Portugal. Math. (N.S.)
Vol. 65, Fasc. 2, 2008, 285–309

Portugaliae Mathematica

The Fourier–Borel transform between spaces of entire functions of a given type and order

Vinícius V. Fávaro*

(Communicated by Luís Barreira)

Abstract. In this article we study the Fourier–Borel transform between the dual of $\mathrm{Exp}^{k}_{\tilde{N},(s;(r,q)),A}(E)$, the space of entire functions on E of $(s;(r,q))$-quasi-nuclear order k and $(s;(r,q))$-quasi-nuclear type strictly less than A, and the space $\mathrm{Exp}^{k'}_{(s',m(r';q')),0,(\theta(k)A)^{-1}}(E')$ of entire functions on E' of $(s',m(r';q'))$-summing order k and $(s',m(r';q'))$-summing type less than or equal to $(\theta(k)A)^{-1}$. This mapping identifies algebraically and topologically these two spaces. On the dual space it is considered the strong topology. This generalizes results of Matos [4] and Martineau [3].

Mathematics Subject Classification (2000). 46G20, 46G25.

Keywords. Fourier–Borel transform, quasi-nuclear polynomial, $(s;m(r,q))$-summing polynomial.

1. Introduction and notation

The results of this article are a generalization of results obtained by Matos in [4], and the main results are the algebraic isomorphisms given by the Fourier–Borel transforms in Theorems 3.4, 3.5 and 3.9 as well as the topological isomorphism given by the Fourier–Borel transform in Theorem 4.5.

See Matos [5] for the theory used to define these spaces and the duality results which are the key to prove the theorems of the Fourier–Borel transforms. The notation follows Matos, [4], [5]: if E is a complex Banach space, then $\mathscr{H}(E)$ is the vector space of all entire functions in E; $\mathscr{P}_{(s;m(r,q))}({}^{n}E)$ is the Banach space of all n-homogeneous polynomials in E that are $(s;m(r,q))$-summing in 0, with the norm $\|\cdot\|_{(s,m(r;q))}$; and $\mathscr{P}_{\tilde{N},(s;(r,q))}({}^{n}E)$ is the Banach space of all $(s;(r,q))$-quasi-nuclear n-homogeneous polynomials in E, with the norm $\|\cdot\|_{\tilde{N},(s;(r,q))}$ for all $j \in \mathbb{N}$ where $\mathbb{N} = \{0,1,2,\ldots\}$.

*Supported by Fapesp, process 04/13520-3.

In the definitions involving $(s, m(r; q))$-summing polynomials we consider $0 < q \leq r \leq +\infty$ and $s \in [1, +\infty]$, and in the definitions involving $(s; (r, q))$-quasi-nuclear polynomials we consider $s \leq q$, $r \leq q$ and $s, r, q \in [1, +\infty]$.

2. Spaces of entire functions of a given type and order

Definition 2.1. If $\rho > 0$ and $k \geq 1$, we denote by $\mathscr{B}^k_{(s,m(r;q)),\rho}(E)$ the complex vector space of all $f \in \mathscr{H}(E)$ such that $\hat{d}^j f(0) \in \mathscr{P}_{(s,m(r;q))}({}^jE)$ for all $j \in \mathbb{N}$ and

$$\|f\|_{(s,m(r;q)),k,\rho} = \sum_{j=0}^{\infty} \rho^{-j} \left(\frac{j}{ke}\right)^{j/k} \left\| \frac{1}{j!} \hat{d}^j f(0) \right\|_{(s,m(r;q))} < +\infty,$$

normed by $\|\cdot\|_{(s,m(r;q)),k,\rho}$. We denote by $\mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho}(E)$ the complex vector space of all $f \in \mathscr{H}(E)$ such that $\hat{d}^j f(0) \in \mathscr{P}_{\tilde{N},(s;(r,q))}({}^jE)$ for all $j \in \mathbb{N}$ and

$$\|f\|_{\tilde{N},(s;(r,q)),k,\rho} = \sum_{j=0}^{\infty} \rho^{-j} \left(\frac{j}{ke}\right)^{j/k} \left\| \frac{1}{j!} \hat{d}^j f(0) \right\|_{\tilde{N},(s;(r,q))} < +\infty,$$

with norm $\|\cdot\|_{\tilde{N},(s;(r,q)),k,\rho}$.

Proposition 2.2. *For each $\rho > 0$ and $k \geq 1$, $\mathscr{B}^k_{\tilde{N},(s,(r,q)),\rho}(E)$ and $\mathscr{B}^k_{(s,m(r;q)),\rho}(E)$ are Banach spaces.*

Proof. The proof is analogous to that in Matos [4] for the spaces $\mathscr{B}^k_{N,\rho}(E)$ and $\mathscr{B}^k_{\rho}(E)$. □

Definition 2.3. If $A \in (0, +\infty)$ and $k \geq 1$, we denote by $\mathrm{Exp}^k_{(s,m(r;q)),A}(E)$ and $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$ the complex vector spaces $\bigcup_{\rho<A} \mathscr{B}^k_{(s,m(r;q)),\rho}(E)$ and $\bigcup_{\rho<A} \mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho}(E)$ with the locally convex inductive limit topologies, respectively. We consider the complex vector spaces $\mathrm{Exp}^k_{(s,m(r;q)),0,A}(E) = \bigcap_{\rho>A} \mathscr{B}^k_{(s,m(r;q)),\rho}(E)$ and $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),0,A}(E) = \bigcap_{\rho>A} \mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho}(E)$ with the projective limit topologies.

If $A = +\infty$ and $k \geq 1$, we consider the complex vector spaces

$$\mathrm{Exp}^k_{(s,m(r;q)),\infty}(E) = \bigcup_{\rho>0} \mathscr{B}^k_{(s,m(r;q)),\rho}(E)$$

and $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),\infty}(E) = \bigcup_{\rho>0} \mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho}(E)$ with the locally convex inductive limit topologies, and if $A = 0$ and $k \geq 1$, we consider the complex vector spaces $\mathrm{Exp}^k_{(s,m(r;q)),0}(E) = \bigcap_{\rho>0} \mathscr{B}^k_{(s,m(r;q)),\rho}(E)$ and

$$\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),0}(E) = \bigcap_{\rho>0} \mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho}(E)$$

with the projective limit topologies.

The next two results give characterizations of these spaces.

Proposition 2.4. *If $f \in \mathscr{H}(E)$ is such that $\hat{d}^j f(0) \in \mathscr{P}_{(s,m(r;q))}({}^jE)$ for all $j \in \mathbb{N}$, then the following holds:*

(a) *For each $A \in (0,+\infty]$,*

$$f \in \mathrm{Exp}^k_{(s,m(r;q)),A}(E) \quad \textit{if and only if} \quad \limsup_{j\to\infty} \left(\frac{j}{ke}\right)^{1/k} \left\|\frac{1}{j!}\hat{d}^j f(0)\right\|^{1/j}_{(s,m(r;q))} < A.$$

(b) *For each $A \in [0,+\infty)$,*

$$f \in \mathrm{Exp}^k_{(s,m(r;q)),0,A}(E) \quad \textit{if and only if} \quad \limsup_{j\to\infty} \left(\frac{j}{ke}\right)^{1/k} \left\|\frac{1}{j!}\hat{d}^j f(0)\right\|^{1/j}_{(s,m(r;q))} \le A.$$

Proposition 2.5. *If $f \in \mathscr{H}(E)$ is such that $\hat{d}^j f(0) \in \mathscr{P}_{\tilde{N},(s;(r,q))}({}^jE)$ for all $j \in \mathbb{N}$, then:*

(a) *For each $A \in (0,+\infty]$,*

$$f \in \mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E) \quad \textit{if and only if} \quad \limsup_{j\to\infty} \left(\frac{j}{ke}\right)^{1/k} \left\|\frac{1}{j!}\hat{d}^j f(0)\right\|^{1/j}_{\tilde{N},(s;(r,q))} < A.$$

(b) *For each $A \in [0,+\infty)$,*

$$f \in \mathrm{Exp}^k_{\tilde{N},(s;(r,q)),0,A}(E) \quad \textit{if and only if} \quad \limsup_{j\to\infty} \left(\frac{j}{ke}\right)^{1/k} \left\|\frac{1}{j!}\hat{d}^j f(0)\right\|^{1/j}_{\tilde{N},(s;(r,q))} \le A.$$

Due to these two results the elements of $\mathrm{Exp}^k_{(s,m(r;q)),A}(E)$ are called *entire functions of $(s,m(r;q))$-summing order k and $(s,m(r;q))$-summing type strictly less than A*, and the elements of $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$ are called *entire functions of $(s;(r,q))$-quasi-nuclear order k and $(s;(r,q))$-quasi-nuclear type strictly less than A.* For $A = +\infty$ we omit “strictly less than A”.

We call the elements of $\mathrm{Exp}^k_{(s,m(r;q)),0,A}(E)$ *entire functions of $(s,m(r;q))$-summing order k and $(s,m(r;q))$-summing type less than or equal to A*, and the elements of $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),0,A}(E)$ are called *entire functions of $(s;(r,q))$-quasi-nuclear order k and $(s;(r,q))$-quasi-nuclear type less than or equal to A.*

Proposition 2.6. (a) *For each $A \in (0,+\infty]$ and $k > 1$, $\mathrm{Exp}^k_{(s,m(r;q)),A}(E)$ and $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$ are DF-spaces.*

(b) *For each* $A \in [0, +\infty)$ *and* $k > 1$, $\mathrm{Exp}^k_{(s,m(r;q)),0,A}(E)$ *and* $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),0,A}(E)$ *are Fréchet spaces.*

Proof. For part (a), let $(a_n)_{n=1}^{\infty}$ be a strictly increasing sequence of positive real numbers converging to A. Hence $\bigcup_{\rho<A} \mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho}(E) = \bigcup_{n=1}^{\infty} \mathscr{B}^k_{\tilde{N},(s;(r,q)),a_n}(E)$ and the inductive limit topologies given by $\mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho}(E)$, $\rho < A$ and $\mathscr{B}^k_{\tilde{N},(s;(r,q)),a_n}(E)$, $n \in \mathbb{N}$, are equal. Since the inductive limit of a sequence of DF-spaces is a DF-space, we have that $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$ is a DF-space. The proof for $\mathrm{Exp}^k_{(s,m(r;q)),A}(E)$ is analogous.

For part (b), since $\mathrm{Exp}^k_{(s,m(r;q)),0,A}(E)$ and $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),0,A}(E)$ are projective limits of Banach spaces, they are complete locally convex spaces. Now let $(b_n)_{n=1}^{\infty}$ be a strictly decreasing sequence of positive real numbers converging to A. Then the metrizability of $\mathrm{Exp}^k_{(s,m(r;q)),0,A}(E)$ and $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),0,A}(E)$ follows from the fact that the topologies of these spaces and the topologies generated by $\|\cdot\|_{(s,m(r;q)),b_n}$ and $\|\cdot\|_{\tilde{N},(s;(r,q)),b_n}$, $n \in \mathbb{N}$, coincide. □

Now we construct similar spaces of entire functions of infinite order.

Definition 2.7. If $A \in [0, +\infty)$, we denote by $\mathscr{H}_{b(s,m(r;q))}(B_{1/A}(0))$ the complex vector space of all $f \in \mathscr{H}(B_{1/A}(0))$ such that $\hat{d}^j f(0) \in \mathscr{P}_{(s,m(r;q))}({}^jE)$ for all $j \in \mathbb{N}$ and

$$\limsup_{j\to\infty} \left\| \frac{1}{j!} \hat{d}^j f(0) \right\|_{(s,m(r;q))}^{1/j} \le A,$$

endowed with the locally convex topology generated by the family of seminorms $(p^{\infty}_{(s,m(r;q)),\rho})_{\rho>A}$, where

$$p^{\infty}_{(s,m(r;q)),\rho}(f) = \sum_{j=0}^{\infty} \rho^{-j} \left\| \frac{1}{j!} \hat{d}^j f(0) \right\|_{(s,m(r;q))}.$$

We denote by $\mathscr{H}_{\tilde{N}b,(s;(r,q))}(B_{1/A}(0))$ the complex vector space of all $f \in \mathscr{H}(B_{1/A}(0))$ such that $\hat{d}^j f(0) \in \mathscr{P}_{\tilde{N},(s;(r,q))}({}^jE)$ for all $j \in \mathbb{N}$ and

$$\limsup_{j\to\infty} \left\| \frac{1}{j!} \hat{d}^j f(0) \right\|_{\tilde{N},(s;(r,q))}^{1/j} \le A,$$

endowed with the locally convex topology generated by the family of seminorms $(p^{\infty}_{\tilde{N},(s;(r,q)),\rho})_{\rho>A}$, where

$$p^{\infty}_{\tilde{N},(s;(r,q)),\rho}(f) = \sum_{j=0}^{\infty} \sum_{j=0}^{\infty} \rho^{-j} \left\| \frac{1}{j!} \hat{d}^j f(0) \right\|_{\tilde{N},(s;(r,q))}.$$

We denote $\mathscr{H}_{b(s,m(r;q))}(B_{1/A}(0))$ by $\mathrm{Exp}^{\infty}_{(s,m(r;q)),0,A}(E)$ and $\mathscr{H}_{\tilde{N}b,(s;(r,q))}(B_{1/A}(0))$ by $\mathrm{Exp}^{\infty}_{\tilde{N},(s;(r,q)),0,A}(E)$, and we also write $\mathrm{Exp}^{\infty}_{(s,m(r;q)),0}(E) = \mathrm{Exp}^{\infty}_{(s,m(r;q)),0,0}(E)$ and $\mathrm{Exp}^{\infty}_{\tilde{N},(s;(r,q)),0}(E) = \mathrm{Exp}^{\infty}_{\tilde{N},(s;(r,q)),0,0}(E)$.

Proposition 2.8. *If $A \in [0,+\infty)$, then*

$$\mathscr{H}_{b(s,m(r;q))}(B_{1/A}(0)) \quad \text{and} \quad \mathscr{H}_{\tilde{N}b,(s;(r,q))}(B_{1/A}(0))$$

are Fréchet spaces.

Proof. Since both $(p^{\infty}_{\tilde{N},(s;(r,q)),a_n})_{n=1}^{\infty}$ and $(p^{\infty}_{\tilde{N},(s;(r,q)),\rho})_{\rho>A}$ generate the same topology, where $(a_n)_{n=1}^{\infty}$ is a strictly decreasing sequence of positive real numbers converging to A, we have that $\mathscr{H}_{\tilde{N}b,(s;(r,q))}(B_{1/A}(0))$ is a locally convex and metrizable topological vector space. Now the completeness follows as mentioned in Proposition 2.2.

For $\mathscr{H}_{b(s,m(r;q))}(B_{1/A}(0))$ the proof is analogous. □

Now we use the definitions of the spaces $\mathscr{H}_{b(s,m(r;q))}(B_{1/A}(0))$ and $\mathscr{H}_{\tilde{N}b,(s;(r,q))}(B_{1/A}(0))$ to construct new spaces as follows:

Let $L = \bigcup_{\rho<A} \mathscr{H}_{\tilde{N}b,(s;(r,q))}(B_{1/\rho}(0))$ and define the following relation:

$$f \sim g \iff \text{there is } \rho \in (0,A) \text{ such that } f|_{B_{1/\rho}(0)} = g|_{B_{1/\rho}(0)}.$$

It is obvious that $\sim$ is an equivalence relation. We denote by $L/\sim$ the set of all equivalence classes of elements of L and by $[f]$ the equivalence class which has f as one representant. Now we define the following operations in $L/\sim$:

$$[f] + [g] = [f|_{B_{1/\rho}(0)} + g|_{B_{1/\rho}(0)}],$$

where $\rho \in (0,A)$ is such that $f|_{B_{1/\rho}(0)}, g|_{B_{1/\rho}(0)} \in \mathscr{H}_{\tilde{N}b,(s;(r,q))}(B_{1/\rho}(0))$, and

$$\lambda[f] = [\lambda f], \qquad \lambda \in \mathbb{C}.$$

With these two operations $L/\sim$ is a vector space. The case $(s,m(r;q))$ is analogous.

For each $\rho \in (0,A)$, let $i_\rho : \mathscr{H}_{\tilde{N}b,(s;(r,q))}(B_{1/\rho}(0)) \to L/\sim$ be given by $i_\rho(f) = [f]$.

Definition 2.9. If $A \in (0,+\infty]$, we define $\mathscr{H}_{\tilde{N}b,(s;(r,q))}(\overline{B_{1/A}(0)}) = L/\sim$ with the locally convex inductive limit topology generated by the $(i_\rho)_{\rho\in(0,A)}$.

In the same way we construct the space $\mathscr{H}_{b(s,m(r;q))}(\overline{B_{1/A}(0)})$.

Now we define the following spaces:

Definition 2.10. If $\rho > 0$, let $\mathscr{H}^{\infty}_{(s,m(r;q))}(B_{1/\rho}(0))$ be the complex vector space of all $f \in \mathscr{H}(B_{1/\rho}(0))$ such that $\hat{d}^j f(0) \in \mathscr{P}_{(s,m(r;q))}({}^jE)$ for all $j \in \mathbb{N}$ and

$$\sum_{j=0}^{\infty} \rho^{-j} \left\| \frac{1}{j!} \hat{d}^j f(0) \right\|_{(s,m(r;q))} < +\infty,$$

which is a Banach space with the norm $p^{\infty}_{(s,m(r;q)),\rho}$. Moreover, let $\mathscr{H}^{\infty}_{\tilde{N},(s;(r,q))}(B_{1/\rho}(0))$ be the complex vector space of all $f \in \mathscr{H}(B_{1/\rho}(0))$ such that $\hat{d}^j f(0) \in \mathscr{P}_{\tilde{N},(s;(r,q))}({}^jE)$ for all $j \in \mathbb{N}$ and

$$\sum_{j=0}^{\infty} \rho^{-j} \left\| \frac{1}{j!} \hat{d}^j f(0) \right\|_{\tilde{N},(s;(r,q))} < +\infty,$$

which is a Banach space as well with the norm $p^{\infty}_{\tilde{N},(s;(r,q)),\rho}$.

As in Definition 2.9, we consider an equivalence relation in $L = \bigcup_{\rho<A} \mathscr{H}^{\infty}_{(s,m(r;q))}(B_{1/\rho}(0))$, and for $A \in (0, +\infty]$ we define $\mathrm{Exp}^{\infty}_{(s,m(r;q)),A}(E) = L/\sim = \bigcup_{\rho<A} \mathscr{H}^{\infty}_{(s,m(r;q))}(B_{1/\rho}(0))/\sim$ with the locally convex inductive limit topology. We also define $\mathrm{Exp}^{\infty}_{\tilde{N},(s;(r,q)),A}(E) = \bigcup_{\rho<A} \mathscr{H}^{\infty}_{\tilde{N}b,(s;(r,q))}(B_{1/\rho}(0))/\sim$ with the locally convex inductive limit topology.

The next result assures that Definitions 2.9 and 2.10 are equivalent:

Proposition 2.11. *The spaces* $\mathscr{H}_{\tilde{N}b,(s;(r,q))}(\overline{B_{1/A}(0)})$ *and* $\mathrm{Exp}^{\infty}_{\tilde{N},(s;(r,q)),A}(E)$ *coincide algebraically and are topologically isomorphic, and the same holds for the spaces* $\mathscr{H}_{b(s,m(r;q))}(\overline{B_{1/A}(0)})$ *and* $\mathrm{Exp}^{\infty}_{(s,m(r;q)),A}(E)$.

Proof. Straightforward. □

Due to Proposition 2.11 we denote the spaces $\mathscr{H}_{b(s,m(r;q))}(\overline{B_{1/A}(0)})$ and $\mathscr{H}_{\tilde{N}b,(s;(r,q))}(\overline{B_{1/A}(0)})$ by $\mathrm{Exp}^{\infty}_{(s,m(r;q)),A}(E)$ and $\mathrm{Exp}^{\infty}_{\tilde{N},(s;(r,q)),A}(E)$, respectively.

Since $\lim_{k\to\infty}\left(\frac{j}{ke}\right)^{1/k} = 1$, we may use notations $\|\cdot\|_{(s,m(r;q)),\infty,\rho}$ and $\|\cdot\|_{\tilde{N},(s;(r,q)),\infty,\rho}$ for $p^{\infty}_{(s,m(r;q)),\rho}$ and $p^{\infty}_{\tilde{N},(s;(r,q)),\rho}$, respectively.

Proposition 2.12. (a) *For each* $A \in (0, +\infty]$,

$$\mathrm{Exp}^{\infty}_{(s,m(r;q)),A}(E) \quad \textit{and} \quad \mathrm{Exp}^{\infty}_{\tilde{N},(s;(r,q)),A}(E)$$

are DF-spaces.

(b) *For each* $A \in [0, +\infty)$, $\mathrm{Exp}^{\infty}_{(s,m(r;q)),0,A}(E)$ *and* $\mathrm{Exp}^{\infty}_{\tilde{N},(s;(r,q)),0,A}(E)$ *are Fréchet spaces.*

Proof. Note that (b) is Proposition 2.8, and the proof for (a) follows as in Proposition 2.6. □

Proposition 2.13. (a) *If* $k \in [1, +\infty]$ *and* $A \in (0, +\infty]$, *then the Taylor series at* 0 *for each element of* $\mathrm{Exp}^k_{(s,m(r;q)),A}(E)$ (*respectively,* $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$) *converges to the element in the topology of the space.*

(b) *If* $k \in [1, +\infty]$ *and* $A \in [0, +\infty)$, *then the Taylor series at* 0 *for each element of* $\mathrm{Exp}^k_{(s,m(r;q)),0,A}(E)$ (*respectively,* $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),0,A}(E)$) *converges to the element in the topology of the space.*

Proof. In each case it is enough to consider the difference $f - \sum_{j=0}^{n} \frac{1}{j!} \hat{d}^j f(0)$ and the corresponding norm of the space in question. □

Proposition 2.14. (a) *If* $k \in (1, +\infty]$ *and* $A \in (0, +\infty]$, *then* e^{φ} *belongs to* $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$ *for all* $\varphi \in E'$.

(b) *If* $k \in (1, +\infty]$ *and* $A \in [0, +\infty)$, *then* e^{φ} *belongs to* $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),0,A}(E)$ *for all* $\varphi \in E'$.

(c) *If* $k = 1$ *and* $A \in (0, +\infty]$, *then* e^{φ} *belongs to* $\mathrm{Exp}^1_{\tilde{N},(s;(r,q)),A}(E)$ *for all* $\varphi \in E'$ *such that* $\|\varphi\| < A$.

(d) *If* $k = 1$ *and* $A \in (0, +\infty]$, *then* e^{φ} *belongs to* $\mathrm{Exp}^1_{\tilde{N},(s;(r,q)),0,A}(E)$ *for all* $\varphi \in E'$ *such that* $\|\varphi\| \leq A$.

Proof. This follows from the definitions and characterizations of the spaces and the fact that $\|\hat{d}^j(e^{\varphi})(0)\| = \|\varphi\|^j = \|\hat{d}^j(e^{\varphi})(0)\|_{\tilde{N},(s;(r,q))}$. The proof of the last equality can be found in Matos [5], p. 162. □

Proposition 2.15. *Let* $r \leq s$. *Then:*

(a) *If* $k \in (1, +\infty]$ *and* $A \in (0, +\infty]$, *then* e^{φ} *belongs to* $\mathrm{Exp}^k_{(s,m(r;q)),A}(E)$ *for all* $\varphi \in E'$.

(b) *If* $k \in (1, +\infty]$ *and* $A \in [0, +\infty)$, *then* e^{φ} *belongs to* $\mathrm{Exp}^k_{(s,m(r;q)),0,A}(E)$ *for all* $\varphi \in E'$.

(c) *If* $k = 1$ *and* $A \in (0, +\infty]$, *then* e^{φ} *belongs to* $\mathrm{Exp}^1_{(s,m(r;q)),A}(E)$, *for all* $\varphi \in E'$ *such that* $\|\varphi\| < A$.

(d) *If* $k = 1$ *and* $A \in (0, +\infty]$, *then* e^{φ} *belongs to* $\mathrm{Exp}^1_{(s,m(r;q)),0,A}(E)$ *for all* $\varphi \in E'$ *such that* $\|\varphi\| \leq A$.

Proof. Since $r \leq s$, we have $\varphi^n \in \mathscr{P}_{(s,m(r;q))}({}^nE)$ and $\|\hat{d}^n(e^{\varphi})(0)\|_{(s,m(r;q))} = \|\varphi^n\|_{(s,m(r;q))} = \|\varphi\|^n$ for each $n \in \mathbb{N}$. Now the proof follows along the same lines as that of Proposition 2.14. □

Proposition 2.16. (1) *The vector subspace generated by all* e^{φ}, $\varphi \in E'$, *is dense in:*

(a) $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$ *if* $k \in (1, +\infty]$ *and* $A \in (0, +\infty]$;

(b) $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),0,A}(E)$ *if* $k \in (1, +\infty]$ *and* $A \in [0, +\infty)$.

(2) *The vector subspace generated by* e^{φ}, $\varphi \in E'$, $\|\varphi\| < A$, *is dense in* $\mathrm{Exp}^{1}_{\tilde{N},(s;(r,q)),A}(E)$ *if* $A \in (0, +\infty)$.

(3) *The vector subspace generated by* e^{φ}, $\varphi \in E'$, $\|\varphi\| \leq A$, *is dense in* $\mathrm{Exp}^{1}_{\tilde{N},(s;(r,q)),0,A}(E)$ *if* $A \in (0, +\infty)$.

Proof. Let g be the closure of $\mathrm{Exp}^{k}_{\tilde{N},(s;(r,q)),A}(E)$. By Proposition 2.13 and by the fact that $\mathscr{P}_f({}^jE)$ is dense in $\mathscr{P}_{\tilde{N},(s;(r,q))}({}^jE)$ for all $j \in \mathbb{N}$ (see Matos [5]), it is enough to show that $\mathscr{P}_f({}^jE) \subseteq g$ for each $j \in \mathbb{N}$. Therefore it is enough to show that $\varphi^n \in g$ for all $n \in \mathbb{N}$ and $\varphi \in E'$. We proceed by induction on n. For $\lambda \in \mathbb{C}$, $\lambda \neq 0$, we have

$$e^{\lambda\varphi} = \sum_{j=0}^{\infty} \frac{1}{j!}\lambda^j \varphi^j$$

converging in $\mathrm{Exp}^{k}_{\tilde{N},(s;(r,q)),A}(E)$ and

$$\lim_{|\lambda|\to 0} \left\| \frac{e^{\lambda\varphi}-1}{\lambda} - \varphi \right\|_{\tilde{N},(s;(r,q)),k,\rho} = \lim_{|\lambda|\to 0} |\lambda| \left\| \sum_{j=2}^{\infty} \frac{1}{j!}\lambda^{j-2}\varphi^j \right\|_{\tilde{N},(s;(r,q)),k,\rho}.$$

Since

$$\left\| \sum_{j=2}^{\infty} \frac{1}{j!}\lambda^{j-2}\varphi^j \right\|_{\tilde{N},(s;(r,q)),k,\rho} \leq \left\| \sum_{j=2}^{\infty} \frac{1}{j!}\varphi^j \right\|_{\tilde{N},(s;(r,q)),k,\rho}$$

whenever $|\lambda| \leq 1$, we have the above limit equal to zero. Hence, $\varphi \in g$ for all $\varphi \in E'$. Now suppose that $\varphi^j \in g$ for $j \leq n-1$. Then

$$\lim_{|\lambda|\to 0} \left\| \frac{1}{\lambda^n}\left(e^{\lambda\varphi} - \sum_{j=0}^{n-1} \frac{1}{j!}\lambda^j\varphi^j \right) - \varphi^n \right\|_{\tilde{N},(s;(r,q)),k,\rho}$$
$$= \lim_{|\lambda|\to 0} |\lambda| \left\| \sum_{j=n+1}^{\infty} \frac{1}{j!}\lambda^{j-n}\varphi^j \right\|_{\tilde{N},(s;(r,q)),k,\rho}.$$

Since

$$\left\| \sum_{j=n+1}^{\infty} \frac{1}{j!}\lambda^{j-n}\varphi^j \right\|_{\tilde{N},(s;(r,q)),k,\rho} \leq \left\| \sum_{j=n+1}^{\infty} \frac{1}{j!}\varphi^j \right\|_{\tilde{N},(s;(r,q)),k,\rho}$$

whenever $|\lambda| \leq 1$, we have the above limit equal to zero. Hence, $\varphi^n \in g$ for all $\varphi \in E'$. The proof for the other cases is analogous. □

3. The Fourier–Borel transforms

Definition 3.1. For $k \in (1, +\infty]$ and $A \in (0, +\infty]$, the *Fourier–Borel transform* FT of $T \in [\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)]'$ is the function on E' defined by $FT(\varphi) = T(e^\varphi) \in \mathbb{C}$.

For $k \in (1, +\infty]$ and $A \in [0, +\infty)$, *the Fourier–Borel transform* FT of $T \in [\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),0,A}(E)]'$ is the function on E' defined by $FT(\varphi) = T(e^\varphi) \in \mathbb{C}$.

For $k = 1$ and $A \in (0, +\infty]$, *the Fourier–Borel transform* FT of $T \in [\mathrm{Exp}^1_{\tilde{N},(s;(r,q)),A}(E)]'$ is the function on $B_A(0) \subset E'$ defined by $FT(\varphi) = T(e^\varphi) \in \mathbb{C}$.

In all cases the function FT is well defined by Proposition 2.14.

Notation 3.2. As usual we set $A^{-1} = \frac{1}{A}$ for $A \in (0, +\infty)$. If $A = 0$, we set $A^{-1} = +\infty$ and if $A = +\infty$, we set $A^{-1} = 0$. For $k \in (1, +\infty)$, we denote by k' its conjugate, that is, $\frac{1}{k} + \frac{1}{k'} = 1$. For $k = 1$ we set $k' = +\infty$, and for $k = +\infty$ we set $k' = 1$. We define $\theta(k) = \frac{k}{(k-1)^{(k-1)/k}}$ for $k \in (1, +\infty)$. Since $\lim_{k \to 1} \theta(k) = 1 = \lim_{k \to \infty} \theta(k)$, we set $\theta(1) = \theta(\infty) = 1$.

To prove the next results we need a duality result obtained by Matos in [5]:

Lemma 3.3. *If E' has the λ-bounded approximation property, then the topological dual of $\mathcal{P}_{\tilde{N},(s;(r,q))}(^n E)$ is isometrically isomorphic to $\mathcal{P}_{(s';m(r',q'))}(^n E')$ through the mapping*

$$\mathcal{B}(\Psi)(\varphi) = \Psi(\varphi^n)$$

for all $\varphi \in E'$ and Ψ in the required dual.

Theorem 3.4. *If E' has the λ-bounded approximation property, then the mapping*

$$F : [\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)]' \to \mathrm{Exp}^{k'}_{(s',m(r';q')),0,(\theta(k)A)^{-1}}(E'),$$

given by $FT(\varphi) = T(e^\varphi)$ for all $T \in [\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)]'$, $\varphi \in E'$ (if $k \in (1, +\infty]$) or $\varphi \in E'$ with $\|\varphi\| < A$ (if $k = 1$), establishes an algebraic isomorphism between these spaces for all $A \in (0, +\infty]$.

Proof. First we consider $k \in (1, +\infty)$. Let $T \in [\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)]'$. Then for each $\rho \in (0, A)$ there is $C(\rho) > 0$ such that

$$|T(f)| \le C(\rho) \|f\|_{\tilde{N},(s;(r,q)),k,\rho} = C(\rho) \sum_{j=0}^{\infty} \rho^{-j} \left(\frac{j}{ke}\right)^{j/k} \left\| \frac{1}{j!} \hat{d}^j f(0) \right\|_{\tilde{N},(s;(r,q))}$$

for all $f \in \mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$. In particular, for $P \in \mathcal{P}_{\tilde{N},(s;(r,q))}(^j E)$ we have

$$|T(P)| \le C(\rho)\rho^{-j}\left(\frac{j}{ke}\right)^{j/k} \|P\|_{\tilde{N},(s;(r,q))}.$$

Let $T_j = T|_{\mathscr{P}_{\tilde{N},(s;(r,q))}(^jE)}$. By Lemma 3.3 there is $\mathscr{B}T_j \in \mathscr{P}_{(s',m(r';q'))}(^jE')$ with $\mathscr{B}T_j(\varphi) = T_j(\varphi^j)$ for all $\varphi \in E'$ and $\|T_j\| = \|\mathscr{B}T_j\|_{(s',m(r';q'))}$. Hence

$$\|\mathscr{B}T_j\|_{(s',m(r';q'))} = \|T_j\| \le C(\rho)\rho^{-j}\left(\frac{j}{ke}\right)^{j/k} \tag{1}$$

for each $\rho \in (0, A)$, and we may write

$$FT(\varphi) = T(e^{\varphi}) = \sum_{j=0}^{\infty} \frac{1}{j!} T(\varphi^j) = \sum_{j=0}^{\infty} \frac{1}{j!} \mathscr{B}T_j(\varphi) \tag{2}$$

for all $\varphi \in E'$. By (1) we have

$$\begin{aligned}
&\limsup_{j\to\infty}\left(\frac{j}{k'e}\right)^{1/k'} \frac{1}{(j!)^{1/j}} \|\mathscr{B}T_j\|^{1/j}_{(s',m(r';q'))} \\
&\quad \le \limsup_{j\to\infty} (C(\rho))^{1/j} \frac{1}{\rho}\left(\frac{j}{ke}\right)^{1/k}\left(\frac{j}{k'e}\right)^{1/k'}\left(\frac{1}{j!}\right)^{1/j} \\
&\quad = \frac{1}{\rho}\left(\frac{1}{k}\right)^{1/k}\left(\frac{1}{k'}\right)^{1/k'} = \frac{1}{\rho\theta(k)}
\end{aligned}$$

for all $\rho \in (0, A)$. Hence

$$\limsup_{j\to\infty}\left(\frac{j}{k'e}\right)^{1/k'}\left(\frac{1}{j!}\right)^{1/j} \|\mathscr{B}T_j\|^{1/j}_{(s',m(r';q'))} \le \frac{1}{A\theta(k)} < +\infty$$

and since

$$\limsup_{j\to\infty}\left(\frac{j}{k'e}\right)^{1/k'} = +\infty$$

we have

$$\limsup_{j\to\infty}\left(\frac{1}{j!}\right)^{1/j} \|\mathscr{B}T_j\|^{1/j}_{(s',m(r';q'))} = 0.$$

Since $\|\mathscr{B}T_j\| \le \|\mathscr{B}T_j\|_{(s',m(r';q'))}$, we have that the radius of convergence of (2) is $+\infty$. Therefore, by Proposition 2.4 we have $FT \in \mathrm{Exp}^{k'}_{(s',m(r';q')),0,(\theta(k)A)^{-1}}(E')$.

Now we prove that F is surjective. Consider $H \in \mathrm{Exp}^{k'}_{(s',m(r';q')),0,(\theta(k)A)^{-1}}(E')$. For each $\rho \in (0, A)$, there is $C(\rho) > 0$ such that

$$\left(\frac{j}{k'e}\right)^{j/k'}\frac{1}{j!}\|\hat{d}^jH(0)\|_{(s',m(r';q'))}\le C(\rho)\frac{1}{\left(\rho\theta(k)\right)^j}$$

for all $j\in\mathbb{N}$. Let $T_j\in[\mathscr{P}_{\tilde{N},(s;(r,q))}(^jE)]'$ such that $\mathscr{B}T_j=\hat{d}^jH(0)$, $\|T_j\|=\|\hat{d}^jH(0)\|_{(s',m(r';q'))}$, then

$$\frac{1}{j!}\|T_j\|\le C(\rho)\frac{1}{\rho^j}\frac{1}{\left(\theta(k)\right)^j}\left(\frac{k'e}{j}\right)^{j/k'}. \tag{3}$$

For $f\in \mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$ we define

$$T_H(f)=\sum_{j=0}^{\infty}\frac{1}{j!}T_j\left(\hat{d}^jf(0)\right).$$

Now, we prove that $T_H\in[\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)]'$ and $FT_H=H$. By (3) and the definition of T_H we have

$$\frac{1}{j!}\|T_j\|\,\|\hat{d}^jf(0)\|_{\tilde{N},(s;(r,q))}\le C(\rho)\frac{1}{\rho^j}\left(\frac{j}{ke}\right)^{j/k}\left(\frac{e}{j}\right)^j\|\hat{d}^jf(0)\|_{\tilde{N},(s;(r,q))}.$$

Since $\lim_{j\to\infty}\frac{e}{j}(j!)^{1/j}=1$ for each $\varepsilon>0$, there is $C(\varepsilon)>0$ such that

$$\left(\frac{e}{j}\right)^j\le C(\varepsilon)(1+\varepsilon)^j\frac{1}{j!}\qquad\text{for all } j\in\mathbb{N}.$$

Hence

$$\frac{1}{j!}\|T_j\|\,\|\hat{d}^jf(0)\|_{\tilde{N},(s;(r,q))}\le C(\rho)C(\varepsilon)\left(\frac{1+\varepsilon}{\rho}\right)^j\left(\frac{j}{ke}\right)^{j/k}\left\|\frac{\hat{d}^jf(0)}{j!}\right\|_{\tilde{N},(s;(r,q))}$$

and

$$\begin{aligned}|T_H(f)|&\le C(\rho)C(\varepsilon)\sum_{j=0}^{\infty}\left(\frac{\rho}{1+\varepsilon}\right)^{-j}\left(\frac{j}{ke}\right)^{j/k}\left\|\frac{\hat{d}^jf(0)}{j!}\right\|_{\tilde{N},(s;(r,q))}\\&=C(\rho)C(\varepsilon)\|f\|_{\tilde{N},(s;(r,q)),k,\rho/(1+\varepsilon)}\end{aligned}$$

for all $f\in\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$, $\varepsilon>0$ and $\rho\in(0,A)$. Thus $T_H\in[\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)]'$ and it is easy to see that $FT_H=H$.

Now we prove the case $k=1$. Proceeding in the same way and using the same notations as in the case $k\in(1,+\infty)$ we have

$$FT(\varphi)=T(e^{\varphi})=\sum_{j=0}^{\infty}\frac{1}{j!}T(\varphi^j)=\sum_{j=0}^{\infty}\frac{1}{j!}\mathscr{B}T_j(\varphi)$$

for all $\varphi \in E'$ such that $\|\varphi\| < A$ and

$$\limsup_{j\to\infty}\left\|\frac{1}{j!}\mathscr{B}T_j\right\|_{(s',m(r';q'))}^{1/j} \leq \frac{1}{A},$$

which implies that

$$\limsup_{j\to\infty}\left\|\frac{1}{j!}\mathscr{B}T_j\right\|^{1/j} \leq \frac{1}{A}$$

because $\|\mathscr{B}T_j\| \leq \|\mathscr{B}T_j\|_{(s',m(r';q'))}$. Hence, by Definition 2.7 we have $FT \in \mathscr{H}_{b(s',m(r';q'))}(B_A(0)) = \mathrm{Exp}^{\infty}_{(s',m(r';q')),0,1/A}(E')$.

Now we prove that F is surjective. Consider $H \in \mathrm{Exp}^{\infty}_{(s',m(r';q')),0,1/A}(E')$. Then

$$\limsup_{j\to\infty}\left\|\frac{\hat{d}^jH(0)}{j!}\right\|_{(s',m(r';q'))}^{1/j} \leq \frac{1}{A}. \tag{4}$$

By Lemma 3.3 there is $T_j \in [\mathscr{P}_{\tilde{N},(s;(r,q))}(^jE)]'$ such that $\mathscr{B}T_j = \hat{d}^jH(0)$ and $\|T_j\| = \|\hat{d}^jH(0)\|_{(s',m(r';q'))}$. For $f \in \mathrm{Exp}_{\tilde{N},(s;(r,q)),A}(E)$, we define

$$T_H(f) = \sum_{j=0}^{\infty}\frac{1}{j!}T_j\big(\hat{d}^jf(0)\big).$$

Then we have

$$\begin{aligned}\left|\frac{1}{j!}T_j\big(\hat{d}^jf(0)\big)\right| &\leq \frac{1}{j!}\|T_j\|\,\|\hat{d}^jf(0)\|_{\tilde{N},(s;(r,q))}\\ &= \frac{1}{j!}\|\hat{d}^jH(0)\|_{(s',m(r';q'))}\|\hat{d}^jf(0)\|_{\tilde{N},(s;(r,q))}.\end{aligned}$$

By (4), for each $\rho \in (0,A)$, there is $C(\rho) > 0$ such that

$$\frac{1}{j!}\|\hat{d}^jH(0)\|_{(s',m(r';q'))} \leq C(\rho)\frac{1}{\rho^j} \qquad \text{for all } j \in \mathbb{N}.$$

Hence

$$|T_H(f)| \leq \sum_{j=0}^{\infty}C(\rho)\frac{1}{\rho^j}\|\hat{d}^jf(0)\|_{\tilde{N},(s;(r,q))} = C(\rho)\|f\|_{\tilde{N},(s;(r,q)),\rho}$$

for each $\rho \in (0,A)$ and $f \in \mathrm{Exp}_{\tilde{N},(s;(r,q)),A}(E)$. Therefore

$$T_H \in [\mathrm{Exp}_{\tilde{N},(s;(r,q)),A}(E)]',$$

and it is easy to see that $FT_H = H$.

Now we prove the case $k = +\infty$. Proceeding as before and using the same notations, we have that for $T \in [\mathrm{Exp}^{\infty}_{\tilde{N},(s;(r,q)),A}(E)]' = \left[\mathscr{H}_{\tilde{N},(s;(r,q))}\left(\bar{B}_{1/A}(0)\right)\right]'$,

$$FT(\varphi) = T(e^{\varphi}) = \sum_{j=0}^{\infty} \frac{1}{j!} T(\varphi^j) = \sum_{j=0}^{\infty} \frac{1}{j!} \mathscr{B}T_j(\varphi)$$

for all $\varphi \in E'$ and

$$\limsup_{j\to\infty} \|\mathscr{B}T_j\|^{1/j}_{(s',m(r';q'))} \leq \frac{1}{A}.$$

Since $\limsup_{j\to\infty}\left(\frac{1}{j!}\right)^{1/j} = 0$, we have

$$\limsup_{j\to\infty}\left(\frac{1}{j!}\right)^{1/j} \|\mathscr{B}T_j\|^{1/j}_{(s',m(r';q'))} = 0,$$

and this implies that $FT \in \mathrm{Exp}_{(s',m(r';q')),0,1/A}(E')$.

Finally, let us prove that F is surjective. If $H \in \mathrm{Exp}_{(s',m(r';q')),0,1/A}(E')$, then

$$\limsup_{j\to\infty} \|\hat{d}^j H(0)\|^{1/j}_{(s',m(r';q'))} \leq \frac{1}{A}. \tag{5}$$

By Lemma 3.3 there is $T_j \in [\mathscr{P}_{\tilde{N},(s;(r,q))}({}^jE)]'$ such that $\mathscr{B}T_j = \hat{d}^j H(0)$ and $\|T_j\| = \|\hat{d}^j H(0)\|_{(s',m(r';q'))}$. For $f \in \mathrm{Exp}^{\infty}_{\tilde{N},(s;(r,q)),A}(E)$, we define

$$T_H(f) = \sum_{j=0}^{\infty} \frac{1}{j!} T_j\left(\hat{d}^j f(0)\right)$$

and obtain

$$\begin{aligned}\left|\frac{1}{j!} T_j\left(\hat{d}^j f(0)\right)\right| &\leq \frac{1}{j!}\|T_j\|\,\|\hat{d}^j f(0)\|_{\tilde{N},(s;(r,q))}\\ &= \frac{1}{j!}\|\hat{d}^j H(0)\|_{(s',m(r';q'))}\|\hat{d}^j f(0)\|_{\tilde{N},(s;(r,q))}.\end{aligned}$$

By (5), for each $\rho \in (0, A)$, there is $C(\rho) > 0$ such that

$$\|\hat{d}^j H(0)\|_{(s',m(r';q'))} \leq C(\rho)\frac{1}{\rho^j} \qquad \text{for all } j \in \mathbb{N}.$$

Thus these two above inequalities imply that

$$
\begin{aligned}
|T_H(f)| &\le \sum_{j=0}^{\infty} \left| \frac{1}{j!} T_j\big(\hat{d}^j f(0)\big) \right| \\
&\le \sum_{j=0}^{\infty} C(\rho) \frac{1}{j!\rho^j} \|\hat{d}^j f(0)\|_{\tilde{N},(s;(r,q))} = C(\rho)\|f\|_{\tilde{N},(s;(r,q)),\infty,\rho}
\end{aligned}
$$

for each $\rho \in (0, A)$ and each $f \in \mathrm{Exp}^{\infty}_{\tilde{N},(s;(r,q)),A}(E)$. Then

$$T_H \in [\mathrm{Exp}^{\infty}_{\tilde{N},(s;(r,q)),A}(E)]',$$

and it is easy to see that $FT_H = H$.

It is clear that F is linear and injective by Proposition 2.16. □

Theorem 3.5. *If E' has the λ-bounded approximation property, then the mapping*

$$F : [\mathrm{Exp}^{k}_{\tilde{N},(s;(r,q)),0,A}(E)]' \to \mathrm{Exp}^{k'}_{(s',m(r';q')),(\theta(k)A)^{-1}}(E'),$$

given by $FT(\varphi) = T(e^{\varphi})$ for all $T \in [\mathrm{Exp}^{k}_{\tilde{N},(s;(r,q)),0,A}(E)]'$ and $\varphi \in E'$, establishes an algebraic isomorphism between these spaces for all $k \in (1, +\infty]$ and $A \in [0, +\infty)$.

Proof. The proof is similar to that of Theorem 3.4. □

Remark 3.6. If $T \in [\mathrm{Exp}^{1}_{\tilde{N},(s;(r,q)),0,A}(E)]'$, by Proposition 2.14 the natural definition for the Fourier–Borel transform of T would be $FT(\varphi) = T(e^{\varphi})$ for all $\varphi \in E'$ such that $\|\varphi\| \le A$. However we will prove that we can define FT for all $\varphi \in E'$ in a "bigger" set in such way that it agrees with the previous definition for $\varphi \in E'$ with $\|\varphi\| \le A$. This is Proposition 3.8 below.

Definition 3.7. If $k = 1$, $A \in [0, +\infty)$ and E' has the λ-bounded approximation property, then *the Fourier–Borel transform FT* of $T \in [\mathrm{Exp}^{1}_{\tilde{N},(s;(r,q)),0,A}(E)]'$ is the function defined by

$$FT(\varphi) = \sum_{j=0}^{\infty} \frac{1}{j!} \mathscr{B}T_j(\varphi)$$

for all $\varphi \in E'$ such that the series converges absolutely. Here $T_j = T|_{\mathscr{P}_{\tilde{N},(s;(r,q))}(^jE)}$, $\mathscr{B}T_j \in P_{(s',m(r';q'))}(^jE')$ is given by $\mathscr{B}T_j(\varphi) = T_j(\varphi^j)$ for all $\varphi \in E'$, and $\|T_j\| = \|\mathscr{B}T_j\|_{(s',m(r';q'))}$ by Lemma 3.3.

Proposition 3.8. *If E' has the λ-bounded approximation property, $A \in [0, +\infty)$ and $T \in [\mathrm{Exp}^{1}_{\tilde{N},(s;(r,q)),0,A}(E)]'$, then there is $\rho > A$ such that $FT \in \mathscr{H}^{\infty}_{(s',m(r';q'))}(B_{\rho}(0))$, where $B_{\rho}(0)$ is the open ball in E'.*

Proof. If $T \in [\mathrm{Exp}^1_{\tilde{N},(s;(r,q)),0,A}(E)]'$, there are $\delta > A$ and $C(\delta) > 0$ such that

$$|T(f)| \leq C(\delta)\|f\|_{\tilde{N},(s;(r,q)),\delta} = C(\delta)\sum_{j=0}^{\infty}\delta^{-j}\|\hat{d}^j f(0)\|_{\tilde{N},(s;(r,q))}$$

for all $f \in \mathrm{Exp}^1_{\tilde{N},(s;(r,q)),0,A}(E)$. Hence for $P \in \mathscr{P}_{\tilde{N},(s;(r,q))}({}^jE)$ we have

$$|T(P)| \leq C(\delta)\delta^{-j} j!\|P\|_{\tilde{N},(s;(r,q))}.$$

Thus

$$\|\mathscr{B}T_j\|_{(s',m(r';q'))} = \|T_j\| \leq C(\delta) j! \delta^{-j} \qquad \text{for all } j \in \mathbb{N}$$

and

$$\limsup_{j\to\infty}\left\|\frac{\mathscr{B}T_j}{j!}\right\|^{1/j}_{(s',m(r';q'))} \leq \frac{1}{\delta}.$$

Let $\rho \in (A,\delta)$. Then

$$\limsup_{j\to\infty}\left\|\frac{\mathscr{B}T_j}{j!}\right\|^{1/j}_{(s',m(r';q'))} < \frac{1}{\rho},$$

and this implies that $FT \in \mathscr{H}(B_\rho(0))$. Furthermore,

$$\sum_{j=0}^{\infty}\rho^j\left\|\frac{1}{j!}\mathscr{B}T_j\right\|_{(s',m(r';q'))} < +\infty.$$

Thus $FT \in \mathscr{H}^{\infty}_{(s',m(r';q'))}(B_\rho(0))$. □

Theorem 3.9. *If E' has the λ-bounded approximation property, then the mapping*

$$F : T \in [\mathrm{Exp}^1_{\tilde{N},(s;(r,q)),0,A}(E)]' \to FT \in \mathrm{Exp}^{\infty}_{(s',m(r';q')),1/A}(E')$$

establishes an algebraic isomorphism between the two spaces spaces, for $A \in [0,+\infty)$. Here we identify the class $[FT]$ with its representative FT.

Proof. By definition of $\mathrm{Exp}^{\infty}_{(s',m(r';q')),1/A}(E')$ and Proposition 3.8 we have that FT belongs to $\mathrm{Exp}^{\infty}_{(s',m(r';q')),1/A}(E')$ for all $T \in [\mathrm{Exp}^1_{\tilde{N},(s;(r,q)),0,A}(E)]'$.

Now suppose that $T \in [\mathrm{Exp}^1_{\tilde{N},(s;(r,q)),0,A}(E)]'$ is such that $FT = 0$. By Proposition 3.8, there is $\rho > A$ such that $FT(\varphi) = 0$ for all $\varphi \in E'$ with $\|\varphi\| < \rho$, that is, $\mathscr{B}T_j(\varphi) = 0$ for all $\varphi \in B_\rho(0) \subseteq E'$ and $j \in \mathbb{N}$. Hence $\mathscr{B}T_j = 0$ and $\|T_j\| = \|\mathscr{B}T_j\|_{(s',m(r';q'))} = 0$. Thus $T|_{\mathscr{P}_{\tilde{N},(s;(r,q))}({}^jE)} = 0$ for all $j \in \mathbb{N}$, and by Proposition

2.13 we have $T(f)=0$ for all $f \in \mathrm{Exp}^{1}_{\tilde{N},(s;(r,q)),0,A}(E)$ because $\hat{d}^j f(0) \in \mathscr{P}_{\tilde{N},(s;(r,q))}(^jE)$ for all $j \in \mathbb{N}$. Therefore $T=0$ and consequently F is injective. The linearity of F is clear. Now let $H \in \mathrm{Exp}^{\infty}_{(s',m(r';q')),1/A}(E')$. Then there is $\rho > A$ such that $H \in \mathscr{H}^{\infty}_{(s',m(r';q'))}\big(B_\rho(0)\big)$. Thus

$$\limsup_{j\to\infty} \left\| \frac{\hat{d}^j H(0)}{j!} \right\|^{1/j}_{(s',m(r';q'))} \leq \frac{1}{\rho}.$$

Therefore, for all $\varepsilon > 0$, there is $C(\varepsilon) > 0$ such that

$$\frac{1}{j!}\|\hat{d}^j H(0)\|_{(s',m(r';q'))} \leq C(\varepsilon)\left(\frac{1+\varepsilon}{\rho}\right)^j \quad \text{for all } j \in \mathbb{N}.$$

By Lemma 3.3, there is $T_j \in [\mathscr{P}_{\tilde{N},(s;(r,q))}(^jE)]'$ such that $\mathscr{B}T_j = \hat{d}^j H(0)$ and $\|T_j\| = \|\hat{d}^j H(0)\|_{(s',m(r';q'))}$. Hence

$$\begin{aligned}\frac{1}{j!}\big|T_j\big(\hat{d}^j f(0)\big)\big| &\leq \frac{1}{j!}\|\hat{d}^j H(0)\|_{(s',m(r';q'))}\|\hat{d}^j f(0)\|_{\tilde{N},(s;(r,q))} \\ &\leq C(\varepsilon)\left(\frac{1+\varepsilon}{\rho}\right)^j \|\hat{d}^j f(0)\|_{\tilde{N},(s;(r,q))} \quad \text{for all } j \in \mathbb{N}.\end{aligned}$$

For $f \in \mathrm{Exp}^{1}_{\tilde{N},(s;(r,q)),0,A}(E)$, we define

$$T_H(f) = \sum_{j=0}^{\infty} \frac{1}{j!} T_j\big(\hat{d}^j f(0)\big).$$

Then

$$|T_H(f)| \leq C(\varepsilon)\sum_{j=0}^{\infty}\left(\frac{1+\varepsilon}{\rho}\right)^j \|\hat{d}^j f(0)\|_{\tilde{N},(s;(r,q))} = C(\varepsilon)\|f\|_{\tilde{N},(s;(r,q)),\rho/(1+\varepsilon)}$$

for all $f \in \mathrm{Exp}^{1}_{\tilde{N},(s;(r,q)),0,A}(E)$ and $\varepsilon > 0$ such that $\frac{\rho}{1+\varepsilon} > A$. Therefore, $T_H \in [\mathrm{Exp}^{1}_{\tilde{N},(s;(r,q)),0,A}(E)]'$ and $FT_H = H$. □

4. Bounded sets

In this section we give characterizations of the bounded sets of $\mathrm{Exp}^{k}_{\tilde{N},(s;(r,q)),A}(E)$ and $\mathrm{Exp}^{k}_{(s,m(r;q)),A}(E)$ for $k \in [1,+\infty]$, $A \in (0,+\infty]$ ($\mathrm{Exp}^{k}_{\tilde{N},(s;(r,q)),0,A}(E)$ and $\mathrm{Exp}^{k}_{(s,m(r;q)),0,A}(E)$ for $k \in [1,+\infty]$, $A \in [0,+\infty)$). We shall use these characterizations to prove that the Fourier–Borel transforms are topological isomorphisms in some cases.

We denote by $\mathscr{S}_A$ the family of all sequences $\alpha = (\alpha_j)_{j=0}^{\infty}$ of real numbers $\alpha_j \geq 0$ such that $\limsup_{j\to\infty} \alpha_j^{1/j} \leq A$.

Proposition 4.1. *For* $k \in [1, +\infty)$, $A \in (0, +\infty]$ *and* $\alpha \in \mathscr{S}_{1/A}$, *the seminorms* $p_{\tilde{N},(s;(r,q)),k,\alpha}$ *and* $p_{(s,m(r;q)),k,\alpha}$ *defined by*

$$p_{\tilde{N},(s;(r,q)),k,\alpha}(f) = \sum_{j=0}^{\infty} \alpha_j \left(\frac{j}{ke}\right)^{j/k} \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{\tilde{N},(s;(r,q))},$$

$$p_{(s,m(r;q)),k,\alpha}(f) = \sum_{j=0}^{\infty} \alpha_j \left(\frac{j}{ke}\right)^{j/k} \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{(s,m(r;q))}$$

are continuous in $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$ *and* $\mathrm{Exp}^k_{(s,m(r;q)),A}(E)$, *respectively.*

For $k = +\infty$, $A \in (0, +\infty]$ *and* $\alpha \in \mathscr{S}_{1/A}$ *the seminorms* $p_{\tilde{N},(s;(r,q)),\infty,\alpha}$ *and* $p_{(s,m(r;q)),\infty,\alpha}$ *defined by*

$$p_{\tilde{N},(s;(r,q)),\infty,\alpha}(f) = \sum_{j=0}^{\infty} \alpha_j \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{\tilde{N},(s;(r,q))},$$

$$p_{(s,m(r;q)),\infty,\alpha}(f) = \sum_{j=0}^{\infty} \alpha_j \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{(s,m(r;q))}$$

are continuous in $\mathrm{Exp}^\infty_{\tilde{N},(s;(r,q)),A}(E)$ *and* $\mathrm{Exp}^\infty_{(s,m(r;q)),A}(E)$, *respectively.*

Proof. For each $\rho \in (0, A)$, we have $\frac{1}{A} < \frac{1}{\rho}$ and since $\alpha \in \mathscr{S}_{1/A}$, there is $C(\rho) > 0$ such that $\alpha_j \leq C(\rho)\frac{1}{\rho^j}$ for all $j \in \mathbb{N}$. Thus, for $k \in [1, +\infty)$ we get

$$p_{\tilde{N},(s;(r,q)),k,\alpha}(f) \leq C(\rho) \sum_{j=0}^{\infty} \frac{1}{\rho^j} \left(\frac{j}{ke}\right)^{j/k} \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{\tilde{N},(s;(r,q))} = C(\rho) \|f\|_{\tilde{N},(s;(r,q)),k,\rho},$$

$$p_{(s,m(r;q)),k,\alpha}(f) \leq C(\rho) \sum_{j=0}^{\infty} \frac{1}{\rho^j} \left(\frac{j}{ke}\right)^{j/k} \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{(s,m(r;q))} = C(\rho) \|f\|_{(s,m(r;q)),k,\rho}.$$

For $k = +\infty$ we obtain that

$$p_{\tilde{N},(s;(r,q)),\infty,\alpha}(f) \leq C(\rho) \sum_{j=0}^{\infty} \frac{1}{\rho^j} \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{\tilde{N},(s;(r,q))} = C(\rho) p^\infty_{\tilde{N},(s;(r,q)),\rho}(f),$$

$$p_{(s,m(r;q)),\infty,\alpha}(f) \leq C(\rho) \sum_{j=0}^{\infty} \frac{1}{\rho^j} \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{(s,m(r;q))} = C(\rho) p^\infty_{(s,m(r;q)),\rho}(f).$$

Now these four inequalities and the properties of the inductive limit topology imply the continuity of these seminorms in the respective spaces. □

Proposition 4.2. *For* $k \in [1, +\infty)$ *and* $A \in (0, +\infty]$, *a subset* B *of* $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$ *or* $\mathrm{Exp}^k_{(s,m(r;q)),A}(E)$ *is bounded if and only if there is* $\rho \in (0, A)$ *such that*

$$\limsup_{j\to\infty} \left(\frac{j}{ke}\right)^{1/k} \left(\sup_{f\in B} \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{\tilde{N},(s;(r,q))}\right)^{1/j} \leq \rho \tag{6}$$

or

$$\limsup_{j\to\infty} \left(\frac{j}{ke}\right)^{1/k} \left(\sup_{f\in B} \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{(s,m(r;q))}\right)^{1/j} \leq \rho, \tag{7}$$

respectively.

For $k = +\infty$ *and* $A \in (0, +\infty]$, *a subset* B *of* $\mathrm{Exp}^\infty_{\tilde{N},(s;(r,q)),A}(E)$ *or* $\mathrm{Exp}^\infty_{(s,m(r;q)),A}(E)$ *is bounded if and only if there is* $\rho \in (0, A)$ *such that*

$$\limsup_{j\to\infty} \left(\sup_{f\in B} \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{\tilde{N},(s;(r,q))}\right)^{1/j} \leq \rho \tag{8}$$

or

$$\limsup_{j\to\infty} \left(\sup_{f\in B} \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{(s,m(r;q))}\right)^{1/j} \leq \rho, \tag{9}$$

respectively.

Proof. It is a known result of the theory of topological vector spaces that a subset L of a locally convex space X is bounded if an only if each continuous seminorm in X is bounded in L (see Grothendieck [1], p. 25). We use this result to prove that if one of the conditions (6), (7), (8), (9) holds, then B is bounded in its corresponding space. Let p be a continuous seminorm in its corresponding space, then for each $\delta \in (0, A)$ there is $C(\delta) > 0$ such that

$$p(f) \leq C(\delta)\|f\|_{\tilde{N},(s;(r,q)),k,\delta} \qquad (\text{for all } f \in \mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E))$$

or

$$p(f) \leq C(\delta)\|f\|_{(s,m(r;q)),k,\delta} \qquad (\text{for all } f \in \mathrm{Exp}^k_{(s,m(r;q)),A}(E))$$

for each $k \in [1, +\infty]$. In particular,

$$\sup_{f\in B} p(f) \leq C(\rho) \sup_{f\in B} \|f\|_{\tilde{N},(s;(r,q)),k,\rho} \tag{19}$$

$$\sup_{f\in B} p(f) \leq C(\rho) \sup_{f\in B} \|f\|_{(s,m(r;q)),k,\rho}. \tag{11}$$

For $k \in [1,+\infty)$, we have from (6) and (7) that

$$\sup_{f\in B} \|f\|_{\tilde{N},(s;(r,q)),k,\rho} \leq \sum_{j=0}^{\infty} \frac{1}{\rho^j}\left(\frac{j}{ke}\right)^{j/k} \sup_{f\in B} \left\|\frac{\hat{d}^j f(0)}{j!}\right\|_{\tilde{N},(s;(r,q))} < +\infty,$$

$$\sup_{f\in B} \|f\|_{(s,m(r;q)),k,\rho} \leq \sum_{j=0}^{\infty} \frac{1}{\rho^j}\left(\frac{j}{ke}\right)^{j/k} \sup_{f\in B} \left\|\frac{\hat{d}^j f(0)}{j!}\right\|_{(s,m(r;q))} < +\infty.$$

Hence B is bounded in $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$.

Now we suppose that B is bounded in $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$ (for $\mathrm{Exp}^k_{(s,m(r;q)),A}(E)$ the proof is analogous). Since these spaces are inductive limits of a sequence of DF-spaces of type $\mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho_n}(E)$, where $(\rho_n)_{n=1}^{\infty}$ is a strictly increasing sequence of positive real numbers converging to A, B is contained in the closure of a bounded subset of some $\mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho_n}(E)$ (this is also a result of the theory of topological vector spaces and can be found in Grothendieck [1], p. 171, Proposition 5). Without loss of generality we suppose that B is contained in the closed unit ball of $\mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho}(E)$, where $\rho \in (0,A)$. Now to get our result it is enough to show that the closure, for the topology in $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$, of this ball is contained in a ball of some $\mathscr{B}^k_{\tilde{N},(s;(r,q)),\delta}(E)$. Let $B_{\mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho}(E)} = \{f \in \mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho}(E); \|f\|_{\tilde{N},(s;(r,q)),k,\rho} \leq 1\}$ be the unit ball of $\mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho}(E)$. If g belongs to $\overline{B_{\mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho}(E)}}$ (closure of $B_{\mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho}(E)}$ in $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$), then there is a net $(g_i)_{i\in I}$ in $B_{\mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho}(E)}$ converging to g in the topology of $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$. Thus,

$$\rho^{-j}\left(\frac{j}{ke}\right)^{j/k} \sup_{i\in I} \left\|\frac{\hat{d}^j g_i(0)}{j!}\right\|_{\tilde{N},(s;(r,q))} \leq 1 \quad \text{for all } j \in \mathbb{N}. \tag{12}$$

For each $j \in \mathbb{N}$ we define the seminorms $p_{\tilde{N},(s;(r,q)),k,\alpha}$ by

$$p_{\tilde{N},(s;(r,q)),k,\alpha}(f) = \rho^{-j}\left(\frac{j}{ke}\right)^{j/k} \left\|\frac{\hat{d}^j f(0)}{j!}\right\|_{\tilde{N},(s;(r,q))},$$

where $\alpha_j = \rho^{-j}$ and $\alpha_l = 0$ for $l \neq j$. Hence by Proposition 4.1 we have that $p_{\tilde{N},(s;(r,q)),k,\alpha}$ is a continuous seminorm in $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$. By (12) we have

$$\rho^{-j}\left(\frac{j}{ke}\right)^{j/k} \left\|\frac{\hat{d}^j g(0)}{j!}\right\|_{\tilde{N},(s;(r,q))} \leq 1 \quad \text{for all } j \in \mathbb{N}.$$

Thus

$$\left(\frac{j}{ke}\right)^{j/k} \frac{\sup}{g \in B_{\mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho}(E)}} \left\| \frac{\hat{d}^j g(0)}{j!} \right\|_{\tilde{N},(s;(r,q))} \leq \rho^j \quad \text{for all } j \in \mathbb{N}.$$

Hence, if $\delta \in (\rho, A)$ we have

$$\sup_{g \in B} \|g\|_{\tilde{N},(s;(r,q)),k,\delta} \leq \sum_{j=0}^{\infty} \frac{1}{\delta^j} \left(\frac{j}{ke}\right)^{j/k} \sup_{g \in B} \left\| \frac{\hat{d}^j g(0)}{j!} \right\|_{\tilde{N},(s;(r,q))} \leq \sum_{j=0}^{\infty} \left(\frac{\rho}{\delta}\right)^j = \frac{1}{1 - \frac{\rho}{\delta}}.$$

The case $k = +\infty$ is analogous. □

Corollary 4.3. *For $k \in [1, +\infty]$ and $A \in (0, +\infty]$, a subset B of $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$ or $\mathrm{Exp}^k_{(s,m(r;q)),A}(E)$ is bounded if and only if there is $\rho \in (0, A)$ such that B is contained and bounded in $\mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho}(E)$ or $\mathscr{B}^k_{(s,m(r;q)),\rho}(E)$, respectively.*

Proof. This follows immediately using Proposition 4.2. □

Proposition 4.4. *For $k \in [1, +\infty)$ and $A \in [0, +\infty)$, a subset B of $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),0,A}(E)$ or $\mathrm{Exp}^k_{(s,m(r;q)),0,A}(E)$ is bounded if and only if*

$$\limsup_{j \to \infty} \left(\frac{j}{ke}\right)^{1/k} \left(\sup_{f \in B} \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{\tilde{N},(s;(r,q))} \right)^{1/j} \leq A \tag{13}$$

or

$$\limsup_{j \to \infty} \left(\frac{j}{ke}\right)^{1/k} \left(\sup_{f \in B} \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{(s,m(r;q))} \right)^{1/j} \leq A, \tag{14}$$

respectively.

For $k = +\infty$ and $A \in [0, +\infty)$, a subset B of $\mathrm{Exp}^{\infty}_{\tilde{N},(s;(r,q)),0,A}(E)$ or $\mathrm{Exp}^{\infty}_{(s,m(r;q)),0,A}(E)$ is bounded if and only if

$$\limsup_{j \to \infty} \left(\sup_{f \in B} \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{\tilde{N},(s;(r,q))} \right)^{1/j} \leq A \tag{15}$$

or

$$\limsup_{j \to \infty} \left(\sup_{f \in B} \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{(s,m(r;q))} \right)^{1/j} \leq A, \tag{16}$$

respectively.

Proof. If B is a bounded subset of $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),0,A}(E)$ for $k \in [1,+\infty)$, then B is bounded in $\mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho}(E)$ for all $\rho > A$ (see Grothendieck [1], p. 24, Proposition 11). Thus,

$$\limsup_{j\to\infty}\left(\frac{j}{ke}\right)^{1/k}\left(\sup_{f\in B}\left\|\frac{\hat{d}^j f(0)}{j!}\right\|_{\tilde{N},(s;(r,q))}\right)^{1/j} \le A.$$

For $\mathrm{Exp}^k_{(s,m(r;q)),A}(E)$ it is analogous. The case $k = +\infty$ is analogous too.

Now suppose that (13) holds. Then for $\varepsilon > 0$ there is $C(\varepsilon) > 0$ such that

$$\left(\frac{j}{ke}\right)^{j/k}\sup_{f\in B}\left\|\frac{\hat{d}^j f(0)}{j!}\right\|_{\tilde{N},(s;(r,q))} \le C(\varepsilon)(A+\varepsilon)^j \quad \text{for all } j \in \mathbb{N}.$$

If $\rho > A$, we have

$$\rho^{-j}\left(\frac{j}{ke}\right)^{j/k}\sup_{f\in B}\left\|\frac{\hat{d}^j f(0)}{j!}\right\|_{\tilde{N},(s;(r,q))} \le C(\varepsilon)\left(\frac{A+\varepsilon}{\rho}\right)^j \quad \text{for all } j \in \mathbb{N}.$$

Let $\varepsilon > 0$ such $\rho > A + \varepsilon$. Then

$$\begin{aligned}\sup_{f\in B}\|f\|_{\tilde{N},(s;(r,q)),k,\rho} &\le \sum_{j=0}^{\infty}\frac{1}{\rho^j}\left(\frac{j}{ke}\right)^{j/k}\sup_{f\in B}\left\|\frac{\hat{d}^j f(0)}{j!}\right\|_{\tilde{N},(s;(r,q))} \\ &\le C(\varepsilon)\sum_{j=0}^{\infty}\left(\frac{A+\varepsilon}{\rho}\right)^j < +\infty.\end{aligned}$$

Hence B is bounded in $\mathscr{B}^k_{\tilde{N},(s;(r,q)),\rho}(E)$ for each $\rho > A$, and so is bounded in $\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),0,A}(E)$.

For $\mathrm{Exp}^k_{(s,m(r;q)),A}(E)$ as well as the case $k = +\infty$ the argument is analogous. □

Now we are able to prove that the Fourier–Borel transforms are topological isomorphisms.

Theorem 4.5. *If E' has the λ-bounded approximation property, then the Fourier–Borel transform F is a topological isomorphism between the spaces $[\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)]'_\beta$ and $\mathrm{Exp}^{k'}_{(s',m(r';q')),0,(\theta(k)A)^{-1}}(E')$ for all $k \in [1,+\infty]$ and $A \in (0,+\infty]$. Here β denotes the strong topology on the dual.*

Proof. By the Open Mapping Theorem it is enough to show that F^{-1} is continuous, because $[\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)]'_\beta$ and $\mathrm{Exp}^{k'}_{(s',m(r';q')),0,(\theta(k)A)^{-1}}(E')$ are Fréchet spaces. Note that $[\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)]'_\beta$ is a Fréchet space since it is the strong dual of a DF-space.

We will prove that F^{-1} is continuous by showing that for each continuous seminorm q in $[\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)]'_\beta$, there are $C > 0$ and a continuous seminorm p in $\mathrm{Exp}^{k'}_{(s',m(r';q')),0,(\theta(k)A)^{-1}}(E')$ such that $q(F^{-1}(H)) = q(T_H) \le Cp(H)$ for all $H \in \mathrm{Exp}^{k'}_{(s',m(r';q')),0,(\theta(k)A)^{-1}}(E')$ (here T_H is the same we used several times before). We know that the strong topology on the dual is generated by a family of seminorms $p_B(S) = \sup_{f\in B}|S(f)|$, where $S \in [\mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)]'_\beta$ and $B \subseteq \mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$ is a bounded subset.

Let $B \in \mathrm{Exp}^k_{\tilde{N},(s;(r,q)),A}(E)$ be a bounded subset for $k \in (1, +\infty)$. By Proposition 4.2 there is $\rho \in (0, A)$ such that

$$\limsup_{j\to\infty}\left(\frac{j}{ke}\right)^{1/k}\left(\sup_{f\in B}\left\|\frac{\hat{d}^j f(0)}{j!}\right\|_{\tilde{N},(s;(r,q))}\right)^{1/j} \le \rho.$$

Thus, for each $\varepsilon > 0$ there is $C(\varepsilon) > 0$ such that

$$\left(\frac{j}{ke}\right)^{j/k}\sup_{f\in B}\left\|\frac{\hat{d}^j f(0)}{j!}\right\|_{\tilde{N},(s;(r,q))} \le C(\varepsilon)(\rho+\varepsilon)^j$$

for all $j \in \mathbb{N}$. Then

$$\begin{aligned}\sup_{f\in B}|F^{-1}(H)(f)| &= \sup_{f\in B}|T_H(f)| \\ &\le \sum_{j=0}^{\infty}\|\hat{d}^j H(0)\|_{(s',m(r';q'))}\sup_{f\in B}\left\|\frac{\hat{d}^j f(0)}{j!}\right\|_{\tilde{N},(s;(r,q))} \\ &\le C(\varepsilon)\sum_{j=0}^{\infty}(\rho+\varepsilon)^j\left(\frac{ke}{j}\right)^{j/k} j!\left\|\frac{\hat{d}^j H(0)}{j!}\right\|_{(s',m(r';q'))}. \qquad (17)\end{aligned}$$

Since

$$\left(\frac{ke}{j}\right)^{j/k} = (\theta(k))^j\left(\frac{j}{k'e}\right)^{j/k'}\frac{e^j}{j^j}$$

we have

$$\begin{aligned}&\sum_{j=0}^{\infty}(\rho+\varepsilon)^j\left(\frac{ke}{j}\right)^{j/k} j!\left\|\frac{\hat{d}^j H(0)}{j!}\right\|_{(s',m(r';q'))} \\ &\quad\le \sum_{j=0}^{\infty}(\theta(k))^j(\rho+\varepsilon)^j\frac{j!}{j^j}e^j\left(\frac{j}{k'e}\right)^{j/k'}\left\|\frac{\hat{d}^j H(0)}{j!}\right\|_{(s',m(r';q'))}, \qquad (18)\end{aligned}$$

and since

$$\lim_{j\to\infty} \frac{e}{j}(j!)^{1/j} = 1,$$

there is $D(\varepsilon) > 0$ such that

$$\frac{j!}{j^j} e^j \le D(\varepsilon)(1+\varepsilon)^j \tag{19}$$

for all $j \in \mathbb{N}$. Therefore,

$$\sup_{f\in B} |F^{-1}(H)(f)| \le C(\varepsilon)D(\varepsilon)\|H\|_{(s',m(r';q')),k',1/\theta(k)(\rho+\varepsilon)(1+\varepsilon)}.$$

Now, choosing $\varepsilon > 0$ such that $(\rho+\varepsilon)(1+\varepsilon) < A$, we have $\frac{1}{\theta(k)(\rho+\varepsilon)(1+\varepsilon)} > \frac{1}{\theta(k)A}$ and the continuity of F^{-1} follows.

Proceeding the same way we have

$$\sup_{f\in B} |F^{-1}(H)(f)| \le C(\varepsilon)\|H\|_{(s',m(r';q')),\,\infty,\,1/(\rho+\varepsilon)}$$

if $k = 1$, and

$$\sup_{f\in B} |F^{-1}(H)(f)| \le C(\varepsilon)\|H\|_{(s',m(r';q')),1/(\rho+\varepsilon)}$$

if $k = +\infty$. Then, choosing $\varepsilon > 0$ so that $(\rho+\varepsilon) < A$, we have $\frac{1}{\rho+\varepsilon} > \frac{1}{A}$, and the continuity of F^{-1} follows. □

It is an open problem whether the Fourier–Borel transform, in the next case, is a topological isomorphism. But it is possible to prove that F^{-1} is continuous.

Theorem 4.6. *If E' has the λ-bounded approximation property, $k \in [1,+\infty]$ and $A \in [0,+\infty)$, then*

$$F^{-1} : \mathrm{Exp}^{k'}_{(s',m(r';q')),(\theta(k)A)^{-1}}(E') \to [\mathrm{Exp}^{k}_{\tilde{N},(s;(r,q)),0,A}(E)]'_{\beta}$$

is continuous. Again β denotes the strong topology on the dual.

Proof. Let $B \in \mathrm{Exp}^{k}_{\tilde{N},(s;(r,q)),0,A}(E)$ be a bounded subset. By Proposition 4.4 we have

$$\limsup_{j\to\infty} \left(\frac{j}{ke}\right)^{1/k} \left(\sup_{f\in B} \left\|\frac{\hat{d}^j f(0)}{j!}\right\|_{\tilde{N},(s;(r,q))}\right)^{1/j} \le A.$$

For $\rho > A$ there is $C(\rho) > 0$ such that

$$\left(\frac{j}{ke}\right)^{j/k} \sup_{f\in B}\left\|\frac{\hat{d}^j f(0)}{j!}\right\|_{\tilde{N},(s;(r,q))} \leq C(\rho)\rho^j$$

for all $j \in \mathbb{N}$. Thus

$$\begin{aligned}
\sup_{f\in B}|F^{-1}(H)(f)| &\leq \sum_{j=0}^{\infty}\|\hat{d}^j H(0)\|_{(s',m(r';q'))}\sup_{f\in B}\left\|\frac{\hat{d}^j f(0)}{j!}\right\|_{\tilde{N},(s;(r,q))} \\
&\leq C(\rho)\sum_{j=0}^{\infty}\rho^j\left(\frac{ke}{j}\right)^{j/k} j!\left\|\frac{\hat{d}^j H(0)}{j!}\right\|_{(s',m(r';q'))} \\
&= C(\rho)\sum_{j=0}^{\infty}(\rho\theta(k))^j\left(\frac{j}{k'e}\right)^{j/k'}\frac{j!}{j^j}e^j\left\|\frac{\hat{d}^j H(0)}{j!}\right\|_{(s',m(r';q'))}.
\end{aligned}$$

By (19) we have

$$\begin{aligned}
\sup_{f\in B}|F^{-1}(H)(f)| &\leq C(\rho)\sum_{j=0}^{\infty}(\rho\theta(k))^j\left(\frac{j}{k'e}\right)^{j/k'}\frac{j!}{j^j}e^j\left\|\frac{\hat{d}^j H(0)}{j!}\right\|_{(s',m(r';q'))} \\
&\leq C(\rho)D(\varepsilon)\sum_{j=0}^{\infty}(\rho(1+\varepsilon)\theta(k))^j\left(\frac{j}{k'e}\right)^{j/k'}\left\|\frac{\hat{d}^j H(0)}{j!}\right\|_{(s',m(r';q'))} \\
&= C(\rho)D(\varepsilon)\|H\|_{(s',m(r';q')),k',1/\theta(k)\rho(1+\varepsilon)}
\end{aligned}$$

for all $\varepsilon > 0$ and $\rho > A$. Hence

$$\sup_{f\in B}|F^{-1}(H)(f)| \leq C(\rho)D(\varepsilon)\|H\|_{(s',m(r';q')),k',r},$$

for all $r < \frac{1}{\theta(k)A}$, which proves that F^{-1} is continuous.

The proof for $k = 1$ and $k = +\infty$ follows the same pattern. □

Acknowledgements. The author thanks Mário C. Matos for introducing him to this subject, for his helpful advice during the preparation of this paper and his permanent encouragement. The author also wants to thank X. Mujica for the careful reading of the first version of this paper and also for pointing out several misprints of that version. Finally, the author wishes to express his gratitude to the referee for his/her useful comments and suggestions.

References

[1] A. Grothendieck, *Topological vector spaces*. Gordon and Breach Science Publishers, New York 1973. Zbl 0275.46001 MR 0372565

[2] C. P. Gupta, Convolution operators and holomorphic mappings on a Banach Space. Seminaire d'Analyse Moderne 2, Université de Sherbrooke, Département de Mathématiques, Sherbrooke 1969. Zbl 0243.47016

[3] A. Martineau, Equations différentielles d'ordre infini. *Bull. Soc. Math. France* **95** (1967), 109–154. Zbl 0167.44202 MR 1507968

[4] M. C. Matos, On the Fourier-Borel transformation and spaces of entire functions in a normed space. In *Functional analysis, holomorphy and approximation theory, II* (Rio de Janeiro, 1981), North-Holland Math. Stud. 86, North-Holland, Amsterdam 1984, 139–169. Zbl 0568.46036 MR 0771327

[5] M. C. Matos, Absolutely summing mappings, nuclear mappings and convolution equations. Relatório de pesquisa 03/07, IMECC-UNICAMP, 2007. http://www.ime.unicamp.br/rel_pesq/2007/rp03-07.html

Received March 27, 2007; revised June 14, 2007

V. V. Fávaro, IMECC, UNICAMP, Caixa postal 6065, 13083-859 Campinas, SP, Brasil
E-mail: vvfavaro@ime.unicamp.br