Portugal. Math. (N.S.)
Vol. 65, Fasc. 2, 2008, 275–283 **Portugaliae Mathematica**

Wedderburn quasialgebras

Helena Albuquerque[1], Alberto Elduque[2] and José M. Pérez-Izquierdo[3]

(Communicated by Arnaldo Garcia)

Abstract. The Wedderburn–Artin Theorem for G-graded quasialgebras is proved. This provides new examples of quasialgebras.

Mathematics Subject Classification (2000). 17A01, 16W50.
Keywords. Nonassociative algebras, group graded algebras, Wedderburn algebras.

1. Introduction

Octonions are one of the best known examples of nonassociative algebras. In [3] the nonassociativity of octonions was interpreted as inherited from being an algebra in a quasitensor category. These categories have tensor products $(V \otimes W) \otimes Z \cong V \otimes (W \otimes Z)$, but these isomorphisms are not necessarily the trivial vector space isomorphisms. Nonassociative algebras coming from quasitensor categories have recently received attention in the context of noncommutative geometry and gauge theory [7].

Given a group G, a *G-graded quasialgebra* is a G-graded algebra $A = \bigoplus_{a \in G} A^a$, over a unital commutative and associative ring k, such that the product of any three homogeneous elements in A satisfies the weak associative condition

$$(xy)z = \phi(|x|, |y|, |z|)x(yz) \tag{1}$$

for some cocycle ϕ of G with values in $k^\times$ ($|x|$ denotes the degree of x). That is,

---

[1]Supported by CMUC (Centro de Matemática da Universidade de Coimbra)-FCT.
[2]Supported by the Ministerio de Educación y Ciencia and FEDER (MTM 2004-08115-C04-02), and by the Diputación General de Aragón (Grupo de Investigación de Álgebra).
[3]Supported by the CONACyT (U44100-F) and the ERASMUS programme.

$$\phi(a,e,b) = 1,$$
$$\phi(a,b,c)\phi(a,bc,d)\phi(b,c,d) = \phi(ab,c,d)\phi(a,b,cd),$$

for any $a,b,c,d \in G$, where e denotes the neutral element of G.

A quasialgebra is called a division quasialgebra if the left and right multiplications by any nonzero homogeneous elements are bijective. For unital quasialgebras, the identity element 1 belongs to A^e, and the division property is equivalent to the condition of any nonzero homogeneous element having a left and a right inverse. Division quasialgebras were considered in [5] and in [1]. The octonion algebra is a $\mathbb{Z}_2^3$-graded division quasialgebra; see [3] for details.

In [2] a classical study of the structure of quasialgebras was initiated. In that paper the authors focus upon the case that the grading group is $\mathbb{Z}_2$. A full description of $\mathbb{Z}_2$-graded quasialgebras $A = A_{\bar{0}} \oplus A_{\bar{1}}$ in the case that $A_{\bar{0}}$ is semisimple and $A_{\bar{1}}$ is a unital $A_{\bar{0}}$-bimodule was given.

By analogy to the classical case [9], we define a *Wedderburn quasialgebra* as a unital quasialgebra that has no nonzero nilpotent graded ideals and satisfies the descending chain condition on graded left ideals. The goal of this paper is to provide an analogous of the Wedderburn–Artin Theorem for quasialgebras. The results are formulated in terms of a nonassociative version of the usual matrix algebra related with the quasi-associative linear algebra introduced in [3] and extend those in [4].

In this paper G will denote a finite group and A a Wedderburn quasialgebra $A = \bigoplus_{a\in G} A^a$ with attached cocycle ϕ, over a unital commutative and associative ring k.

2. A^e is a Wedderburn algebra

Since the product of homogeneous elements in a quasialgebra is associative up to scalar multiples, it will be convenient to define $x \cdot y = \operatorname{span}\langle xy\rangle$. For homogeneous elements this 'product' is associative.

Lemma 2.1. *Let A be a Wedderburn quasialgebra. Then A^e is a Wedderburn algebra.*

Proof. Let $I_1 \supseteq I_2 \supseteq \cdots$ be a decreasing chain of left ideals of A^e, and consider the left graded ideals $\hat{I}_i = AI_i$ of A. Since A is a Wedderburn quasialgebra, the decreasing chain $\hat{I}_1 \supseteq \hat{I}_2 \supseteq \cdots$ stops. Thus $\hat{I}_n = \hat{I}_{n+i}$ for all $i \geq 0$, and $I_n = \hat{I}_n^e = \hat{I}_{n+i}^e = I_{n+i}$ for all $i \geq 0$.

Let I be an ideal of A^e with $I^2 = 0$. We will prove that $\hat{I} = (AI)A = A(IA)$ is a graded ideal of A satisfying $\hat{I}^{|G|+1} = 0$. Let $n = |G|$. Then $\hat{I}^{n+1}$ is spanned by elements in $\underline{x} = x_{a_0} \cdot x_0 \ldots x_{a_n} \cdot x_n \cdot x_{a_{n+1}}$ with $x_{a_i} \in A^{a_i}$ and $x_i \in I$. If $a_i a_{i+1} \ldots a_j = e$ with $1 \leq i \leq j \leq n$, then $x_{a_i} \cdot x_i \ldots x_{a_j} \subseteq A^e$,

$$x_{i-1} \cdot (x_{a_i} \ldots x_{a_j} \cdot x_j) \subseteq I^2 = 0$$

and $\underline{x} = 0$. Therefore, whenever e appears in $\{a_1, a_1 a_2, \ldots, a_1 \ldots a_n\}$ the corresponding generator in $\underline{x}$ vanishes. If e does not appear, then two elements collapse to the same value, say $a_1 \ldots a_i = a_1 \ldots a_j$ with $i \leq j$. But this means that $a_{i+1} \ldots a_j = e$, so $\underline{x} = 0$ again. □

3. The basic quasialgebra of A

The Wedderburn–Artin Theorem asserts that $A^e = \bigoplus_{i=1}^r B_i$ with $B_i \cong M_{n_i}(D_i)$ for some division algebras D_i. The quasi-associativity of A implies that A^a is a unital A^e-bimodule.

If e_i is the identity element of B_i for any i, then $1 = e_1 + \cdots + e_r$ and $A^a = 1A^a 1 = \bigoplus_{i,j=1}^r e_i A^a e_j$. Let A^a_{ij} be $e_i A^a e_j$, which is a unital B_i-B_j-bimodule. Hence $A^a = \bigoplus_{i,j=1}^r A^a_{ij}$. Other components B_k act trivially on A^a_{ij}, i.e., $B_k A^a_{ij} = 0$ if $k \neq i$ and $A^a_{ij} B_k = 0$ if $k \neq j$. The product $A^a \otimes_{A^e} A^b \to A^{ab}$ is a homomorphism of A^e-bimodules, so

$$A^a_{ij} A^b_{kl} = 0 \ (j \neq k) \quad \text{and} \quad A^a_{ij} A^b_{jl} \subseteq A^{ab}_{il}.$$

Let V_i be the irreducible left B_i-module (unique up to isomorphism) and $V_i^* = \mathrm{Hom}_{D_i}(V_i, D_i)$ its dual. As in [2], Proposition 9, up to isomorphism we can write A^a_{ij} as

$$A^a_{ij} = V_i \otimes_{D_i} W^a_{ij} \otimes_{D_j} V_j^*$$

for some D_i-D_j-bimodule W^a_{ij}. Since $V_j^* \otimes_{A^e} V_j \cong D_j$ as D_j-bimodules, it follows that the product $A^a_{ij} \otimes_{A^e} A^b_{jl} \to A^{ab}_{il}$ becomes an element of $\mathrm{Hom}_{B_i\text{-}B_l}(V_i \otimes_{D_i} \otimes W^a_{ij} \otimes_{D_j} W^b_{jl} \otimes_{D_l} \otimes V_l^*, V_i \otimes_{D_i} \otimes W^{ab}_{il} \otimes_{D_l} \otimes V_l^*)$. With the same arguments as in [2], proof of Theorem 10, for any D_i-D_l-bimodules W and W' the map

$$\mathrm{Hom}_{B_i-B_l}(V_i \otimes_{D_i} \otimes W \otimes_{D_l} \otimes V_l^*, V_i \otimes_{D_i} \otimes W' \otimes_{D_l} \otimes V_l^*) \to \mathrm{Hom}_{D_i-D_l}(W, W'),$$
$$\mathrm{id} \otimes f \otimes \mathrm{id} \leftrightarrow f,$$

is a bijection (since the image of an element $v_i \otimes w \otimes \varphi_l$ under an element in $\mathrm{Hom}_{B_i\text{-}B_l}(V_i \otimes_{D_i} \otimes W \otimes_{D_l} \otimes V_l^*, V_i \otimes_{D_i} \otimes W' \otimes_{D_l} \otimes V_l^*)$ is annihilated by the annihilator of v_i on the left and the annihilator of φ_l on the right, and hence this image belongs to $v_i \otimes W' \otimes \varphi_l$).

Thus, the product $A_{ij}^a \otimes A_{kl}^b \to A_{il}^{ab}$ induces a product $W_{ij}^a \otimes W_{kl}^b \to W_{il}^{ab}$ so that

$$(v_i \otimes w_{ij}^a \otimes \varphi_j)(v_k \otimes w_{kl}^b \otimes \varphi_l) = \delta_{jk} v_i \otimes \big(w_{ij}^a \varphi_j(v_j)\big) w_{kl}^b \otimes \varphi_l,$$

for any $1 \le i, j, k, l \le r$, $a, b \in G$, $v_s \in V_s$ $(s = i, k)$, $\varphi_t \in V_t^*$, $w_{st} \in W_{st}^a$ $(t = j, l)$ and $w_{ij}^a \in W_{ij}^a$, $w_{kl}^b \in W_{kl}^b$.

Definition 3.1. *The algebra $W = \bigoplus_{a \in G}(\bigoplus_{i,j=1}^r W_{ij}^a)$ with the previous product is called the basic quasialgebra of A.*

It is easy to check that the basic quasialgebra of A is a quasialgebra with the same cocycle ϕ as A. Some properties of W are collected in the following proposition.

Proposition 3.2. *Under the hypotheses above, the following holds:*

i) *$W_{ii}^e = D_i$ and $W_{ij}^e = 0$ if $i \neq j$.*

ii) *If $W_{ij}^a \neq 0$ then $W_{ij}^a W_{ji}^{a^{-1}} = D_i$. In particular, $W_{ij}^a \neq 0$ implies that $W_{ji}^{a^{-1}} \neq 0$.*

iii) *If $W_{ij}^a \neq 0$ then $W_{ij}^a = D_i w = w D_j$ for any nonzero element $w \in W_{ij}^a$.*

iv) *If $W_{ij}^a \neq 0$ then $W_{ik}^a = 0 = W_{lj}^a$ for any $k \neq j$ and $l \neq i$.*

v) *If $W_{ij}^a \neq 0$ and $W_{jl}^b \neq 0$ then $W_{ij}^a W_{jl}^b = W_{il}^{ab} \neq 0$.*

Proof. Part i) is obvious.

ii) Since $W_{ij}^a W_{ji}^{a^{-1}} \subseteq D_i$ if $W_{ij}^a W_{ji}^{a^{-1}} \neq 0$, the equality trivially holds. Hence, in order to prove ii) we are left with the case in which $W_{ij}^a W_{ji}^{a^{-1}} = 0$. If $W_{ki}^b W_{ij}^a W_{jl}^c \neq 0$ for some $bac = e$ then $k = l$ and $W_{ki}^b W_{ij}^a W_{jk}^c = D_k$, so $W_{ki}^b W_{ij}^a W_{jk}^c W_{ki}^b = W_{ki}^b \neq 0$. However, $bac = e$ implies that $cb = a^{-1}$, so $W_{jk}^c W_{ki}^b \subseteq W_{ji}^{a^{-1}}$ and $0 \neq W_{ki}^b = W_{ki}^b W_{ij}^a W_{jk}^c W_{ki}^b \subseteq W_{ki}^b W_{ij}^a W_{ji}^{a^{-1}}$. In particular $W_{ij}^a W_{ji}^{a^{-1}} \neq 0$, contrary to our assumption. Therefore, $W_{ki}^b W_{ij}^a W_{jl}^c = 0$ whenever $bac = e$. The e component of the graded ideal $I = (AA_{ij}^a)A$ vanishes. Moreover, given $|G| + 1$ elements $x_{a_i} \in I^{a_i}$, $i = 0, \dots, |G|$, some sequence $a_i, \dots, a_j$ $(i \le j)$ satisfisfies $a_i \dots a_j = e$ and hence the product of $x_{a_0}, \dots, x_{a_{|G|}}$ in any order of association is zero. The nonzero graded ideal I is then nilpotent, which is not possible.

iii) Given $w, w' \in W_{ij}^a$ and $\overline{w} \in W_{ji}^{a^{-1}}$, we have

$$(w\overline{w})w' = \phi(a, a^{-1}, a) w(\overline{w} w').$$

If $w\overline{w} = d_i \neq 0$, then $w(\overline{w} d_i^{-1}) = 1_{D_i}$ (the identity element of D_i). Thus, we choose w, $\overline{w}$ with $w\overline{w} = 1_{D_i}$, and then we get that $w' \in wD_j$. Thus $W_{ij}^a = wD_j$ is a one-dimensional D_j vector space. A dimension argument makes the equality valid for any nonzero w. Similarly $W_{ij}^a = D_i w$.

iv) $W_{ik}^{a} = D_i W_{ik}^{a} = W_{ij}^{a}(W_{ji}^{a^{-1}} W_{ik}^{a}) \subseteq W_{ij}^{a} W_{jk}^{e} = 0$ if $k \neq j$. In the same way we obtain that $W_{lj}^{a} = 0$ if $l \neq i$.

v) If $W_{ij}^{a} W_{jl}^{b} \neq 0$, then the statement follows by dimension counting. If $W_{ij}^{a} W_{jl}^{b} = 0$, then

$$W_{jl}^{b} = D_j W_{jl}^{b} = (W_{ji}^{a^{-1}} W_{ij}^{a}) W_{jl}^{b} = W_{ji}^{a^{-1}} (W_{ij}^{a} W_{jl}^{b}) = 0,$$

which contradicts the hypothesis. □

Definition 3.3. *Given $1 \leq i, j \leq r$, we say that i and j are related if $W_{ij}^{a} \neq 0$ for some $a \in G$.*

Parts i), ii) and v) of the preceding proposition show that to be related is an equivalence relation. We denote the different equivalence classes by $C_1, \dots, C_s$.

Proposition 3.4. *Let $C \in \{C_1, \dots, C_s\}$ and $I_C = \sum_{a \in G} \sum_{i,j \in C} W_{ij}^{a}$. Then I_C is a minimal graded ideal of W, which is simple as a quasialgebra. Moreover, $W = \bigoplus_{i=1}^{s} I_{C_i}$.*

Proof. Let $W_{ij}^{a} \subseteq I_C$. If $W_{kl}^{b} W_{ij}^{a} \neq 0$, then $l = i$ and k and j are related, so $j, k \in C$ and $W_{kl}^{b} W_{ij}^{a} = W_{kj}^{ba} \subseteq I_C$ by Proposition 3.2. This shows that $WI_C \subseteq I_C$. In the same way we obtain that $I_C W \subseteq I_C$.

Given a nonzero graded ideal $J \subseteq W$ with $J \subseteq I_C$, there exists $J_{ij}^{a} \neq 0$, with $i, j \in C$. By dimension counting, we have $J_{ij}^{a} = W_{ij}^{a}$, so $W_{ij}^{a} W_{ij}^{a^{-1}} = D_i \subseteq J$. Fix any $0 \neq W_{kl}^{b} \subseteq I_C$. Since k is related to i, there exists $0 \neq W_{ki}^{c} = W_{ki}^{c} D_i \subseteq J$. Again $D_k \subseteq J$, and $W_{kl}^{b} = D_k W_{kl}^{b} \subseteq J$. Therefore, $J = I_C$.

Let $J \subseteq I_C$ be a nonzero graded ideal of I_C. If $l \notin C$, then $W_{kl}^{b} J = 0 = J W_{lk}^{c}$ for all k, b and c. Hence J is an ideal of W and so $J = I_C$. Thus I_C is simple. □

Corollary 3.5. *$A_C = \sum_{a \in G} \sum_{i,j \in C} V_i \otimes_{D_i} W_{ij}^{a} \otimes_{D_j} \otimes V_j^{*}$ is a minimal graded ideal of A, which is simple as a quasialgebra. Moreover, $A = \bigoplus_{i=1}^{s} A_{C_i}$.*

4. Classifying the basic quasialgebra

Corollary 3.5 reduces the study of Wedderburn quasialgebras to the study of those that in addition satisfy that for any $1 \leq i, j \leq r$ there exists $a \in G$ with $A_{ij}^{a} \neq 0$. The basic quasialgebra W of A satisfies the same property.

Let R be a quasialgebra with cocycle ϕ and let $|1|, \dots, |r|$ be elements in the grading group G. In the set of all matrices $M_r^{\phi}(R) = \text{span}\langle x_{ij} = E_{ij} x \mid x \in R$ and $1 \leq i, j \leq r \rangle$ we define the gradation (cf. [8], I.5)

$$|x_{ij}| = |i|\,|x|\,|j|^{-1} \qquad \text{for homogeneous } x,$$

and a product

$$x_{ij}y_{kl} = \delta_{jk}\frac{\phi(|i|,|x|\,|j|^{-1},|j|\,|y|\,|l|^{-1})}{\phi(|x|\,|j|^{-1},|j|,|y|\,|l|^{-1})}\frac{\phi(|x|,|j|^{-1},|j|)}{\phi(|x|,|y|,|l|^{-1})}(xy)_{il}. \tag{2}$$

A straightforward computation gives the following extension of Proposition 7.3 in [3]. (Think of x_{ij} as $v_i(x\varphi_j)$.)

Proposition 4.1. *$M_r^{\phi}(R)$ is a quasialgebra with cocycle ϕ.*

Let A be a simple Wedderburn quasialgebra with $A^e = \bigoplus_{i=1}^r B_i$ as before, and let W be its basic quasialgebra. By Corollary 3.5, for any $i \neq j$ there exists an element $a \in G$ such that $A_{ij}^a \neq 0$ (or $W_{ij}^a \neq 0$).

Our goal is to prove that $W \cong M_r^{\phi}(\hat{D})$ for some division quasialgebra $\hat{D}$.

The division quasialgebra. Proposition 3.2 shows, in particular, that $\hat{D} = \bigoplus_{a\in G} W_{11}^a$ is a division quasialgebra. This is the division quasialgebra we are looking for. Note that $N = N_{11} = \{a \in G \mid W_{11}^a \neq 0\}$ is a subgroup of G and $\hat{D} = \bigoplus_{a\in N} W_{11}^a$.

The grading elements. Let $N_{ij} = \{a \in G \mid W_{ij}^a \neq 0\}$. Our hypotheses on A imply that $N_{ij} \neq \emptyset$. Given $a \in N_{ij}$ and $0 \neq b \in N_{ik}$, it follows that $b^{-1} \in N_{ki}$ and $b^{-1}a \in N_{kj}$, and so $N_{ij} = bN_{kj}$. Similarly, $N_{ij} = N_{ik}c$ for any $c \in N_{kj}$.

Fix $a_{i1} \in N_{i1}$, $i = 1, \dots, r$ with $a_{11} = e$. Clearly

$$N_{ij} = a_{i1}N_{1j} = a_{i1}Na_{j1}^{-1}.$$

The elements in G that provide the gradation are $|i| = a_{i1}$.

Now fix $w_{i1} \in W_{i1}^{|i|}$ with $w_{11} = 1_{D_1}$ and take $w_{1i} \in W_{1i}^{|i|^{-1}}$ such that $w_{1i}w_{i1} = 1$. Observe that

$$w_{i1}(\hat{D}w_{1j}) = w_{i1}\Big(\bigoplus_{a\in N} W_{11}^a\Big)w_{1j} = w_{i1}\Big(\bigoplus_{a\in N} W_{1j}^{a|j|^{-1}}\Big) = \bigoplus_{a\in N_{ij}} W_{ij}^a.$$

Then $W = \bigoplus_{i,j=1}^r w_{i1}(\hat{D}w_{1j})$.

Given $d \in \hat{D}$ define $d_{ij} = w_{i1}(dw_{1j})$. If d is homogeneous then

$$|d_{ij}| = |i|\,|d|\,|j|^{-1}.$$

Proposition 4.2. *The map*

$$M_r^{\phi}(\hat{D}) \to W,$$
$$E_{ij}d \mapsto d_{ij},$$

is an isomorphism of quasialgebras.

Proof. Clearly the map is bijective and preserves the gradation, so we only have to check that the product of the elements d_{ij} is given by an analogue of (2). For homogeneous $d, d' \in \hat{D}$ we have

$$\begin{aligned} d_{ij}d'_{jl} &= \big(w_{i1}(dw_{1j})\big)\big(w_{j1}(d'w_{1l})\big) \\ &= \frac{\phi(|i|, |d|\,|j|^{-1}, |j|\,|d'|\,|l|^{-1})}{\phi(|d|\,|j|^{-1}, |j|, |d'|\,|l|^{-1})} w_{i1}\big(((dw_{1j})w_{j1})(d'w_{1l})\big) \\ &= \frac{\phi(|i|, |d|\,|j|^{-1}, |j|\,|d'|\,|l|^{-1})}{\phi(|d|\,|j|^{-1}, |j|, |d'|\,|l|^{-1})} \frac{\phi(|d|, |j|^{-1}, |j|)}{\phi(|d|, |d'|, |l|^{-1})} w_{i1}\big((dd')w_{1l}\big), \\ &= \frac{\phi(|i|, |d|\,|j|^{-1}, |j|\,|d'|\,|l|^{-1})}{\phi(|d|\,|j|^{-1}, |j|, |d'|\,|l|^{-1})} \frac{\phi(|d|, |j|^{-1}, |j|)}{\phi(|d|, |d'|, |l|^{-1})} (dd')_{il}, \end{aligned}$$

and the result follows. □

The maps $\sigma_{i1} : D_i \to D_1$ given by $w_{i1}d = \sigma_{i1}(d)w_{i1}$ are isomorphisms of algebras, which allow to identify D_i with $D = D_1$. Thus,

$$w_{i1}\alpha = \alpha w_{i1} \qquad \text{for all } \alpha \in D.$$

In particular, since $(w_{1j}\alpha)w_{j1} = w_{1j}(\alpha w_{j1}) = w_{1j}(w_{j1}\alpha) = (w_{1j}w_{j1})\alpha = \alpha = (\alpha w_{1j})w_{j1}$, it follows that $w_{1j}\alpha = \alpha w_{1j}$ too, and hence that $d_{ij}\alpha = (d\alpha)_{ij}$ and $\alpha d_{ij} = (\alpha d)_{ij}$ for any $\alpha \in D$, $d \in \hat{D}$, and $i, j = 1, \dots, r$.

5. Wedderburn–Artin Theorem

Note first that, given any natural numbers $n_1, \dots, n_r$, the matrix algebra $M_{n_1+\cdots+n_r}(k)$ has a basis consisting of the elements

$$E_{ij}^{kl} = E_{n_1+\cdots+n_{i-1}+k, n_1+\cdots+n_{j-1}+l}$$

where, as usual, E_{ij} denotes the matrix with 1 in the (i, j)-entry and 0's elsewhere. These basic elements multiply as follows:

$$E_{ij}^{kl}E_{mn}^{pq} = \delta_{jm}\delta_{lp}E_{in}^{kq}.$$

As in the previous section, consider a quasialgebra R with cocycle ϕ, and given some grading elements $|1|, \dots, |r| \in G$, take natural numbers $n_1, \dots, n_r$ and consider

$$\tilde{M}^{\phi}_{n_1,\dots,n_r}(R) = \operatorname{span}\langle x_{ij}^{kl} = E_{ij}^{kl}x \mid x \in R, 1 \le i, j \le r, 1 \le k \le n_i, 1 \le l \le n_j \rangle.$$

This is G-graded by

$$|x_{ij}^{kl}| = |i|\,|x|\,|j|^{-1} \qquad \text{for homogeneous } x,$$

and it is endowed with the multiplication given by

$$x_{ij}^{kl} y_{mn}^{pq} = \delta_{jm}\delta_{lp} \frac{\phi(|i|, |x|\,|j|^{-1}, |j|\,|y'|\,|n|^{-1})}{\phi(|x|\,|j|^{-1}, |j|, |y|\,|n|^{-1})} \frac{\phi(|x|, |j|^{-1}, |j|)}{\phi(|x|, |y|, |n|^{-1})} (xy)_{in}^{kq}.$$

Again, $\tilde{M}^{\phi}_{n_1,\dots,n_r}(R)$ is a quasialgebra with cocycle ϕ, as in the previous section.

Let us return to a simple Wedderburn quasialgebra A with cocycle ϕ. Let $A_{ij} = \bigoplus_{a\in N_{ij}} A_{ij}^a$ with $N_{ij} = \{a \in G \mid A_{ij}^a \neq 0\}$. We write A_{ij} as $A_{ij} = V_i \otimes_D W_{ij} \otimes_D V_j^*$ with $W_{ij} = w_{i1}(\hat{D}w_{1j})$, where D_i and D_j are identified with D through σ_{i1} and σ_{j1}.

For any $i = 1, \dots, r$, let $n_i = \dim_{D_i} V_i$, and let $\{v_1^i, \dots, v_{n_i}^i\}$ and $\{\varphi_1^i, \dots, \varphi_{n_i}^i\}$ be dual bases in V_i and V_i^*. Then the linear map determined by

$$A \to \tilde{M}^{\phi}_{n_1,\dots,n_r}(\hat{D}),$$
$$v_k^i \otimes d_{ij} \otimes \varphi_l^j \mapsto E_{ij}^{kl} d,$$

is shown to be an isomorphism by a straightforward computation. Thus we obtain our main result:

Theorem 5.1. *Let A be a Wedderburn quasialgebra. Then A is isomorphic to a finite direct sum of quasialgebras of the form $\tilde{M}^{\phi}_{n_1,\dots,n_r}(\hat{D})$.*

References

[1] H. Albuquerque, A. Elduque, and J. M. Pérez-Izquierdo, Alternative quasialgebras. *Bull. Austral. Math. Soc.* **63** (2001), 257–268. Zbl 0999.17007 MR 1823712

[2] H. Albuquerque, A. Elduque, and J. M. Pérez-Izquierdo, $\mathbb{Z}_2$-quasialgebras. *Comm. Algebra* **30** (2002), 2161–2174. Zbl 1035.17006 MR 1904631

[3] H. Albuquerque and S. Majid, Quasialgebra structure of the octonions. *J. Algebra* **220** (1999), 188–224. Zbl 0999.17006 MR 1713433

[4] H. Albuquerque and A. P. Santana, Quasiassociative algebras with a simple Artinian null part. *Algebra i Logika* **43** (2004), 551–564; English transl. *Algebra Logic* **43** (2004), 307–315. Zbl 1101.17001 MR 2112058

[5] H. Albuquerque and A. P. Santana, A note on quasiassociative algebras. In *The J. A. Pereira da Silva birthday schrift*, Textos Mat. Sér. B 32, Universidade de Coimbra, Coimbra 2002, 5–16. Zbl 1038.17002 MR 1969431

[6] H. Albuquerque and S. Majid, Z_n-quasialgebras. In *Matrices and group representations* (Coimbra, 1998), Textos Mat. Sér. B 19, Universidade de Coimbra, Coimbra 1999, 57–64. Zbl 1041.17501 MR 1773570

[7] S. Majid, Gauge theory on nonassociative spaces. *J. Math. Phys.* **46** (2005), no. 10, 103519. Zbl 1111.81101 MR 2178619

[8] C. Năstăsescu and F. van Oystaeyen, *Graded ring theory*. North-Holland Math. Library 28, North-Holland, Amsterdam 1982. Zbl 0494.16001 MR 0676974

[9] D. S. Passman, *A course in ring theory*. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1991. Zbl 0783.16001 MR 1096302

Received December 27, 2006; revised May 29, 2007

H. Albuquerque, Departamento de Matemática da Universidade de Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal
E-mail: lena@mat.uc.pt

A. Elduque, Departamento de Matemáticas, Universidad de Zaragoza, 50009 Zaragoza, Spain
E-mail: elduque@unizar.es

J. M. Pérez-Izquierdo, Departamento de Matemáticas y Computación, Universidad de La Rioja, 26004 Logroño, Spain
E-mail: jm.perez@unirioja.es