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On radicals and polynomial rings

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Abstract. For any class $\mathscr{M}$ of rings, it is shown that the class $\mathscr{E}_\ell(\mathscr{M})$ of all rings each non-zero homomorphic image of which contains either a non-zero left ideal in $\mathscr{M}$ or a proper essential left ideal is a radical. Some characterizations and properties of these radicals are presented. It is also shown that, for radicals γ under certain constraints, one can obtain a strictly decreasing chain of radicals $\gamma = \gamma_{(1)} \supset \gamma_{(2)} \supset \cdots \supset \gamma_{(n)} \supset \cdots$ where, for each positive integer n, $\gamma_{(n)}$ is the radical consisting of all rings A such that $A[x_1, \ldots, x_n]$ is in γ, thus giving a negative answer to a question posed by Gardner. Moreover, classes $\mathscr{M}$ of rings are constructed such that there exist several such radicals γ in the interval $[\mathscr{E}_\ell(0), \mathscr{E}_\ell(\mathscr{M})]$.

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1. Introduction

All rings considered in this note are associative. Let us recall that a (Kurosh–Amitsur) radical γ is a class of rings which is closed under homomorphisms, extensions (I and A/I in γ imply that A in γ) and has the inductive property (if $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_\alpha \subseteq \cdots$ is an ascending chain of ideals of A such that each I_α is in γ, then also $\bigcup I_\alpha \in \gamma$). The unique largest γ-ideal $\gamma(A)$ of A is then called the γ-radical of A. A class $\mathscr{M}$ of rings is said to be regular if every non-zero ideal of a ring in $\mathscr{M}$ has a non-zero homomorphic image in $\mathscr{M}$. Starting from a regular class $\mathscr{M}$ of rings, the upper radical operator $\mathfrak{U}$ yields a radical class:

$$\mathfrak{U}\mathscr{M} = \{A \mid A \text{ has no non-zero homomorphic image in } \mathscr{M}\}.$$

The definitions and significant radical theoretic properties used throughout this text may be found in [2], [9]. As usual, $I \lhd A$ indicates that I is an ideal of the ring A. A ring (ideal) belonging to a class $\mathscr{M}$ will be called an $\mathscr{M}$-ring ($\mathscr{M}$-ideal).

2. Upper radicals and essential left ideals

For any class $\mathscr{M}$ of rings, we show that the class $\mathscr{E}_\ell(\mathscr{M})$ of all rings each non-zero homomorphic image of which contains either a non-zero left ideal in $\mathscr{M}$ or a proper essential left ideal is a radical. For certain classes $\mathscr{M}$ of rings, $\mathscr{E}_\ell(\mathscr{M})$ is characterized as an upper radical (for example, the Thierrin radical) and we also show that the collection of all the radicals $\mathscr{E}_\ell(\mathscr{M})$ forms a meet sublattice of the lattice of all radicals.

Let us recall that a simple ring means a ring without nontrivial ideals. A ring without nontrivial left ideals shall be called a left simple ring. We remind the reader that such rings are simple zero-rings or division rings. An essential left ideal of a ring A is a left ideal which has non-zero intersection with every non-zero ideal of A.

It is well known [4] that if a ring A has a homomorphic image with a proper essential ideal, then A has a proper essential ideal. We now show, in a similar way, that the same holds for left-sided ideals.

Lemma 1. *If A has a homomorphic image with a proper essential left ideal, then A has a proper essential left ideal.*

Proof. Let A/I be a non-zero homomorphic image of A with L/I a proper essential left ideal of A/I. Then I is a proper ideal of A and $L \neq A$. To show that L is essential in A, let $0 \neq J \lhd A$. If $J \subseteq L$, then $J \cap L = J \neq 0$, as desired. If $J \nsubseteq L$, then $J \nsubseteq I$ and $0 \neq (J+I)/I \lhd A/I$ so that $[(J+I)/I] \cap [L/I] \neq 0$. Thus, there exists $a+b \in J+I$ and $l \in L$ such that $(a+b)+I = l+I \neq I$. This implies that $a+b+b_1 = l+b_2$ for some $b_1, b_2 \in I$. Since $I \subseteq L$, we have $b+b_1 \in L$ and $l+b_2 \in L$. Therefore, $a = (l+b_2)-(b+b_1) \in L$ and so $a \in J \cap L$. If $a = 0$, then $b+b_1 = l+b_2$, which implies that $l \in I$. However, this is impossible since $l+I \neq I$. Hence $J \cap L \neq 0$, as required. □

Theorem 2. *$\mathscr{E}_\ell(\mathscr{M})$ is a radical class for any class $\mathscr{M}$ of rings.*

Proof. Clearly, $\mathscr{E}_\ell(\mathscr{M})$ is a homomorphically closed class of rings. Now suppose that $A \notin \mathscr{E}_\ell(\mathscr{M})$. Then A has a non-zero homomorphic image $\bar{A}$ which has neither a non-zero left ideal in $\mathscr{M}$ nor a proper essential left ideal. Let $\bar{K}$ be a non-zero $\mathscr{E}_\ell(\mathscr{M})$-ideal of $\bar{A}$. By [3], Theorem 5, $\bar{K}$ must be a direct summand of $\bar{A}$ and then also, $\bar{K}$ is a homomorphic image of $\bar{A}$. Since $\bar{K} \in \mathscr{E}_\ell(\mathscr{M})$, it must have a proper essential left ideal or a non-zero left ideal in $\mathscr{M}$. But, if $\bar{K}$ has a proper essential left ideal, then Lemma 1 forces $\bar{A}$ to have a proper essential left ideal, contrary to the choice of $\bar{A}$. Also, if $\bar{K}$ has a non-zero left ideal in $\mathscr{M}$, then this left ideal of $\bar{K}$ is actually a left ideal of $\bar{A}$, since $\bar{K}$ is a direct summand of $\bar{A}$. Again this is impossible. Thus $\bar{A}$ has no non-zero ideal in $\mathscr{E}_\ell(\mathscr{M})$. □

Let $\mathscr{M}$ be any class of rings. In the sequel, $\mathscr{M}_1$ stands for the class of all simple rings contained in $\mathscr{M}$, and $\mathscr{M}_0$ denotes the class of all left simple rings contained in $\mathscr{M}$. The class of rings complementary to the class $\mathscr{M}$ is denoted by $\mathscr{M}'$.

Proposition 3. *For any class $\mathscr{M}$ of rings, $\mathscr{E}_\ell(\mathscr{M}) = \mathscr{E}_\ell(\mathscr{M}_0) = \mathscr{E}_\ell(\mathscr{M}_1)$.*

Proof. Obviously, $\mathscr{E}_\ell(\mathscr{M}_0) \subseteq \mathscr{E}_\ell(\mathscr{M}_1) \subseteq \mathscr{E}_\ell(\mathscr{M})$. Let $A \in \mathscr{E}_\ell(\mathscr{M})$ and let $\bar{A}$ be any non-zero homomorphic image of A. If $\bar{A}$ has no proper essential left ideal, then, according to [3], Theorem 5, $\bar{A}$ is a direct sum of left simple rings and hence contains a non-zero $\mathscr{M}_0$-ideal. Thus $A \in \mathscr{E}_\ell(\mathscr{M}_0)$. $\square$

Theorem 4. *For any class $\mathscr{M}$ of left simple rings, $\mathfrak{U}\mathscr{M} = \mathscr{E}_\ell(\mathscr{M}_0') = \mathscr{E}_\ell(\mathscr{M}_1')$.*

Proof. Suppose $A \notin \mathfrak{U}\mathscr{M}$. Then $\bar{A} \in \mathscr{M}$ for some non-zero homomorphic image of A. So, $\bar{A}$ has neither a proper essential left ideal nor a non-zero left ideal in $\mathscr{M}_1'$. Hence, $A \notin \mathscr{E}_\ell(\mathscr{M}_1')$. On the other hand, if $A \notin \mathscr{E}_\ell(\mathscr{M}_1')$, then A has a non-zero homomorphic image $\bar{A}$ without a proper essential left ideal or a non-zero left ideal in $\mathscr{M}_1'$. Then $\bar{A}$ is a direct sum of simple rings having only trivial left ideals, each of which can be considered an ideal of $\bar{A}$. These simple rings are obviously in $\mathscr{M}$. Moreover, each of these summands is a homomorphic image of $\bar{A}$ and thus of A. Therefore $A \notin \mathfrak{U}\mathscr{M}$. $\square$

Corollary 5. *If $\mathscr{M}$ is a homomorphically closed class of rings, then $\mathfrak{L}\mathscr{M} = \mathscr{E}_\ell(\mathscr{M})$ if and only if $\mathfrak{L}\mathscr{M} = \mathfrak{U}\mathscr{M}_0'$, where $\mathfrak{L}\mathscr{M}$ denotes the lower radical determined by $\mathscr{M}$.*

Proof. This is clear since $\mathscr{E}_\ell(\mathscr{M}) = \mathscr{E}_\ell(\mathscr{M}_1) = \mathscr{E}_\ell(\mathscr{M}_0) = \mathfrak{U}\mathscr{M}_0'$. $\square$

Corollary 6. *If $\mathscr{M}$ is a regular class of rings, then there exists a class $\mathscr{N}$ of rings such that $\mathfrak{U}\mathscr{M} = \mathscr{E}_\ell(\mathscr{N})$ if and only if $\mathfrak{U}\mathscr{M} = \mathfrak{U}\mathscr{M}_0$.*

Proof. If $\mathfrak{U}\mathscr{M} = \mathscr{E}_\ell(\mathscr{N})$ for some class $\mathscr{N}$ of rings, then $\mathscr{E}_\ell(\mathscr{N}_0) = \mathscr{E}_\ell(\mathscr{N}) = \mathfrak{U}\mathscr{M}$, where $\mathscr{E}_\ell(\mathscr{N}_0) = \mathfrak{U}\mathscr{N}_0'$. Hence $\mathfrak{U}\mathscr{M} = \mathfrak{U}\mathscr{N}_0'$. But $\mathscr{N}_0' \subseteq \mathscr{M}_0$, for $B \in \mathscr{N}_0'$ implies that $B \notin \mathfrak{U}\mathscr{N}_0' = \mathfrak{U}\mathscr{M}$, whence $B \in \mathscr{M}$. Thus $\mathfrak{U}\mathscr{M}_0 \subseteq \mathfrak{U}\mathscr{N}_0'$. Moreover, $\mathscr{M}_0 \subseteq \mathscr{N}_0'$. In fact, A left simple and $A \notin \mathscr{N}_0'$ implies that $A \in \mathscr{N}_0 \subseteq \mathfrak{U}\mathscr{M}$. Therefore $A \notin \mathscr{M}$ and consequently $A \notin \mathscr{M}_0$. We have now that $\mathfrak{U}\mathscr{M} = \mathfrak{U}\mathscr{M}_0$.

Conversely, $\mathfrak{U}\mathscr{M} = \mathfrak{U}\mathscr{M}_0 = \mathscr{E}_\ell(\mathscr{M}_0')$. $\square$

Corollary 7. *Let $\mathscr{M}$ be a regular class of rings which contains non-zero left simple rings. Then there exists a class $\mathscr{N}$ of rings such that $\mathfrak{U}\mathscr{M} = \mathscr{E}_\ell(\mathscr{N})$ if and only if every non-zero $\mathscr{M}$-ring has a left simple non-zero homomorphic image in $\mathscr{M}$.*

Remark 8. Let γ be a hypernilpotent radical (that is, all nilpotent rings are γ-rings) and let $\mathscr{S}\gamma$ be the semisimple class of γ. If $\mathscr{M}$ is any class of rings such that $\mathscr{M}_0 = \gamma_0$ then we have, by Theorem 4,

$$\mathscr{U}\gamma_0' = \mathscr{E}_\ell(\gamma_0) = \mathscr{E}_\ell(\mathscr{M}),$$

where γ_0', the class of left simple rings in the complementary class of γ, clearly coincides with the class of division rings in $\mathscr{S}\gamma$.

Notice that if $\mathscr{D}$ is the class of all division rings and $\mathscr{U}\mathscr{D} \subseteq \gamma$, then γ_0' coincides with the class of division rings in $\mathscr{S}\gamma$ (hence γ_0' is a special class of rings) and $\mathscr{E}_\ell(\mathscr{M})$ is a special radical.

Example 9. If $\mathscr{D}$ again denotes the class of all division rings, and $\mathscr{Z}$ is the class of all simple zero-rings, then

$$\mathscr{E}_\ell(\mathscr{Z}) = \mathscr{E}_\ell(\mathscr{D}_0') = \mathfrak{U}\mathscr{D}$$

and $\mathscr{E}_\ell(\mathscr{D}) = \mathscr{E}_\ell(\mathscr{Z}_0') = \mathfrak{U}\mathscr{Z}$.

Example 10. Consider the subclass $\mathscr{E}_\ell(0)$ of all rings each non-zero homomorphic image of which has a proper essential left ideal. Obviously, left simple rings cannot belong to $\mathscr{E}_\ell(0)$. For $\mathscr{S} = \{A \mid A$ is either a division ring or a prime order zero ring$\}$ and $\mathscr{T} = \{$all direct sums of members of $\mathscr{S}\}$, we have:

$$\mathscr{E}_\ell(0) = \mathfrak{U}\mathscr{S} = \mathfrak{U}\mathscr{T} = \mathfrak{U}\mathscr{D} \cap \mathfrak{U}\mathscr{Z}.$$

In fact, from Theorem 4 and the definition of $\mathscr{E}_\ell(0)$, it follows that $\mathscr{E}_\ell(0) = \mathfrak{U}\mathscr{T}$. Also, $\mathscr{S} \subseteq \mathscr{T}$, so $\mathfrak{U}\mathscr{T} \subseteq \mathfrak{U}\mathscr{S}$. Conversely, if $A \in \mathfrak{U}\mathscr{S}$ then R, having no non-zero image in $\mathscr{S}$, cannot have an image in $\mathscr{T}$, so that $A \in \mathfrak{U}\mathscr{T}$ and so $\mathscr{E}_\ell(0) = \mathfrak{U}\mathscr{T} = \mathfrak{U}\mathscr{S}$. It is easily seen that $\mathfrak{U}\mathscr{T} = \mathfrak{U}\mathscr{D} \cap \mathfrak{U}\mathscr{Z}$.

In order to prove the next proposition, we require the following result.

Lemma 11. *The following conditions are equivalent for an abelian group G:*

(i) *Every non-zero homomorphic image of G has a proper essential subgroup;*

(ii) *G is divisible.*

Proof. (i) implies (ii). Suppose $pG \neq G$ for some prime p. Then the cyclic group $\mathbb{Z}(p)$ of order p is a homomorphic image of G/pG and therefore of G. But $\mathbb{Z}(p)$ has no proper essential subgroup; a contradiction. Thus $pG = G$ for every prime p and so G is divisible.

(ii) implies (i). If G is non-zero and divisible, then $G = A \oplus B$ where $A \cong \mathbb{Q}$ or $A \cong \mathbb{Z}(p^\infty)$ for some prime p. If $0 \neq a \in A$, then $\langle a \rangle \oplus B$ is a proper essential subgroup of G, where $\langle a \rangle$ is the cyclic subgroup of G generated by a. Now we just observe that all homomorphic images of G are also divisible. $\square$

Proposition 12. *The radical $\mathscr{E}_\ell(0)$ is polynomially extensible; that is, $A \in \mathscr{E}_\ell(0)$ implies that $A[x] \in \mathscr{E}_\ell(0)$.*

Proof. Let $A \in \mathscr{E}_\ell(0)$ and consider an arbitrary non-zero homomorphic image $A[x]/I$ of the polynomial ring $A[x]$. If $A \not\subseteq I$, then $0 \neq (A+I)/I \cong A/(A \cap I)$ and $(A+I)/I$ has a proper essential left ideal. But $(A+I)/I$ is a homomorphic image of $A[x]/I$. In fact, $\varphi : A[x]/I \to (A+I)/I$ defined by $\varphi(a_0 + a_1x + \cdots + a_nx^n + I) = a_0 + I$ is a ring epimorphism. Hence, by Lemma 1, $A[x]/I$ has a proper essential left ideal. If $A \subseteq I$, then $a_0 + b_1x + b_2x^2 + \cdots \in I$ for all $a_0 \in A$, $b_i \in A^2$, so $A[x]/I$ is a homomorphic image of $\{(c_1 + A^2)x + (c_2 + A^2)x^2 + \cdots \mid c_i \in A\} \cong A/A^2 \oplus A/A^2 \oplus \cdots$. Now $A \neq A^2$, since otherwise $I = A[x]$. Since $A \in \mathscr{E}_\ell(0)$ it follows that $0 \neq A/A^2 \in \mathscr{E}_\ell(0)$. By the previous lemma, A/A^2 is a divisible zero-ring and hence also $A[x]/I$. By the first part of the proof, $A[x]/I$ has a proper essential left ideal. $\square$

Proposition 13. (1) *For any class $\mathscr{M}$ of rings, $\mathscr{E}_\ell(\mathscr{E}_\ell(\mathscr{M})) = \mathscr{E}_\ell(\mathscr{M})$.*

(2) *For any family $\{\mathscr{M}_i \mid i \in \Lambda\}$ of classes of rings, we have*

$$\bigcap_{i\in\Lambda} \mathscr{E}_\ell(\mathscr{M}_i) = \mathscr{E}_\ell\Big(\bigcap_{i\in\Lambda} \mathscr{M}_i\Big).$$

Proof. (1) It is clear that $\mathscr{E}_\ell(\mathscr{M}) \subseteq \mathscr{E}_\ell(\mathscr{E}_\ell(\mathscr{M}))$. On the other hand, suppose that $A \in \mathscr{E}_\ell(\mathscr{E}_\ell(\mathscr{M}))$ and let $\bar{A}$ be any non-zero homomorphic image of A. Suppose that $\bar{A}$ has no proper essential left ideals. Then $\bar{A}$ has a non-zero left ideal $\bar{L} \in \mathscr{E}_\ell(\mathscr{M})$ and, by [3], Theorem 5, $\bar{L}$ is a direct summand of $\bar{A}$; that is, $\bar{A} = \bar{L} \oplus \bar{K}$ for some ideal $\bar{K}$ of $\bar{A}$. But then any left ideal of $\bar{L}$ is also a left ideal of $\bar{A}$. Now, if $\bar{L}$ has a proper essential left ideal, then by

$$\bar{A}/\bar{K} = (\bar{L} + \bar{K})/\bar{K} \simeq \bar{L}/(\bar{L} \cap \bar{K}) \simeq \bar{L}$$

and Lemma 1, $\bar{A}$ has a proper essential left ideal, a contradiction. Hence $\bar{L}$ has a non-zero left ideal in $\mathscr{M}$, which is also a left ideal in $\bar{A}$. Therefore $A \in \mathscr{E}_\ell(\mathscr{M})$.

(2) Clearly, $\mathscr{E}_\ell(\bigcap_{i\in\Lambda}\mathscr{M}_i) \subseteq \bigcap_{i\in\Lambda}\mathscr{E}_\ell(\mathscr{M}_i)$. Let $A \in \bigcap_{i\in\Lambda}\mathscr{E}_\ell(\mathscr{M}_i)$ and suppose that $\bar{A}$ is any non-zero homomorphic image of A. If $\bar{A}$ has no proper essential left ideal, then $\bar{A}$ has a non-zero left ideal $\bar{S} \in \mathscr{M}_1$ which is a left simple ring and a summand of $\bar{A}$. We claim that $\bar{S} \in \mathscr{M}_i$ for each $i \in \Lambda$. If $\bar{S} \notin \mathscr{M}_i$ for some $i \in \Lambda$, then $\bar{S}$, as a homomorphic image of A, has neither a proper essential left ideal nor a non-zero left ideal in $\mathscr{M}_i$, contradicting $A \in \mathscr{E}_\ell(\mathscr{M}_i)$. Therefore, $A \in \mathscr{E}_\ell(\bigcap_{i\in\Lambda}\mathscr{M}_i)$. $\square$

Remark 14. (i) The lattice of all radicals $\mathscr{E}_\ell(\mathscr{M})$ is atomic and coatomic. The atoms are the radicals $\mathscr{E}_\ell(\{P\})$ and the coatoms are the radicals $\mathscr{E}_\ell(\mathscr{S} - \{P\})$, where P is a non-zero left simple ring and $\mathscr{S}$ is the class of all left simple rings.

(ii) The collection of all radicals $\mathscr{E}_\ell(\mathscr{M})$ forms a meet subsemilattice of the lattice of all radicals. Indeed, if $\mathscr{E}_\ell(\mathscr{M}_1)$ and $\mathscr{E}_\ell(\mathscr{M}_2)$ are any two such radical classes, then $\mathscr{E}_\ell(\mathscr{M}_1) \wedge \mathscr{E}_\ell(\mathscr{M}_2) = \mathscr{E}_\ell(\mathscr{M}_1) \cap \mathscr{E}_\ell(\mathscr{M}_2) = \mathscr{E}_\ell(\mathscr{M}_1 \cap \mathscr{M}_2)$.

(iii) [11], Example 5, also shows that this collection of radical classes $\mathscr{E}_\ell(\mathscr{M})$ is not a sublattice of the lattice of all radical classes with respect or $\wedge$ and $\vee$. However, like the collection of the Olson and Jenkins radical classes, this collection also forms a Boolean lattice with respect to the operators $\wedge$ and $\vee'$, where $\mathscr{E}_\ell(\mathscr{M}_1) \vee' \mathscr{E}_\ell(\mathscr{M}_2) = \mathscr{E}_\ell(\mathscr{M}_1 \cup \mathscr{M}_2)$ for arbitrary classes $\mathscr{M}_1$ and $\mathscr{M}_2$ of rings.

3. Polynomial rings and radicals

For a ring A, $A[x]$ and $A[x_1, x_2, \dots, x_n]$ denote, respectively, the ring of polynomials over A in one indeterminate and the ring of polynomials over A in n commuting indeterminates and P_n denotes the class of all polynomial rings $A[x_1, x_2, \dots, x_n]$.

For a radical γ, let $\gamma_{(1)}$ denote the class $\{A \mid A[x] \in \gamma\}$ and let $\gamma^{(1)}$ denote the lower radical $\mathfrak{L}(\gamma \cap P_1)$.

Proposition 15 ([1], Theorem 1). *$\gamma_{(1)}$ is a radical class for any radical class γ.*

Defining $\gamma_{(n)} = \{A \mid A[x_1, \dots, x_n] \in \gamma\}$ and $\gamma^{(n)} = \mathfrak{L}(\{P_n \cap \gamma\})$, we obtain the chains

$$\gamma \supseteq \gamma_{(1)} \supseteq \cdots \supseteq \gamma_{(n)} \supseteq \cdots$$

and

$$\gamma \supseteq \gamma^{(1)} \supseteq \cdots \supseteq \gamma^{(n)} \supseteq \cdots.$$

Clearly,

$$\gamma_{(n)} \subseteq \gamma^{(n)}.$$

In [1], Gardner posed the following question:

Does the chain

$$\gamma = \gamma_{(0)} \supseteq \gamma_{(1)} \supseteq \cdots \supseteq \gamma_{(n)} \supseteq \cdots$$

terminate for every radical class γ? We shall give a negative answer to this question, but first we need some preliminary results.

Proposition 16 ([1]). *Let A be a ring with unity and S a ring with unity and no other non-zero idempotents. Then S belongs to $\mathfrak{L}(\{A\})$ if and only if it is a homomorphic image of A.*

Let A be a ring. A ring I is said to be an accessible subring of A if $I = I_1 \lhd \cdots \lhd I_n = A$ for some natural number n.

Lemma 17. *Let A be a ring with unity and B a ring with unity and no other non-zero idempotents. Then the $\mathfrak{L}(\{A\})$-radical of B is non-zero if and only if B is a homomorphic image of A.*

Proof. Let $\mathfrak{L}(\{A\}) \neq 0$. Then $\mathfrak{L}(\{A\})(B)$ has a non-zero accessible subring I such that I is a homomorphic image of A. Since $\mathfrak{L}(\{A\})(B) \lhd B$, I is an accessible subring of B. Similarly to the proof of Proposition 16, we obtain that $I = B$. Hence B is a homomorphic image of A. The converse is clear. $\square$

Lemma 18. *Let γ be a radical and let I be an accessible subring of a ring A. If $I \in \gamma$, then $\gamma(A) \neq 0$.*

Proof. Suppose that $I = I_1 \lhd \cdots \lhd I_n = A$. Since $I \in \gamma$ and $I \lhd I_2$, $0 \neq \gamma(I) \subseteq \gamma(I_2)$ and, by induction, $0 \neq \gamma(I) \subseteq \gamma(I_2) \subseteq \cdots \subseteq \gamma(I_n) = \gamma(A)$. $\square$

We denote by $\mathscr{C}_p$ the class of all commutative prime rings and $\mathscr{C} = \mathfrak{U}\mathscr{C}_p$. If $\mathscr{U}$ is a class of rings with unity, we put $\mathscr{C}_{\mathscr{U}} = \mathfrak{L}(\mathscr{C} \cup \mathscr{U})$.

Proposition 19. *Let $\mathscr{U}$ be a class of rings with unity and let $A \neq 0$ be a commutative reduced ring with unity and no other non-zero idempotent. Then $\mathscr{C}_{\mathscr{U}}(A) \neq 0$ if and only if A is a homomorphic image of a ring $B \in \mathscr{U}$.*

Proof. Suppose that $\mathscr{C}_{\mathscr{U}}(A) \neq 0$. Since A is a commutative reduced ring, $R = \mathscr{C}_{\mathscr{U}}(A)$ is also a commutative reduced ring. Therefore, for each $0 \neq a \in R$, there exists an ideal K_a of R, which is maximal with respect to the exclusion of a^n for any natural number n. Clearly, R/K_a is a prime commutative ring. Since $0 \neq a \in R$ is arbitrary, R is a subdirect sum of the rings R/K_a. Hence R is a $\mathscr{C}$-semisimple ring. By Lemma 18, R has no non-zero accessible subring which is a homomorphic image of a ring $B \in \mathscr{C}$. Therefore R has an accessible subring I such that I is a homomorphic image B in $\mathscr{U}$. Hence, by Lemma 18, $L(\{B\})(A) \neq 0$, because I is an accessible subring of A. By Lemma 17, A is a homomorphic image of B. The converse is clear. $\square$

We denote by $|A|$ the cardinality of a ring A.

Lemma 20. *Let A be a simple ring with $|A| \geq \aleph_0$. Then $|A| = |B|$ for every non-zero homomorphic image B of $A[x_1, \ldots, x_n]$ and $n \in \mathbb{N}$.*

Proof. Since A is infinite, we have $|A[x_1, \ldots, x_n]| = |A|$. Therefore $|B| \leq |A|$ for every homomorphic image of $A[x_1, \ldots, x_n]$. Let $B = A[x_1, \ldots, x_n]/I \neq 0$.

We show that $A \cap I = 0$. Suppose that $A \cap I \neq 0$. Since $0 \neq A \cap I \lhd A$ and A is a simple ring, $A = A \cap I$. Clearly $A^2 = A$ because A is infinite. Therefore $A[x_1, \ldots, x_n] = A^2[x_1, \ldots, x_n] \subseteq I$. Hence $A[x_1, \ldots, x_n] = I$, a contradiction to $A[x_1, \ldots, x_n]/I \neq 0$. Thus $A \cap I = 0$. Since $(A + I)/I \cong A/(A \cap I) \cong A$, we have $|A| = |(A + I)/I| \leq |A[x_1, \ldots, x_n]/I| = |B|$. Thus $|A| = |B|$. $\square$

Remark 21. Notice that we may find fields $F_1, F_2, \ldots, F_n, \ldots,$ of zero characteristic such that $|F_1| < |F_2| < \cdots < |F_n| < \cdots$ and $|F_1| \geq \aleph_0$. Therefore, we can assume that $F_1 = \mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers.

In what follows, let

$$\begin{aligned}\mathscr{S} = \{F_n[x_1, \ldots, x_n] \mid F_n &\text{ is a field and } F_n[x_1, \ldots, x_n] \\ &\text{is not a homomorphic image of } F_m[x_1, \ldots, x_m] \text{ for any } m \neq n\}\end{aligned}$$

and $\mathscr{F}_{\mathscr{S}} = \mathfrak{L}(\mathfrak{U}\mathscr{F}^1 \cup \mathscr{S})$, where $\mathscr{F}^1$ is the class of all fields. We are now in a position to prove the following result.

Theorem 22. *Let $F_1 = \mathbb{Q}, F_2, \ldots, F_n, \ldots$ be fields such that $|F_i| < |F_{i+1}|$ for each $i = 1, 2, \ldots.$ If γ is any radical such that*

$$\mathscr{S} \subseteq \gamma \subseteq \mathscr{F}_{\mathscr{S}},$$

then $\gamma = \gamma_{(1)} \supset \gamma_{(2)} \supset \cdots \supset \gamma_{(n)} \supset \cdots.$

Proof. By assumption, we have $F_n \in \mathfrak{L}(\mathscr{S})_{(n)} \subseteq \gamma_{(n)} \subseteq (\mathscr{F}_{\mathscr{S}})_{(n)}$. We claim that $F_n \notin \gamma_{(n+1)}$. It is sufficient to show that $F_n[x_1, \ldots, x_n, x_{n+1}] \notin \mathscr{F}_{\mathscr{S}}$. Suppose that $F_n[x_1, \ldots, x_n, x_{n+1}] \in \mathscr{F}_{\mathscr{S}}$. Clearly, $F_n[x_1, \ldots, x_n, x_{n+1}]$ is a commutative reduced ring with unity and no other non-zero idemotents. By an argument similar to the one used in the proof of Proposition 19, $F_n[x_1, \ldots, x_n, x_{n+1}]$ is a homomorphic image of some B_S in S. Let $B_S = F_s[x_1, \ldots, x_s]$ and $F_n[x_1, \ldots, x_n, x_{n+1}] \cong B_S/I$. Then $n = s$. Indeed, if $n \neq s$, then we have, by Lemma 20, $|F_n[x_1, \ldots, x_n, x_{n+1}]| = |F_n|$ and $|F_s| = |F_s[x_1, \ldots, x_s]/I| = |B_S/I|$. Since $F_n[x_1, \ldots, x_n, x_{n+1}] \cong B_S/I$, it follows that $|F_n[x_1, \ldots, x_n, x_{n+1}]| = |B_s/I|$. But $|F_n| \neq |F_s|$, a contradiction. Thus $n = s$ and $F_n[x_1, \ldots, x_n, x_{n+1}]$ is a homomorphic image of $F_n[x_1, \ldots, x_n]$. By [10], Theorem 29, this is impossible. Therefore $F_n \notin \gamma_{(n+1)}$ and so $\gamma_{(n)} \neq \gamma_{(n+1)}$. $\square$

We notice that Theorem 22 is true for any $\mathscr{S}$. For example, take $F_i = \mathbb{Z}_{p(i)}$, where $p(i)$ is a prime number such that $p(i) \neq p(j)$ for $i \neq j$.

Remark 23. Recall that a radical γ is said to be subidempotent if γ consists of idempotent rings. Clearly, $A = F_n[x_1, \dots, x_n]$, where n is a positive integer, is an idempotent ring and hence the radical $\mathfrak{L}(\mathscr{S})$ is subidempotent.

We now consider the following well-known radicals:

- The Baer radical β. This is the upper radical determined by the class of all prime rings.
- The locally nilpotent radical $\mathscr{L}$. This is the radical class of all locally nilpotent rings.
- The Brown-McCoy radical $\mathscr{G}$. This is the upper radical determined by the class of all simple rings with unity.
- The nil radical $\mathscr{N}$. This is the radical class of all nil rings.
- Let $\mathscr{W}$ be the class of all rings A such that for each element $a \in A$, there exist elements $a_1, \dots, a_n, b_1, \dots, b_n \in A$ and $m \in \mathscr{N}$ with $a^m + \sum_{i=1}^{n} a_i[a, b_i] = 0$. In [7], Tumurbat and Wiegandt proved that $\mathscr{W}$ is a radical class and that $\mathscr{W}$ coincides with $\mathscr{G}_{(1)}$.

We notice that $\beta \subset L \subset N \subset W \subset C \subset \mathfrak{U}\mathscr{F}^1$.

Corollary 24. (i) *If γ is one of the radicals*

$$\mathfrak{L}(\beta \cup S), \qquad \mathfrak{L}(\mathscr{L} \cup \mathscr{S}), \qquad \mathfrak{L}(\mathscr{N} \cup \mathscr{S}),$$

or γ is any radical in the interval $\left[\mathfrak{L}\left(\mathscr{E}_\ell(0) \cup \mathscr{S}\right), \mathscr{F}_{\mathscr{S}}\right]$, then

$$\gamma_{(1)} \supset \gamma_{(2)} \supset \cdots \supset \gamma_{(n)} \supset \cdots.$$

(ii) *There exists a radical γ such that $\gamma_{(n)} \neq \gamma_{(n+1)}$. Moreover, $\mathscr{L} \subseteq \bigcap \gamma_{(n)}$.*

Let A be a ring. We denote by $M(A)$ the ring of all infinite matrices over A having only finitely many non-zero elements. We show that the polynomial ring $M(A)[X] \in \bigcap \mathscr{G}_{(n)}$ for any set X of commuting or noncommuting indeterminates. First, however, we need some preliminary results.

Lemma 25. *Let A be a ring with unity and I an ideal of $M(A)$. Then there exists an ideal K of A such that $I = M(K)$.*

Proof. Consider a matrix $B = (b_{ij}) \in I$ and let $(A)_{uv}$ denote the subset of $M(A)$ having non-zero elements only at the (u, v)-entry. For arbitrary indices k and l, we have

$$(A)_{ki}B(A)_{jl} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & Ab_{ij}A & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \subseteq (A)_{kl} \cap I.$$

Since A has a unity,

$$\begin{array}{cc} & \begin{array}{ccccc} & & l & & \end{array} \\ k & \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & b_{ij} & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \end{array} \in I$$

Clearly, $K = \{a \in A \mid a = b_{ij}, (b_{ij}) \in I\} \lhd A$ and $I \subseteq M(K)$. Since

$$\begin{array}{cc} & \begin{array}{ccccc} & & l & & \end{array} \\ k & \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & a & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \end{array} \in I$$

we have $M(K) \subseteq I$. Therefore $M(K) = I$. □

Corollary 26. *Let A be an arbitrary ring and I an ideal of $M(A)$. Then there exist ideals K and J of A such that $M(K) \subseteq I \subseteq M(J)$ and $M(J)^3 \subseteq M(K)$.*

Proof. We denote by A^1 the ring A with an identity adjoined. Clearly, $I \lhd M(A) \lhd M(A^1)$. Let $\langle I \rangle$ be the ideal of $M(A^1)$ generated by I. By Lemma 25, there exists and ideal J of A^1 such that $\langle I \rangle = M(J)$. By Andrunakievich's Lemma, $M(J)^3 \subseteq I$. Clearly, $J \subseteq A$. Since $M(J)^3 \lhd M(A^1)$, $M(J)^3 = M(K)$, where $K \lhd A^1$ and also $K \lhd A$. □

Corollary 27. *Let A be an arbitrary ring and let I be a semiprime ideal of $M(A)$. Then there exists an ideal K of A such that $I = M(K)$.*

Theorem 28. *Let A be an arbitrary ring. If $B \in M(A)$, then there exist $B_1, B_2, \ldots, B_n, A_1, A_2, \ldots, A_n \in M(A)$ and a positive integer m such that*

$$B^m + \sum_{i=1}^{n} B_i[B, A_i] = 0$$

and so $M(A) \in \mathscr{W}$.

Proof. First, we shall show that if $M(A)$ is a semiprime ring, then $M(A)$ has zero center. Suppose that $0 \neq B \in Z\big(M(A)\big)$, where, for a ring T, $Z(T)$ denotes the center of T and

$$B = \begin{pmatrix} b_{11} & \cdots & b_{1n} & 0 & \cdots & 0 \\ \vdots & & \vdots & & & \\ b_{n1} & & b_{nn} & 0 & \cdots & 0 \\ 0 & & 0 & 0 & & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & & 0 & 0 & & 0 \end{pmatrix}.$$

Then $b_{ij} \neq 0$ for some i, j. Let x be an arbitary element of A and let

$$\overline{X} = \begin{array}{cc} \begin{array}{ccccc} & & i & & \end{array} & \\ \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & x & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix} & \begin{array}{c} \\ \\ n+1. \\ \\ \\ \end{array} \end{array}$$

Clearly, $B\overline{X} = 0$. Since $B \in Z\big(M(A)\big)$, $\overline{X}B = 0$ and thus $xb_{ij} = 0$. Since $M(A)$ is semiprime, A is a semiprime ring. But $Ab_{ij} = 0$, a contradiction. Now we show that every non-zero prime homomorphic image of $M(A)$ has zero center. Let $M(A)/I$ be a non-zero prime homomorphic image of $M(A)$. By Corollary 27, $I = M(K)$ for an ideal K of A and so $M(A)/I = M(A)/M(K) \cong M(A/K)$. Since $M(A)/I$ is a prime ring, $Z\big(M(A)/I\big) = 0$. Thus, by [5], $M(A) \in \mathscr{G}_{(1)}$ and, by [7], for arbitrary $B \in M(A)$, there exist $B_1, B_2, \ldots, B_n, A_1, A_2, \ldots, A_n \in M(A)$ and a positive integer m such that $B^m + \sum_{i=1}^{n} B_i[B, A_i] = 0$. □

Corollary 29. *Let A be an arbirary ring. Then $M(A)[x_1, \ldots, x_n] \in \mathscr{W}$ for any positive integer n.*

Proof. In view of $M(A)[x_1] \cong M(A[x_1]) \in \mathscr{W}$, it follows by induction that $M(A)[x_1, \ldots, x_n] \in \mathscr{W}$. □

Theorem 30. *Let A be an arbitrary ring. Then $M(A)[X] \in \bigcap \mathscr{G}_{(n)}$, for any set X of commuting or noncommuting indeterminates.*

Proof. This follows from Corollary 29 and [8], Corollary 2.18(ii). □

Corollary 31. *If $\gamma_\infty = \bigcap \gamma$ for a radical γ, then we have:*

(i) $\mathscr{N} \subset \mathscr{W}_\infty \subset \mathfrak{L}(\mathscr{W} \cup \mathscr{S})_\infty$;

(ii) $\beta = \beta_\infty \subseteq \mathscr{W}_\infty \subseteq \mathscr{W} \subseteq \mathscr{G}$.

Remark 32. (i) All the known examples of radicals γ with $\gamma_{(n)} \neq \gamma_{(n+1)}$ for any positive integer n are not hereditary. We do not know, however, whether for all hereditary radicals the chain terminates.

(ii) We have $\gamma_{(n)} = \gamma_{(n+1)}$ if and only if $\gamma^{(n)} = \gamma^{(n+1)}$; hence there exist many radicals such that $\gamma^{(n+1)} \subset \gamma^{(n)}$ for any positive integer n.

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