(d;q)_{r+s}}{(q^{-j}c;q)_{r+s}}\frac{(q^{-i};q)_r}{(q;q)_r}\frac{[q^{-j}, b'/a'; q]_s}{(q;q)_s(b';q)_s}q^r\left(\frac{a'c}{d}\right)^s\Omega(r).\end{aligned}$$

According to (4a)–(4b), its dual relation reads as the following transformation.

Theorem 10 (New transformation formula).

$$\begin{aligned}&\sum_{i=0}^{m}\sum_{j=0}^{n}\Omega(i)\frac{(d;q)_{i+j}(q^{-m};q)_i}{[q,c;q]_i}\frac{[q^{-n}, a'; q]_j}{[q, b', qd/c; q]_j}q^{i+j}\\ &\quad= \frac{d^n(q/c;q)_n}{(qd/c;q)_n}\sum_{i=0}^{m}\sum_{j=0}^{n}\Omega(i)\frac{(d;q)_{i+j}(q^{-m};q)_i}{(q^{-n}c;q)_{i+j}(q;q)_i}\frac{[q^{-n}, b'/a'; q]_j}{[q, b'; q]_j}q^i\left(\frac{a'c}{d}\right)^j.\end{aligned}$$

For the very particular case $\Omega(m) = (a;q)_m/(b;q)_m$, this theorem yields the q-analogue of the transformation due to Lievens–Jeugt [7], Eq. 11b.

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W. Chu, Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, P. R. China
Current address: Dipartimento di Matematica, Università del Salento, Lecce-Arnesano P. O. Box 193, 73100 Lecce, Italia
E-mail: chu.wenchang@unile.it

C. Jia, Department of Mathematics, Northeast University, Shenyang 110005, P. R. China
E-mail: cangzhijia@yahoo.com.cn