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# The local structure of nonstandard representatives of distributions

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**Abstract.** It is shown that the nonstandard representatives of Schwartz distributions, as introduced by K. D. Stroyan and W. A. J. Luxemburg in their book *Introduction to the theory of infinitesimals* [5], are locally equal to a finite-order derivative of a finite-valued and S-continuous function. By 'equality', we mean a pointwise equality, not an equality in a distributional sense. This proves a conjecture by M. Oberguggenberger in *Z. Anal. Anwendungen* 10 (1991), 263–264. Moreover, the representatives of the zero-distribution are locally equal to a finite-order derivative of a function assuming only infinitesimal values. These results also unify the nonstandard theory of distributions by K. D. Stroyan and W. A. J. Luxemburg with the theory by R. F. Hoskins and J. Sousa Pinto in *Portugal. Math.* 48 (1991), 195–216.

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## 1. Introduction

**1.1. Stroyan and Luxemburg's theory of distributions.** In [5], §10.4, K. D. Stroyan and W. A. J. Luxemburg introduced their nonstandard theory of Schwartz distributions. We give a brief account of the definitions and properties in this theory needed in the sequel. The notations in this section will be used throughout the whole paper (some are different from Stroyan and Luxemburg's). The nonstandard language used is Robinson's book [4].

We will often identify a standard entity $A$ with its image ${}^{\sigma}A := \{{}^{*}x \mid x \in A\}$ when no confusion is possible.

Let $\Omega$ be an open subset of $\mathbb{R}^n$. Let $\mathscr{C}^\infty(\Omega)$ be the space of all $\Omega \to \mathbb{C}$-functions possessing continuous derivatives of any order. Let $\mathscr{D}(\Omega)$ be the space of all test-functions on $\Omega$, i.e., all $\mathscr{C}^\infty(\Omega)$-functions with compact support

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contained in $\Omega$ and $\mathscr{D}'(\Omega)$ the space of Schwartz distributions, i.e., continuous linear functionals on $\mathscr{D}(\Omega)$. By $\mathrm{ns}({}^*\Omega)$, we denote the set $\{x \in {}^*\Omega \mid \exists y \in \Omega : x \approx y\}$ of near-standard points of ${}^*\Omega$. By $\mathrm{Fin}({}^*\mathbb{C})$, we denote the set of finite elements of ${}^*\mathbb{C}$. By st we denote the standard part map.

A topological structure is introduced on ${}^*\mathscr{D}(\Omega)$ in the following way. We denote by $\partial^\alpha$ the partial derivative of order $\alpha \in \mathbb{N}^n$. A function $\phi \in {}^*\mathscr{D}(\Omega)$ is called a *finite* element of ${}^*\mathscr{D}(\Omega)$ iff its support is contained in $\mathrm{ns}({}^*\Omega)$ and if $\partial^\alpha\phi(x) \in \mathrm{Fin}({}^*\mathbb{C})$ for all (finite) multi-indices $\alpha \in \mathbb{N}$ and all $x \in {}^*\Omega$. The set of all finite elements of ${}^*\mathscr{D}(\Omega)$ will be denoted by $\mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big)$.

Similarly, $\phi \in {}^*\mathscr{D}(\Omega)$ is called an *infinitesimal* element of ${}^*\mathscr{D}(\Omega)$ iff its support is contained in $\mathrm{ns}({}^*\Omega)$ and if $\partial^\alpha\phi(x) \approx 0$ for all (finite) multi-indices $\alpha \in \mathbb{N}$ and all $x \in {}^*\Omega$. We will write $\phi \approx_{\mathscr{D}} 0$ in this case.

A ${}^*\mathscr{C}^\infty(\Omega)$-function $f$ is called a representative of $T \in \mathscr{D}'(\Omega)$ iff for each $\phi \in \mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big)$,

$$\int_{{}^*\Omega} f\phi \approx ({}^*T)(\phi).$$

It can be shown that every function $f$ in the set

$$D'(\Omega) := \left\{ f \in {}^*\mathscr{C}^\infty(\Omega) \,\middle|\, \int_{{}^*\Omega} f\phi \in \mathrm{Fin}({}^*\mathbb{C}) \text{ for all } \phi \in \mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big) \right\}$$

is a representative of a distribution $T$ by means of the definition $T(\phi) := \mathrm{st} \int_{{}^*\Omega} f\phi$. This unique distribution is called the standard part of $f$ and is denoted by $\mathrm{st}\, f$.

Vice versa, it can be shown that every distribution has a representative in $D'(\Omega)$.

$T \in {}^*\mathscr{D}'(\Omega)$ is called S-continuous iff

$$\big(\forall \phi \in {}^*\mathscr{D}(\Omega)\big) \qquad \big(\phi \approx_{\mathscr{D}} 0 \implies T(\phi) \approx 0\big). \tag{1}$$

It can be shown that every $f \in D'(\Omega)$ is S-continuous as an element of ${}^*\mathscr{D}'(\Omega)$. Stroyan and Luxemburg call the elements of $D'(\Omega)$ finite distributions. To avoid the suggestion that $D'(\Omega)$ should be a subset of the space of distributions, and because of the S-continuity as an element of ${}^*\mathscr{D}'(\Omega)$, we will call them *S-distributions* instead.

**Remark.** A function $f : {}^*\Omega \to {}^*\mathbb{C}$ is called S-continuous iff

$$x \approx y \implies f(x) \approx f(y) \qquad \text{for all } x, y \in {}^*\Omega.$$

To avoid confusion for elements of $D'(\Omega)$, we will refer to the S-continuity in the sense of eq. (1) explicitly as 'S-continuity as a linear functional'.

Two elements $f$, $g$ of $D'(\Omega)$ represent the same distribution iff

$$\int_{^*\Omega} f\phi \approx \int_{^*\Omega} g\phi \qquad \text{for all } \phi \in \mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big).$$

In such case $f$ and $g$ are called $\mathscr{D}'$-infinitely close, and we write $f \approx_{\mathscr{D}'(\Omega)} g$. If $\Omega$ is fixed in the context and no confusion can exist, we often shortly write $f \approx_{\mathscr{D}'} g$.

An S-distribution $f$ is of *order* at most $m \in \mathbb{N}$ on $K \subset\subset \Omega$ iff

$$(\exists C \in \mathbb{R}^+)\big(\forall \phi \in \mathrm{Fin}\big({}^*\mathscr{D}(K)\big)\big)\left(\left|\int_{^*\Omega} f\phi\right| \le C \max_{|\alpha|\le m} \sup_{x\in {}^*K} |\partial^\alpha \phi(x)|\right).$$

The smallest $m \in \mathbb{N}$ for which $f$ is of order at most $m$ is (logically) called the order of $f$.

**1.2. New results.** In their short section on distributions (which they call a 'sketch' themselves), Stroyan and Luxemburg only mention S-distributions of finite order for proving the theorem that every distribution is locally a finite order derivative of a continuous function, by means of the fact (mentioned as an exercise) that any S-distribution of finite order is $\mathscr{D}'$-infinitely close to a finite-order derivative of an S-continuous function $\in D'(\Omega)$. We will show that the order of an S-distribution $f$ is *not* equal to the order of the distribution $\mathrm{st}\, f$. More precisely, we will prove the following result.

**Theorem 1.** *Let $f \in D'(\Omega)$ and $K \subset\subset \Omega$. Then the* (*distributional*) *order of* $\mathrm{st}\, f$ *on $K$ is the smallest $m \in \mathbb{N}$ such that*

$$(\exists C \in \mathbb{R}^+)\big(\forall \phi \in \mathrm{Fin}\big({}^*\mathscr{D}(K)\big)\big)\left(\left|\int_{^*\Omega} f\phi\right| \lessapprox C \max_{|\alpha|\le m} \sup_{x\in {}^*K} |\partial^\alpha \phi(x)|\right).$$

We write $x \lessapprox y$ iff $x < y$ or $x \approx y$ for $x, y \in {}^*\mathbb{R}$.

The difference between these two orders will be the key to give (at least partially) an answer the following questions.

What do S-distributions look like? Is there a qualitative distinction (apart from what is clear from the definition) between S-distributions and ordinary functions in ${}^*\mathscr{C}^\infty(\Omega)$?

How much can two representatives of the same distribution differ? Except from the fact that they are $\mathscr{D}'$-infinitely close, are there qualitative ways in which this difference can be described?

It may be clear from the following example that there is hardly any pointwise way in which different representatives of a given distribution coincide in general.

**Example.** For each $k \in \mathbb{Z}$ and $\omega \in {}^*\mathbb{N}\backslash\mathbb{N}$, the function $\omega^k \sin(\omega x) \in {}^*\mathscr{C}^\infty(\mathbb{R})$ is a representative of the zero-distribution $\big(\in \mathscr{D}'(\mathbb{R})\big)$.

*Proof.* For $k < 0$, $f_k(x) = \omega^k \sin(\omega x) \approx 0$ for all $x \in {}^*\mathbb{R}$, so $f_k \approx_{\mathscr{D}'} 0$. As it is well known that the distributional derivatives coincide with the derivatives of the representatives, also the second derivative $f_k'' = -f_{k+2} \approx_{\mathscr{D}'} 0$. Inductively, $f_k \approx_{\mathscr{D}'} 0$ for all $k \in \mathbb{N}$. □

In the example, the method to find heavily irregular representatives of the zero-distribution was by taking derivatives of a function that assumes infinitesimal values. We will prove that no other irregularities can exist, i.e., that every $f \approx_{\mathscr{D}'} 0$ is (locally) pointwise equal to some finite order derivative of a ${}^*\mathscr{C}^\infty(\Omega)$-function assuming only infinitesimal values:

**Theorem 3.** *Let $f \in {}^*\mathscr{C}^\infty(\Omega)$. Then $f \approx_{\mathscr{D}'(\Omega)} 0$ iff for each $K \subset\subset \Omega$, there exists $\alpha \in \mathbb{N}^n$ and $g \in {}^*\mathscr{D}(\Omega)$ such that $g(x) \approx 0$ for all $x \in {}^*\Omega$ and $f = \partial^\alpha g$ on ${}^*K$.*

Similarly, we will prove that every $f \in D'(\Omega)$ is (locally) pointwise equal to some finite order derivative of an S-continuous and finite-valued ${}^*\mathscr{C}^\infty(\Omega)$-function:

**Theorem 2.** *Let $f \in {}^*\mathscr{C}^\infty(\Omega)$. Then $f \in D'(\Omega)$ iff for each $K \subset\subset \Omega$ there exists $g \in D'(\Omega)$ which is finite-valued and S-continuous on ${}^*K$ and such that $f$ is a finite order derivative of $g$ on ${}^*K$.*

The last of these two assertions was already mentioned (for $\Omega = \mathbb{R}^n$ and omitting the S-continuity) in [3], Prop. 2.10, in the nonstandard language of Nelson, but, as it appears from the correction to [3], it still remained unproved.

Although such theorems are of a fashion similar to the classical local representation theorem of distributions, the distributional order cannot be a measure for the order of the derivative in our representation theorems: already for the zero-distribution, which is trivially of order 0, the order of the derivative may be arbitrary large. Moreover, equalities in a stronger sense than being $\mathscr{D}'$-infinitely close (such as pointwise equalities) become even more relevant when dealing with nonlinear operations that are ill-defined in a distributional sense.

## 2. Results on the order of an S-distribution

As it will play a crucial role in proving our results, we recall a proposition about S-continuity which is proved implicitly in [5] (i.e., there is a general theorem on S-continuity from which this theorem follows partly). Also in the context of

Banach spaces, characterizations for S-continuity for internal linear maps are well-known (see e.g. [6]).

We write $K \subset\subset \Omega$ if $K$ is a compact subset of $\Omega$.

**Proposition 4.** *Let $T \in {}^*\mathscr{D}'(\Omega)$. Then the following are equivalent:*

(1) *$T$ is S-continuous.*

(2) $\big(\forall \phi \in {}^*\mathscr{D}(\Omega)\big)\ \big(\phi \approx_{\mathscr{D}} 0 \implies T(\phi) \in \mathrm{Fin}({}^*\mathbb{C})\big)$.

(3) $\big(\forall \phi \in \mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big)\big)\ \big(T(\phi) \in \mathrm{Fin}({}^*\mathbb{C})\big)$.

(4) $(\forall K \subset\subset \Omega)\ (\exists C \in \mathbb{R})\ (\exists m \in \mathbb{N})\ \big(\forall \phi \in {}^*\mathscr{D}(K)\big)$

$$\Big(|T(\phi)| \le C \max_{|\alpha| \le m} \sup_{x \in {}^*K} |\partial^\alpha \phi(x)|\Big).$$

(5) $(\forall K \subset\subset \Omega)\ (\forall \varepsilon \in \mathbb{R}^+)\ (\exists \delta \in \mathbb{R}^+)\ (\exists m \in \mathbb{N})\ \big(\forall \phi \in {}^*\mathscr{D}(K)\big)$

$$\Big(\max_{|\alpha| \le m} \sup_{x \in {}^*K} |\partial^\alpha \phi(x)| < \delta \implies |T(\phi)| < \varepsilon\Big).$$

*Proof.* $1 \Rightarrow 2$: clear.

$2 \Rightarrow 3$: follows using the fact that $\varepsilon\phi \approx_{\mathscr{D}} 0$ for all $\varepsilon \in {}^*\mathbb{R}$ with $\varepsilon \approx 0$ and for all $\phi \in \mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big)$.

$3 \Rightarrow 4$: let $K \subset\subset \Omega$. Let $m \in {}^*\mathbb{N}\backslash\mathbb{N}$ and $\phi \in {}^*\mathscr{D}(K)$. Let

$$M := \max_{|\alpha| \le m} \sup_{x \in {}^*K} |\partial^\alpha \phi(x)|.$$

If $M \ne 0$, $\frac{1}{M}\phi \in \mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big)$. So $|T(\phi)| = M \underbrace{|T(\phi/M)|}_{\in \mathrm{Fin}({}^*\mathbb{R})}$, and the internal set

$$\Big\{m \in {}^*\mathbb{N} \mid \big(\forall \phi \in {}^*\mathscr{D}(K)\big)\Big(|T(\phi)| \le m \max_{|\alpha| \le m} \sup_{x \in {}^*K} |\partial^\alpha \phi(x)|\Big)\Big\}$$

contains all infinite $m$. By underspill, property 4 holds.

$4 \Rightarrow 5$: clear.

$5 \Rightarrow 1$: follows using the fact that for each $\phi \in \mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big)$ there exists $K \subset\subset \Omega$ such that $\operatorname{supp}\phi \subseteq {}^*K$. □

**Corollary.** 1. *An S-distribution $f$ is of order at most $m \in \mathbb{N}$ on $K \subset\subset \Omega$ iff*

$$(\exists C \in \mathbb{R}^+)\big(\forall \phi \in {}^*\mathscr{D}(K)\big)\bigg(\Big|\int_{{}^*\Omega} f\phi\Big| \le C \max_{|\alpha| \le m} \sup_{x \in {}^*K} |\partial^\alpha \phi(x)|\bigg).$$

2. *Any S-distribution $f$ is of some finite order on any given $K \subset\subset \Omega$.*

*Proof.* 1. This follows from the fact that for each $\phi \in {}^*\mathscr{D}(K)$ there exists $M \in {}^*\mathbb{R}^+$ such that $\phi/M \in \mathrm{Fin}({}^*\mathscr{D}(K))$ (see the proof of the preceding proposition).

2. This follows from the preceding proposition applied to the 'regular' functional $\phi \mapsto \int_{{}^*\Omega} f\phi \in {}^*\mathscr{D}'(\Omega)$. □

Now we prove the following result.

**Theorem 1.** *Let* $f \in D'(\Omega)$ *and* $K \subset\subset \Omega$. *Then the* (*distributional*) *order of* st $f$ *on* $K$ *is the smallest* $m \in \mathbb{N}$ *such that*

$$(\exists C \in \mathbb{R}^+)\big(\forall \phi \in \mathrm{Fin}({}^*\mathscr{D}(K))\big)\left(\left|\int_{{}^*\Omega} f\phi\right| \lessapprox C \max_{|\alpha|\le m} \sup_{x\in {}^*K} |\partial^\alpha \phi(x)|\right).$$

*Proof.* 1. Let the order of $T := \mathrm{st}\, f$ on $K$ be at most $m$, i.e. (by transfer),

$$(\exists C \in \mathbb{R}^+)\big(\forall \phi \in {}^*\mathscr{D}(K)\big)\big(|{}^*T(\phi)| \le C \max_{|\alpha|\le m} \sup_{x\in {}^*K} |\partial^\alpha \phi(x)|\big).$$

Since ${}^*T(\phi) \approx \int_{{}^*\Omega} f\phi$ for $\phi \in \mathrm{Fin}({}^*\mathscr{D}(\Omega))$, we find that the formula in the statement of Theorem 1 holds for this $m$.

2. On the other hand suppose that the formula in the statement of theorem 1 holds for some $m \in \mathbb{N}$. Again by the fact that ${}^*T(\phi) \approx \int_{{}^*\Omega} f\phi$ (with $T = \mathrm{st}\, f$) for $\phi \in \mathrm{Fin}({}^*\mathscr{D}(\Omega))$, we have in particular that

$$(\exists C \in \mathbb{R}^+)\big(\forall \phi \in \mathscr{D}(K)\big)\big(|{}^*T({}^*\phi)| \lessapprox C \max_{|\alpha|\le m} \sup_{x\in {}^*K} |\partial^{\alpha\,*} \phi(x)|\big).$$

Since both sides of the $\lessapprox$-inequality are standard numbers, we actually have a $\le$-inequality, and the (distributional) order of $T$ on $K$ is at most $m$. □

**Corollary.** *The order of an S-distribution $f$ is not smaller than the distributional order of* st $f$.

The following example shows that the difference between the two orders can be arbitrary large.

**Example.** Consider $f(x) = \omega^k \sin(\omega x)$, with $\omega \in {}^*\mathbb{N}\backslash\mathbb{N}$. It has order $k$ on every compact $K \subset\subset \mathbb{R}$. On the other hand, $f \approx_{\mathscr{D}'} 0$ (see the example in Section 1.2), so the order of the corresponding standard distribution is 0.

*Proof.* Let $\phi \in {}^*\mathscr{D}(K)$. For some $R \in \mathbb{R}$, $K \subseteq [-R, R]$. Then by partial integration

$$\int_{{}^*\mathbb{R}} f\phi = (-1)^k \int_{{}^*\mathbb{R}} g(x)\phi^{(k)}(x)\, dx$$

with $g^{(k)} = f$, so we can choose $g(x) \in \{\pm\sin(\omega x), \pm\cos(\omega x)\}$. Thus

$$\left|\int_{^*\mathbb{R}} f\phi\right| \leq 2R \sup_{x\in {}^*K} |g(x)| \sup_{x\in {}^*K} |\phi^{(k)}(x)| \leq 2R \sup_{x\in {}^*K} |\phi^{(k)}(x)|,$$

so the order is at most $k$.

To see that the order is at least $k$, let $\phi_0 \in \mathscr{D}(K)$ with $\int \phi_0 = 1$ and let $\phi(x) := \sin(\omega x)\phi_0(x)$. Then

$$\frac{1}{\omega^k}\int_{^*\mathbb{R}} f\phi = \frac{1}{2}\int_{^*\mathbb{R}} \big(1 - \cos(2\omega x)\big)\phi_0(x)\,dx \approx \frac{1}{2},$$

since $\cos(2\omega x) \approx_{\mathscr{D}'} 0$ (similarly as in Example 1.2). On the other hand, for each $j \in \mathbb{N}$, $\sup_{x\in {}^*K}|\phi^{(j)}(x)| \leq M\omega^j$ for some $M \in \mathbb{R}$, so for this $\phi \in {}^*\mathscr{D}(K)$, $|\int_{^*\mathbb{R}} f\phi| > C\max_{j\leq k-1}\sup_{x\in {}^*K}|\phi^{(j)}(x)|$ for all $C \in \mathbb{R}$. □

## 3. Structure theorems

We will now prepare our main results. First we show that a representative of a distribution has anti-derivatives representing anti-derivatives of the distribution. To our knowledge, such a theorem is not available in the nonstandard literature. For convenience, we only deal with partial derivatives in the first variable.

We introduce the following notation: for $x = (x_1, \ldots, x_n) \in {}^*\mathbb{R}^n$ we will write $\tilde{x}_i := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Similarly, if $i < j$ we write $\tilde{x}_{i,j} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ and so on for $\tilde{x}_{i,j,k}, \ldots$.

**Lemma 5.** *Let $\Omega$ be an open interval (i.e., it is the Cartesian product of $n$ one-dimensional intervals). Let $T \in \mathscr{D}'(\Omega)$ and $f$ be a representative of $T$. Then there exists an S-distribution $g \in D'(\Omega)$ with $\partial_1 g = f$. As a consequence, $g$ determines a distribution $U$ with $\partial_1 U = T$.*

*Proof.* 1. In order to get some insight in the proof, we first consider the one-dimensional case.

Choose $F \in {}^*\mathscr{C}^\infty(\Omega)$ such that $F' = f$ on $\Omega$. We can only expect $F$ to be an S-distribution if the integration constant is well chosen. So we seek $C \in {}^*\mathbb{C}$ such that $\int_{^*\mathbb{R}}(F + C)\phi \in \mathrm{Fin}({}^*\mathbb{C})$ for all $\phi \in \mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big)$. Now fix $\phi_0 \in \mathscr{D}(\Omega)$ with $\int_{\mathbb{R}} \phi_0 = 1$. Then the previous condition specifies to $\int_{^*\mathbb{R}} F{}^*\phi_0 + C \in \mathrm{Fin}({}^*\mathbb{C})$. As a finite change in the constant does not influence the S-distributional character of $F + C$, we can put $C := -\int_{^*\mathbb{R}} F{}^*\phi_0$. Then for any $\phi \in \mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big)$,

$$\int_{^*\mathbb{R}} (F + C)\phi = \int_{^*\mathbb{R}} F(t)\Bigg(\underbrace{\phi(t) - \Big(\int_{^*\mathbb{R}} \phi\Big){}^*\phi_0(t)}_{=:\psi(t)\,\in\,\mathrm{Fin}({}^*\mathscr{D}(\Omega))}\Bigg)\,dt.$$

As $\int_{^*\mathbb{R}} \psi = 0$, it follows that $\psi^{(-1)}(x) := \int_{-\infty}^{x} \psi \in \mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big)$, and by partial integration we have

$$\int_{^*\mathbb{R}} (F + C)\phi = -\int_{^*\mathbb{R}} f\psi^{(-1)} \in \mathrm{Fin}({}^*\mathbb{C})$$

since $f$ is an S-distribution.

2. In the general case we choose an arbitrary anti-derivative $F$ of $f$ in the first variable (on $\Omega$). E.g., if $\Omega = (a_1, b_1) \times \cdots \times (a_n, b_n)$, $a_i, b_i \in \mathbb{R} \cup \{-\infty, +\infty\}$, then for any $a_1 < c < b_1$, $\int_c^{x_1} f(t, \tilde{x}_1)\,dt$ is such an anti-derivative. An anti-derivative is determined up to a function $G(\tilde{x}_1)$. Now it turns out that for a fixed $\phi_0 \in \mathscr{D}((a_1, b_1))$ with $\int_{\mathbb{R}} \phi_0 = 1$, $G(\tilde{x}_1) = -\int_{^*\mathbb{R}} F(t, \tilde{x}_1)\,{}^*\phi_0(t)\,dt$ is a good choice: for any $\phi \in \mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big)$ we have

$$\int_{^*\mathbb{R}^n} \big(F(x) + G(\tilde{x}_1)\big)\phi(x)\,dx = \int_{^*\mathbb{R}^n} F(x)\Bigg(\underbrace{\phi(x) - \bigg(\int_{^*\mathbb{R}} \phi(u, \tilde{x}_1)\,du\bigg)^* \phi_0(x_1)}_{=:\psi(x)}\Bigg)\,dx.$$

As $\Omega$ is an interval, $\psi \in \mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big)$. Moreover, $\int_{^*\mathbb{R}} \psi(t, \tilde{x}_1)\,dt = 0$ for all $\tilde{x}_1 \in {}^*\mathbb{R}^{n-1}$, so $\chi(x) := \int_{-\infty}^{x_1} \psi(t, \tilde{x})\,dt \in \mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big)$ and similarly as in the one-dimensional case we find that $\int_{^*\mathbb{R}^n} \big(F(x) + G(\tilde{x}_1)\big)\phi(x)\,dx \in \mathrm{Fin}({}^*\mathbb{C})$. □

**Lemma 6.** *Let $f \in D'(\Omega)$ of order $\le m$ on an interval $K \subset\subset \Omega$, $m > 0$. Then there exists $g \in D'(\Omega)$ of order $\le m - 1$ on $K$ such that $\partial_1 \ldots \partial_n g = f$ on ${}^*K$.*

*Proof.* Let $K = [a_1, b_1] \times \cdots \times [a_n, b_n]$. We will show that if $f$ satisfies

$$\left|\int_{^*\Omega} f\phi\right| \le C \sup_{x \in {}^*K} |\partial^{(k,\alpha)}\phi(x)| \qquad \text{for all } \phi \in \mathrm{Fin}\big({}^*\mathscr{D}(K)\big)$$

and for some $C \in \mathbb{R}$, $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{n-1}$, then the anti-derivative $g(x) = F(x) + G(\tilde{x}_1)$ in the first variable defined in Lemma 5 satisfies

$$\left|\int_{^*\Omega} g\phi\right| \le C' \max_{j \le l} \sup_{x \in {}^*K} |\partial^{(j,\alpha)}\phi(x)| \qquad \text{for all } \phi \in \mathrm{Fin}\big({}^*\mathscr{D}(K)\big)$$

with $C' \in \mathbb{R}$ and $l = \max(k - 1, 0)$.

Let $\phi \in \mathrm{Fin}\big({}^*\mathscr{D}(K)\big)$. With $\psi, \chi \in \mathrm{Fin}\big({}^*\mathscr{D}(K)\big)$ as in Lemma 5, we have

$$\left|\int_{^*\Omega} g\phi\right| = \left|\int_{^*\Omega} f\chi\right| \le C \sup_{x \in {}^*K} |\partial^{(k,\alpha)}\chi(x)| = C \sup_{x \in {}^*K} |\partial^{(0,\alpha)}\partial_1^k \partial_1^{-1}\psi(x)|.$$

In case $k = 0$, we have for $x \in {}^*K$ that

$$|\partial^{(0,\alpha)}\partial_1^{-1}\psi(x)| = \left|\int_{-\infty}^{x_1} \partial^{(0,\alpha)}\psi(t_1, \tilde{x}_1)\,dt_1\right| \le (b_1 - a_1) \sup_{x \in {}^*K} |\partial^{(0,\alpha)}\psi(x)|,$$

so in any case we have (for some $C', C'' \in \mathbb{R}$, independent of $\phi$)

$$\begin{aligned}\left|\int_{{}^*\Omega} g\phi\right| &\le C' \sup_{x\in{}^*K} |\partial^{(l,\alpha)}\phi(x)| + C' \sup_{x\in{}^*K} \left|D^{l\,*}\phi_0(x_1)\int_{{}^*\mathbb{R}} \partial^{(0,\alpha)}\phi(u,\tilde{x}_1)\,du\right| \\ &\le C'' \max_{j\le l} \sup_{x\in{}^*K} |\partial^{(j,\alpha)}\phi(x)|.\end{aligned}$$

Since $g$ is well defined on ${}^*\Omega'$ for some interval $\Omega' \subseteq \Omega$ with $K \subset\subset \Omega'$, we can use $\phi_0 \in \mathscr{D}(\Omega')$ with $\phi_0 = 1$ on $K$ to ensure that $g^*\phi_0 \in D'(\Omega)$ without changing the values on ${}^*K$.

If we repeatedly apply also the analogous result for the variables $x_2, \ldots, x_n$, we finally conclude that the order of the primitive $(\partial_1 \ldots \partial_n)^{-1} f$ has decreased (if $m > 0$). □

For $K \subset\subset \Omega$ we call $L^\infty(K)$ the space of all (standard) bounded (Lebesgue-measurable) functions $f : \Omega \to \mathbb{C}$ with support contained in $K$.

**Lemma 7.** *Let $K \subset\subset \Omega$ an interval. An S-distribution $f$ is of order zero on $K$ iff*

$$(\exists C \in \mathbb{R}^+)\big(\forall\phi \in {}^*L^\infty(K)\big)\left(\left|\int_{{}^*\Omega} f\phi\right| \le C \sup_{x\in{}^*K} |\phi(x)|\right).$$

*Proof.* Let $f \in \mathscr{C}^\infty(\Omega)$ and $\phi \in L^\infty(K)$. Then by a classical density theorem it is clear that there exists some $h \in \mathscr{D}(K)$ such that

$$\left|\int_\Omega f\phi - \int_\Omega fh\right| \le \sup_{x\in K} |\phi(x)| \qquad \text{and} \qquad \sup_{x\in K} |h(x)| \le 2 \sup_{x\in K} |\phi(x)|.$$

By transfer, we have $\big(\forall f \in {}^*\mathscr{C}^\infty(\Omega)\big)\ \big(\forall\phi \in {}^*L^\infty(K)\big)\ \big(\exists h \in {}^*\mathscr{D}(K)\big)$

$$\left(\left|\int_{{}^*\Omega} f\phi - \int_{{}^*\Omega} fh\right| \le \sup_{x\in{}^*K} |\phi(x)| \text{ and } \sup_{x\in{}^*K} |h(x)| \le 2 \sup_{x\in{}^*K} |\phi(x)|\right).$$

If in particular $f$ is an S-distribution of order 0 on $K$, then

$$(\exists C \in \mathbb{R}^+)\big(\forall h \in {}^*\mathscr{D}(K)\big)\left(\left|\int_{{}^*\Omega} fh\right| \le C \sup_{x\in{}^*K} |h(x)|\right).$$

The result follows by combining these two formulas. □

**Lemma 8.** *Let $f \in D'(\Omega)$. Suppose that $f$ is of order zero on a (standard) interval $K = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset\subset \Omega$. Then*

(1) *there exists $g \in {}^*\mathscr{C}^\infty(\Omega)$ which is bounded on ${}^*K$ by a standard constant and such that $\partial_1 \ldots \partial_n g = f$ on ${}^*K$.*

(2) *there exists $h \in {}^*\mathscr{C}^\infty(\Omega)$ which is S-continuous and bounded by a standard constant on ${}^*K$ and such that $\partial_1^2 \ldots \partial_n^2 h = f$ on ${}^*K$.*

*Proof.* (1) Let $x = (x_1, \ldots, x_n)$ and $t = (t_1, \ldots, t_n)$. For $A \subset \Omega$, we denote the characteristic function of $A$ by $\chi_A$. Then (for $x \in {}^*K$)

$$g(x) := \int_{a_1}^{x_1} dt_1 \ldots \int_{a_n}^{x_n} f(t)\, dt_n = \int_{{}^*\Omega} f \chi_{[a_1, x_1] \times \cdots \times [a_n, x_n]}$$

clearly satisfies $\partial_1 \ldots \partial_n g = f$ on ${}^*K$. Further, applying the previous lemma with $\phi = \chi_{[a_1, x_1] \times \cdots \times [a_n, x_n]} \in {}^*L^\infty(K)$ (if $x \in {}^*K$), we find $C \in \mathbb{R}^+$ such that

$$(\forall x \in {}^*K)\big(|g(x)| \le C \underbrace{\sup_{x \in {}^*K} |\phi(x)|}_{=1}\big).$$

(2) If $g$ satisfies the conditions from part 1, then (for $x \in {}^*K$)

$$h(x) := \int_{a_1}^{x_1} dt_1 \ldots \int_{a_n}^{x_n} g(t)\, dt_n$$

clearly satisfies $\partial_1^2 \ldots \partial_n^2 h = f$ on ${}^*K$. Further, for $\varepsilon \approx 0$ and $\varepsilon > 0$,

$$|h(x_1 + \varepsilon, \tilde{x}_1) - h(x)| = \left| \int_{x_1}^{x_1 + \varepsilon} dt_1 \ldots \int_{a_n}^{x_n} g(t)\, dt_n \right| \le C\varepsilon \prod_{i \ne 1} (b_i - a_i) \approx 0,$$

and similarly for the other variables. So $h(x) \approx h(y)$ as soon as $x \approx y$ ($x, y \in {}^*K$). Further, $|h(x)| \le C \prod_i (b_i - a_i) \in \mathrm{Fin}({}^*\mathbb{C})$ for all $x \in {}^*K$. □

We are now ready to prove one of our main results.

**Theorem 2.** *Let $f \in {}^*\mathscr{C}^\infty(\Omega)$. Then $f \in D'(\Omega)$ iff for each $K \subset\subset \Omega$ there exists a $g \in D'(\Omega)$ which is finite-valued and S-continuous on ${}^*K$ and such that $f$ is a finite order derivative of $g$ on ${}^*K$.*

*Proof.* $\Leftarrow$: follows using the fact that for each $\phi \in \mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big)$, there exists $K \subset\subset \Omega$ such that $\operatorname{supp} \phi \subseteq {}^*K$.

$\Rightarrow$: 1. We first consider the special case where $K \subset\subset \Omega$ is an interval.

Take an interval $K' \subset\subset \Omega$ with $K \subset\subset {}^\circ(K')$, the (topological) interior of $K'$. Since $f$ has a finite order $m$ on $K'$, we find, by repeatedly applying Lemma 6, some $\tilde{g} \in D'(\Omega)$ of order zero on $K'$ such that $(\partial_1 \ldots \partial_n)^m \tilde{g} = f$ on ${}^*K'$. By Lemma 8, we find $h \in {}^*\mathscr{C}^\infty(\Omega)$ which is finite and S-continuous on ${}^*K'$ and such that $(\partial_1 \ldots \partial_n)^{m+2} h = f$ on ${}^*K'$. If $\phi_0 \in \mathscr{D}(K')$ with $\phi_0 = 1$ on $K$, then $g := h\,{}^*\phi_0 \in D'(\Omega)$ has the required properties.

2. We consider the special case where $f(x) = 0$ for all $x \notin \mathrm{ns}({}^*\Omega)$.

Then $f$ can be extended to a ${}^*\mathscr{C}^\infty(\mathbb{R}^n)$-function, setting $f(x) := 0$ if $x \in {}^*\mathbb{R}^n \backslash \mathrm{ns}({}^*\Omega)$. We claim that this extension is in $D'(\mathbb{R}^n)$. There exists $K_0 \subset\subset \Omega$ such that $f(x) = 0$ outside ${}^*K_0$. Choose $\phi_0 \in \mathscr{D}(\Omega)$ with $\phi_0 = 1$ on $K_0$. Then for any $\phi \in \mathrm{Fin}\big({}^*\mathscr{D}(\mathbb{R}^n)\big)$,

$$\int_{{}^*\mathbb{R}^n} f\phi = \int_{{}^*\Omega} f \underbrace{{}^*\phi_0 \phi}_{\in \mathrm{Fin}({}^*\mathscr{D}(\Omega))} \in \mathrm{Fin}({}^*\mathbb{C}).$$

Now let $K \subset\subset \Omega$ be arbitrary. Since $K \subseteq L \subset\subset \mathbb{R}^n$, with $L$ an interval (possibly $L \nsubseteq \Omega$), we conclude from part 1 that there exists a $g \in D'(\mathbb{R}^n)$ which is finite and S-continuous on ${}^*L$ and such that (the extended) $f$ is a finite order derivative of $g$ on ${}^*L$. The restriction of $g$ to ${}^*\Omega$ has the required properties.

3. In the general case, let $K \subset\subset \Omega$. Taking $\phi_0 \in \mathscr{D}(\Omega)$ with $\phi_0 = 1$ on $K$, we apply part 2 on $f\,{}^*\phi_0 \in D'(\Omega)$. □

The other main result will follow from Theorem 2 together with some additional lemmas.

**Lemma 9.** *Let* $\Omega = (a_1, b_1) \times \cdots \times (a_n, b_n) \subseteq \mathbb{R}^n$ *be an open interval (possibly* $a_i = -\infty$, $b_i = +\infty$*). Let* $\tilde{\Omega} := (a_2, b_2) \times \cdots \times (a_n, b_n) \subseteq \mathbb{R}^{n-1}$. *Let* $f \in {}^*\mathscr{C}^\infty(\Omega)$ *be independent of* $x_1$*, so it can be identified with a* ${}^*\mathscr{C}^\infty(\tilde{\Omega})$*-function. Then:*

(1) $f(\tilde{x}_1) \in D'(\Omega) \iff f(\tilde{x}_1) \in D'(\tilde{\Omega})$.

(2) $f(\tilde{x}_1) \approx_{\mathscr{D}'(\Omega)} 0 \iff f(\tilde{x}_1) \approx_{\mathscr{D}'(\tilde{\Omega})} 0$.

*As a consequence, the expression* $f(\tilde{x}_1) \approx_{\mathscr{D}'} 0$ *is unambiguous.*

*Proof.* (1) $\Rightarrow$: Let $f(\tilde{x}_1) \in D'(\Omega)$. Fix $\psi(x_1) \in \mathrm{Fin}\big({}^*\mathscr{D}(a_1, b_1)\big)$ with $\int_{{}^*\mathbb{R}} \psi = 1$. Choose $\phi(\tilde{x}_1) \in \mathrm{Fin}\big({}^*\mathscr{D}(\tilde{\Omega})\big)$ arbitrarily. Then $\psi(x_1)\phi(\tilde{x}_1) \in \mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big)$, so

$$\mathrm{Fin}({}^*\mathbb{C}) \ni \int_{{}^*\Omega} f(\tilde{x}_1)\psi(x_1)\phi(\tilde{x}_1)\,dx = \underbrace{\int_{a_1}^{b_1} \psi(x_1)\,dx_1}_{=1} \int_{{}^*\tilde{\Omega}} f(\tilde{x}_1)\phi(\tilde{x}_1)\,d\tilde{x}_1,$$

which means that $f(\tilde{x}_1) \in D'(\tilde{\Omega})$.

$\Leftarrow$: Let $f(\tilde{x}_1) \in D'(\tilde{\Omega})$. For any $\phi \in \mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big)$ and $c \in \mathrm{ns}\big({}^*(a_1, b_1)\big)$, the map $\tilde{x}_1 \mapsto \phi(c, \tilde{x}_1) \in \mathrm{Fin}\big({}^*\mathscr{D}(\tilde{\Omega})\big)$, so

$$\psi(c) := \int_{{}^*\tilde{\Omega}} f(\tilde{x}_1)\phi(c, \tilde{x}_1)\, d\tilde{x}_1 \in \mathrm{Fin}({}^*\mathbb{C}).$$

Further, for some $K \subset\subset (a_1, b_1)$, if $c$ lies outside ${}^*K$, then $\psi(c) = 0$. So

$$\int_{{}^*\Omega} f(\tilde{x}_1)\phi(x)\, dx = \int_{{}^*K} \psi(x_1)\, dx_1 \in \mathrm{Fin}({}^*\mathbb{C}),$$

which means that $f(\tilde{x}_1) \in D'(\Omega)$.

(2) Similar. □

The following lemmas could be considered as exercises in distribution theory. To our knowledge, they are not widely known. Therefore, we will include a nonstandard version with proof.

**Lemma 10.** *Let $\Omega$ be an open interval.*

*If $f \in D'(\Omega)$ and $\partial^\alpha f \approx_{\mathscr{D}'} 0$, then there exist $g_{ij} \in D'(\Omega)$ such that*

$$f(x) \approx_{\mathscr{D}'} \sum_{i=1}^{n} \sum_{j=0}^{\alpha_i - 1} g_{ij}(\tilde{x}_i) x_i^j.$$

*Proof.* 1. We first show that if $F \in D'(\Omega)$ and $\partial_1 F \approx_{\mathscr{D}'} 0$, then $F$ is $\mathscr{D}'$-infinitely close to a $D'(\Omega)$-function which does not depend on $x_1$.

If we choose $G(\tilde{x}_1)$ as in Lemma 5, we see that for all $\phi \in \mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big)$,

$$\int_{{}^*\mathbb{R}^n} \big(F(x) + G(\tilde{x}_1)\big)\phi(x)\, dx = \int_{{}^*\mathbb{R}^n} (\partial_1 F)(x)\chi(x)\, dx \approx 0$$

with $\chi \in \mathrm{Fin}\big({}^*\mathscr{D}(\Omega)\big)$ as in Lemma 5. So $F(x) \approx_{\mathscr{D}'} -G(\tilde{x}_1)$.

2. Now suppose that $f \in D'(\Omega)$ and

$$\partial_1 f(x) \approx_{\mathscr{D}'} \sum_{i=1}^{n} \sum_{j=0}^{m_i} g_{ij}(\tilde{x}_i) x_i^j \tag{2}$$

for some $g_{ij} \in D'(\Omega)$. We will show that

$$f(x) \approx_{\mathscr{D}'} \sum_{i=1}^{n} \sum_{j=0}^{\tilde{m}_i} \tilde{g}_{ij}(\tilde{x}_i) x_i^j$$

for some $\tilde{g}_{ij} \in D'(\Omega)$, $\tilde{m}_1 = m_1 + 1$, $\tilde{m}_2 = m_2, \ldots, \tilde{m}_n = m_n$.

We notice that the right-hand side of eq. (2) is equal to

$$\partial_1\Big(\sum_{j=0}^{m_1} g_{1j}(\tilde{x}_1)\frac{x_1^{j+1}}{j+1} + \sum_{i=2}^{n}\sum_{j=0}^{m_i}(\partial_1^{-1}g_{ij})(\tilde{x}_i)x_i^j\Big).$$

From the explicit construction of the primitives $\partial_1^{-1}g_{ij}$ in Lemma 5, it is immediate that also they are independent of $x_i$. Then applying part 1 on the difference of both sides in eq. (2), we find that there exists $G(\tilde{x}_1) \in D'(\Omega)$ such that

$$f(x) \approx_{\mathscr{D}'} G(\tilde{x}_1) + \sum_{j=0}^{m_1} g_{1j}(\tilde{x}_1)\frac{x_1^{j+1}}{j+1} + \sum_{i=2}^{n}\sum_{j=0}^{m_i}(\partial_1^{-1}g_{ij})(\tilde{x}_i)x_i^j,$$

which has the required form.

3. Now the theorem follows inductively using part 2 and the analogous formulas for all the other variables (other than $x_1$), also using the fact that if $f \in D'(\Omega)$, then $\partial^\beta f \in D'(\Omega)$ for all $\beta \in \mathbb{N}^n$. □

**Lemma 11.** *Let $\Omega$ be an open interval. Let $f \in {}^*\mathscr{C}^\infty(\Omega)$ be S-continuous and finite-valued on* ns$({}^*\Omega)$, *and suppose that $\partial^\alpha f \approx_{\mathscr{D}'} 0$. Then there exist $g_{ij} \in {}^*\mathscr{C}^\infty(\Omega)$ which are S-continuous and finite-valued on* ns$({}^*\Omega)$ *such that*

$$f(x) \approx_{\mathscr{D}'} \sum_{i=1}^{n}\sum_{j=0}^{\alpha_i-1} g_{ij}(\tilde{x}_i)x_i^j.$$

*Proof.* First notice that a ${}^*\mathscr{C}^\infty(\Omega)$-function which is finite-valued on ns$({}^*\Omega)$ is in $D'(\Omega)$. Let $\Omega = (a_1, b_1) \times \cdots \times (a_n, b_n)$ and $\tilde{\Omega} := (a_2, b_2) \times \cdots \times (a_n, b_n)$. Let $\partial^\alpha f \approx_{\mathscr{D}'} 0$ and let $\alpha =: (\alpha_1, \tilde{\alpha})$, $\tilde{\alpha} \in \mathbb{N}^{n-1}$. By the previous lemma,

$$f(x) \approx_{\mathscr{D}'} \sum_{i=1}^{n}\sum_{j=0}^{\alpha_i-1} h_{ij}(\tilde{x}_i)x_i^j, \tag{3}$$

with $h_{ij} \in D'(\Omega)$. Now consider an arbitrary $c \in \mathrm{ns}^*(a_1, b_1)$. Fix $\psi(x_1) \in \mathscr{D}(\mathbb{R})$ with $\int_{\mathbb{R}} \psi = 1$ and $\psi \geq 0$. Let $\psi_m(x_1) := m\psi(mx_1)$ for all $m \in {}^*\mathbb{N}$. Let $\phi(\tilde{x}_1) \in \mathrm{Fin}\big({}^*\mathscr{D}(\tilde{\Omega})\big)$ be arbitrary. Since

$$\partial^{(0,\tilde{\alpha})}f(x) \approx_{\mathscr{D}'} \sum_{j=0}^{\alpha_1-1} \partial^{\tilde{\alpha}}h_{1j}(\tilde{x}_1)x_1^j,$$

we have for sufficiently large $m \in \mathbb{N}$ (such that $\mathrm{supp}\big(\psi_m(c - x_1)\big) \subset \mathrm{ns}^*(a_1, b_1)$) that

$$\int_{^*\Omega} \partial^{(0,\tilde\alpha)} f(x)\psi_m(c-x_1)\phi(\tilde x_1)\,dx \approx \sum_{j=0}^{\alpha_1-1}\int_{^*\Omega}\partial^{\tilde\alpha} h_{1j}(\tilde x_1)x_1^j\psi_m(c-x_1)\phi(\tilde x_1)\,dx. \quad (4)$$

By Robinson's sequential lemma, this also holds for some $\omega\in{}^*\mathbb{N}\backslash\mathbb{N}$. If $\tilde x_1\in \mathrm{ns}({}^*\tilde\Omega)$, the map $x_1\to f(x)$ is S-continuous on $\mathrm{ns}\big({}^*(a_1,b_1)\big)$. Then

$$\begin{aligned}\left|\int_{a_1}^{b_1} f(x)\psi_\omega(c-x_1)\,dx_1 - f(c,\tilde x_1)\right| &= \left|\int_{a_1}^{b_1}\big(f(x)-f(c,\tilde x_1)\big)\psi_\omega(c-x_1)\,dx_1\right|\\ &\le \sup_{x_1\in\operatorname{supp}\psi_\omega}|f(x)-f(c,\tilde x_1)|\approx 0\end{aligned}$$

for all $\tilde x_1\in\mathrm{ns}({}^*\tilde\Omega)$, since $\operatorname{supp}\psi_\omega$ contains only infinitesimals and $\int_{^*\mathbb{R}}|\psi_\omega|=1$. In particular, they are $\mathscr{D}'$-infinitely close. So also

$$\int_{a_1}^{b_1}\partial^{(0,\tilde\alpha)} f(x)\psi_\omega(c-x_1)\,dx_1 \approx_{\mathscr{D}'} \partial^{\tilde\alpha} f(c,\tilde x_1).$$

On the other hand,

$$\int_{^*\Omega}\partial^{\tilde\alpha} h_{1j}(\tilde x_1)x_1^j\psi_\omega(c-x_1)\phi(\tilde x_1)\,dx = \underbrace{\int_{^*\tilde\Omega}\partial^{\tilde\alpha} h_{1j}(\tilde x_1)\phi(\tilde x_1)\,d\tilde x_1}_{\in\mathrm{Fin}(^*\mathbb{C})}\underbrace{\int_{a_1}^{b_1} x_1^j\psi_\omega(c-x_1)\,dx_1}_{\approx c^j},$$

so we find from eq. (4) that

$$\partial^{\tilde\alpha} f(c,\tilde x_1)\approx_{\mathscr{D}'}\sum_{j=0}^{\alpha_1-1} c^j\partial^{\tilde\alpha} h_{1j}(\tilde x_1)$$

for each $c\in\mathrm{ns}\big({}^*(a_1,b_1)\big)$. Now choose $\alpha_1$ different values $c_i\in\mathrm{ns}\big({}^*(a_1,b_1)\big)$, with $c_i\not\approx c_j$ if $i\ne j$. Then we find a linear system with $\alpha_1$ equations and $\alpha_1$ unknown functions $\partial^{\tilde\alpha} h_{1j}$. The determinant of the system is a Vandermonde determinant equal to $\prod_{i<j}(c_j-c_i)\not\approx 0$. Therefore, each $\partial^{\tilde\alpha} h_{1j}(\tilde x_1)$ is $\mathscr{D}'$-infinitely close to a $\mathrm{Fin}(^*\mathbb{C})$-linear combination of the $\partial^{\tilde\alpha} f(c_j,\tilde x_1)$, which we call $\partial^{\tilde\alpha} g_{1j}(\tilde x_1)$. So $g_{1j}(\tilde x_1)\in{}^*\mathscr{C}^\infty(\Omega)$ are S-continuous and finite-valued on $\mathrm{ns}(^*\Omega)$. By the previous lemma (applied to $\tilde\Omega\subseteq\mathbb{R}^{n-1}$),

$$h_{1j}(\tilde x_1)\approx_{\mathscr{D}'} g_{1j}(\tilde x_1)+\sum_{i=2}^{n}\sum_{k=0}^{\alpha_i-1}\tilde h_{ik}(\tilde x_{1i})x_i^k$$

for some $\tilde h_{ik}\in D'(\Omega)$. Substituting these expressions, together with the analogous expressions for $h_{ij}(\tilde x_i)$ (with $i>1$), in formula (3) yields that

$$f(x) \approx_{\mathscr{D}'} \sum_{i=1}^{n} \sum_{j=0}^{\alpha_i - 1} g_{ij}(\tilde{x}_i) x_i^j + \sum_{1 \le i_1 < i_2 \le n} \sum_{j_1=0}^{\alpha_{i_1}-1} \sum_{j_2=0}^{\alpha_{i_2}-1} h_{i_1 i_2 j_1 j_2}(\tilde{x}_{i_1, i_2}) x_{i_1}^{j_1} x_{i_2}^{j_2}, \tag{5}$$

for some $h_{i_1 i_2 j_1 j_2} \in D'(\Omega)$, since multiplication by $x_i$ preserves the $\approx_{\mathscr{D}'}$-equality.

We now proceed inductively and show that

$$f(x) \approx_{\mathscr{D}'} \sum_{i=1}^{n} \sum_{j=0}^{\alpha_i - 1} g_{ij}(\tilde{x}_i) x_i^j + \sum_{1 \le i_1 < i_2 < i_3 \le n} \sum_{j_1, j_2, j_3} h_{i_1 i_2 i_3 j_1 j_2 j_3}(\tilde{x}_{i_1, i_2, i_3}) x_{i_1}^{j_1} x_{i_2}^{j_2} x_{i_3}^{j_3} \tag{6}$$

for some $g_{ij} \in {}^*\mathscr{C}^\infty(\Omega)$, S-continuous and finite-valued on $\mathrm{ns}({}^*\Omega)$, and some $h_{i_1 i_2 i_3 j_1 j_2 j_3} \in D'(\Omega)$.

The proof is similar. Let $F := f - \sum_{i=1}^{n} \sum_{j=0}^{\alpha_i - 1} g_{ij}(\tilde{x}_i) x_i^j$. Let $\alpha =: (\alpha_1, \alpha_2, \tilde{\alpha})$, $\tilde{\alpha} \in \mathbb{N}^{n-2}$. Let $\tilde{\Omega} := (a_3, b_3) \times \cdots \times (a_n, b_n)$. Then

$$\partial^{(0,0,\tilde{\alpha})} F(x) \approx_{\mathscr{D}'} \sum_{j_1=0}^{\alpha_1 - 1} \sum_{j_2=0}^{\alpha_2 - 1} \partial^{\tilde{\alpha}} h_{1,2,j_1,j_2}(\tilde{x}_{1,2}) x_1^{j_1} x_2^{j_2}.$$

Fixing now $c \in \mathrm{ns}^*(a_1, b_1)$ and $d \in \mathrm{ns}^*(a_2, b_2)$, we choose $\psi_m$ as before, $\phi(\tilde{x}_{1,2}) \in \mathrm{Fin}({}^*\tilde{\Omega})$, multiply the previous expression by $\psi_m(c - x_1)\psi_m(d - x_2)\phi(\tilde{x}_{1,2})$ and integrate over ${}^*\Omega$ to obtain similarly that

$$\partial^{\tilde{\alpha}} F(c, d, \tilde{x}_{1,2}) \approx_{\mathscr{D}'} \sum_{j_1=0}^{\alpha_1 - 1} \sum_{j_2=0}^{\alpha_2 - 1} \partial^{\tilde{\alpha}} h_{1,2,j_1,j_2}(\tilde{x}_{1,2}) c^{j_1} d^{j_2}.$$

Now we substitute $c$ by $\alpha_1$ different values $c_1, \ldots, c_{\alpha_1} \in \mathrm{ns}^*(a_1, b_1)$ and $d$ by $\alpha_2$ different values $d_1, \ldots, d_{\alpha_2} \in \mathrm{ns}^*(a_2, b_2)$, with $c_i \not\approx c_j$ if $i \neq j$ and $d_i \not\approx d_j$ if $i \neq j$. The resulting linear system has $\alpha_1 \alpha_2$ equations and $\alpha_1 \alpha_2$ unknown functions $\partial^{\tilde{\alpha}} h_{1,2,j_1,j_2}$. The matrix of the system is (if the equations and unknowns are written down in a suitable order) the Kronecker product (sometimes also called direct product, see e.g. [2]) of the Vandermonde matrices $(c_i^{j-1})_{i,j=1,\ldots,\alpha_1}$ and $(d_i^{j-1})_{i,j=1,\ldots,\alpha_2}$, with determinant

$$\prod_{i<j} (c_j - c_i)^{\alpha_2} \prod_{i<j} (d_j - d_i)^{\alpha_1} \not\approx 0.$$

Another application of the previous lemma yields that

$$h_{1,2,j_1,j_2}(\tilde{x}_{1,2}) \approx_{\mathscr{D}'} g_{1,2,j_1,j_2}(\tilde{x}_{1,2}) + \sum_{i=3}^{n} \sum_{k=0}^{\alpha_i - 1} \tilde{h}_{ik}(\tilde{x}_{1,2,i}) x_i^k$$

for some $g_{1,2,j_1,j_2} \in {}^*\mathscr{C}^\infty(\Omega)$, S-continuous and finite-valued on $\mathrm{ns}({}^*\Omega)$, and $\tilde{h}_{ik} \in D'(\Omega)$. Substituting these expressions (for all $h_{i_1,i_2,j_1,j_2}$) in formula (5) and absorbing the terms $g_{i_1 i_2 j_1 j_2}(\tilde{x}_{i_1 i_2}) x_{i_1}^{j_1} x_{i_2}^{j_2}$ in the $g_{ij}(\tilde{x}_i) x_i^j$ we arrive at formula (6).

Repeatedly applying this procedure, we conclude that

$$f(x) \approx_{\mathscr{D}'} \sum_{i=1}^{n} \sum_{j=0}^{\alpha_i - 1} g_{ij}(\tilde{x}_i) x_i^j + \sum_{j_1, j_2, \ldots, j_n} c_{j_1,\ldots,j_n} x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n}$$

for some $g_{ij} \in {}^*\mathscr{C}^\infty(\Omega)$, S-continuous and finite-valued on $\mathrm{ns}({}^*\Omega)$, and constant $c_{j_1,\ldots,j_n} \in D'(\Omega)$. Since a constant function belonging to $D'(\Omega)$ is necessarily in $\mathrm{Fin}({}^*\mathbb{C})$ (see Lemma 12), we can absorb the terms $c_{j_1,\ldots,j_n} x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n}$ in the $g_{ij}(\tilde{x}_i) x_i^j$ and finally obtain the required formula. □

Finally, we need a lemma of Robinson's [4], Theorem 5.3.14. Robinson works with real-valued distributions on $\mathbb{R}$. We show that the result can be generalized to our situation.

**Lemma 12.** *Let $T \in \mathscr{D}'(\Omega)$. If there exists a representative $f$ of $T$ which is S-continuous at $a \in \Omega$, then $f(a) \in \mathrm{Fin}({}^*\mathbb{C})$. Moreover, the value $\mathrm{st}\, f(a)$ does not depend on the chosen S-continuous representative.*

*Proof.* Let $\varepsilon \in \mathbb{R}^+$. By S-continuity, there exists $r \in \mathbb{R}^+$ such that $|f(x) - f(a)| \le \varepsilon$ for all $x \in {}^*B(a,r) \subseteq {}^*\Omega$. Now let $\phi \in \mathscr{D}\big(B(a,r)\big)$, real-valued, $\phi(x) \ge 0$ for all $x \in \Omega$ and $\int_\Omega \phi = 1$. Then

$$\left| \int_{{}^*\Omega} f(x)\, {}^*\phi(x)\, dx - f(a) \right| = \left| \int_{{}^*B(a,r)} \big(f(x) - f(a)\big)\, {}^*\phi(x)\, dx \right|$$
$$\le \int_{{}^*B(a,r)} \varepsilon |{}^*\phi(x)|\, dx = \varepsilon.$$

As $f$ represents $T$, we have $|T(\phi) - f(a)| \le 2\varepsilon$. In particular, $f(a) \in \mathrm{Fin}({}^*\mathbb{C})$. For any representative $g$ of $T$, S-continuous at $a$, we have the same inequality (possibly only for some smaller $r \in \mathbb{R}^+$), so $|f(a) - g(a)| \le 4\varepsilon$. As $\varepsilon \in \mathbb{R}^+$ is arbitrary, it follows that $\mathrm{st}\, f(a) = \mathrm{st}\, g(a)$. □

Now we can prove our last main result.

**Theorem 3.** *Let $f \in {}^*\mathscr{C}^\infty(\Omega)$. Then $f \approx_{\mathscr{D}'(\Omega)} 0$ iff for each $K \subset\subset \Omega$ there exists $\alpha \in \mathbb{N}^n$ and $g \in {}^*\mathscr{D}(\Omega)$ such that $g(x) \approx 0$, $\forall x \in {}^*\Omega$ and $f = \partial^\alpha g$ on ${}^*K$.*

*Proof.* 1. $\Rightarrow$: We first consider the case where $K \subset\subset \Omega$ is an interval.

Take an interval $K' \subset\subset \Omega$ with $K \subset\subset {}^\circ(K')$. By Theorem 2, there exists $h \in D'(\Omega)$ which is finite-valued and S-continuous on ${}^*K'$ and such that $\partial^\alpha h = f$ on ${}^*K'$. By Lemma 11 applied on the open interval $\tilde{\Omega} := {}^\circ(K')$, we find in particular that $h$ is $\mathscr{D}'(\tilde{\Omega})$-infinitely close to some $\tilde{h} \in {}^*\mathscr{C}^\infty(\tilde{\Omega})$, which is S-continuous on $\mathrm{ns}({}^*\tilde{\Omega})$. As $\tilde{h}(x) = \sum_{i=1}^{n} \sum_{j=0}^{\alpha_i - 1} g_{ij}(\tilde{x}_i) x_i^j$, we see that $\partial^\alpha \tilde{h} = 0$ on ${}^*\tilde{\Omega}$. Now $h - \tilde{h} \approx_{\mathscr{D}'(\tilde{\Omega})} 0$ and is S-continuous on $\mathrm{ns}({}^*\tilde{\Omega})$, so by Lemma 12, $h(x) - \tilde{h}(x) \approx 0$ for all $x \in \mathrm{ns}({}^*\tilde{\Omega})$. Further, $\partial^\alpha(h - \tilde{h}) = \partial^\alpha h = f$ on ${}^*K$. If $\phi_0 \in \mathscr{D}(\tilde{\Omega})$ with $\phi_0 = 1$ on a neigbourhood of ${}^*K$, then $g := (h - \tilde{h}){}^*\phi_0$ has the required properties.

2. The general case as well as the $\Leftarrow$-part follow in a way similar to the proof of Theorem 2. □

## 4. Application

In [1], R. F. Hoskins and J. Sousa Pinto introduce another nonstandard theory of distributions. In this setting, nonstandard representatives of a distribution are *by definition* locally finite-order derivatives of finite-valued and S-continuous functions. By Theorem 2, it now follows that representatives of distributions in the sense of Hoskins and Sousa Pinto are exactly representatives of distributions in the sense of Stroyan and Luxemburg.

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