Portugal. Math. (N.S.)
Vol. 65, Fasc. 3, 2008, 311–320

Portugaliae Mathematica

On periodic solutions of a nonlinear singular differential equation

Moulay Rchid Sidi Ammi

(Communicated by Luis Barreira)

Abstract. We find sufficient conditions on the rotation number of solutions to a nonlinear singular problem which guarantee the existence of T-periodic solutions.

Mathematics Subject Classification (2000). 31B15, 34C25, 35K60.

Keywords. p-Laplacian, periodic solutions, Poincaré map, rotation number.

1. Introduction

In this work we study the existence of T-periodic solutions to the following initial value problem

$$\left.\begin{aligned} (|x'|^{p-2}x')' + \frac{n}{t}|x'|^{p-2}x' + g\big(x(t)\big) + h(t) = 0, \\ x(0) = x_0, \\ x'(0) = 0, \end{aligned}\right\} \tag{1.1}$$

where x_0 and n are real numbers satisfying $x_0 \neq 0$ and $n \geq 0$. More precisely, we will show that (1.1) has T-periodic solutions provided the functions g and h satisfy certain conditions to be specified below. Most often, existence results of this kind are established via a variational method, where the solutions are obtained as critical points of some energy functional. For example, in [1] and [9], the authors obtain existence results for the semilinear second-order ordinary differential equation when $p = 2$ (i.e., for the operator $-u''$) under a certain growth condition, using the variational approach. Our method of proof, unlike those works, is not variational. Rather, it is based on a phase-plane analysis similar to the methods used in [5, 6], for elliptic problems, and [2], for the Duffing equation. In fact, our results are obtained by combining the time-mapping for the one-dimensional p-Laplacian with the computation of the rotation number of the solutions of the planar system.

Our results generalize the results of [3], in the sense that a more general nonlinearity is allowed (even for the case $p = 2$), and a more general differential operator (the one-dimensional p-Laplacian) is examined. Some partial results in this direction were also obtained in [5] and [8], where the authors prove existence under assumptions some fundamental inequalities relating the asymptotic behaviour of the time mapping of the p-Laplacian with that of the function $\frac{pg(x)}{x^p}$ as $x \to +\infty$.

Our main result is a general existence result for T-periodic solutions of the initial value problem (1.1). We first transform the system into an equivalente planar system. Then we prove that the rotation number of the solutions of the equivalent planar system satisfy a certain condition, which allows us to apply the results of [3], giving existence of T-periodic solutions as fixed points of the associated Poincaré return map.

The assumptions on the functions g and h, which we will refer to throughout the paper, are as follows:

(H1) $h, g : \mathbb{R} \to \mathbb{R}$ are continuous functions and h is T-periodic ($T > 0$ fixed).

(H2) $\lim_{|x| \to +\infty} \frac{\operatorname{sign}(x) g(x)}{|x|^p} = +\infty$.

(H3) $h \in L^\infty(\mathbb{R})$.

(H4) $\lim_{|x| \to +\infty} \tau_G(x) = 0$, where $\tau_G(x) = \int_0^x \frac{p}{p-1} \big(G(x) - G(s)\big)^{-1/p} \, ds$ and G denotes the primitive $G(s) = \int_0^s g(\sigma) \, d\sigma$. Note that $\tau_G(x)$ is well defined for large x.

2. The main result

It will be convenient below to write the initial value problem (1.1) as a first order system. Setting

$$|x'|^{p-2} x' = y,$$

equation (1.1) becomes

$$\left.\begin{aligned} x' &= |y|^{p'-2} y, \\ y' &= -\frac{n}{t} |x'|^{p-2} x' - g\big(x(t)\big) - h(t), \end{aligned}\right\} \tag{2.1}$$

where p' denotes the Hölder conjugate of p (i.e., $\frac{1}{p} + \frac{1}{p'} = 1$).

Lemma 2.1. *Any solution* $z(t) = (x(t), y(t))$ *of* (2.1) *is defined for all* $t \in \mathbb{R}$.

Proof. To prove this lemma, we consider the following Lyapunov function

$$V(x, y) = (G(x(t)) - G_{\min}) + \frac{1}{p'}(|y|^{p'} + 1).$$

Here $G_{\min} = \min\{G(x) \mid x \in \mathbb{R}\}$, which by (H2) is finite.

Differentiating $v(t) = V(x(t), y(t)) = V(z(t))$ and using (H3), we obtain that

$$\begin{aligned} v'(t) &= g(x)x' + |y|^{p'-2}yy' \\ &= |y|^{p'-2}y(y' + g(x)) \\ &= |y|^{p'-2}y\left(-\frac{n}{t}y - h(t)\right) \\ &= -\frac{n}{t}|y|^{p'} - h(t)|y|^{p'-2}y \le M|y|^{p'-1}, \end{aligned}$$

with M some positive constant. Using again the expression of $V(x, y)$, we conclude that

$$\begin{aligned} |v'(t)| &\le M\left(\frac{p}{p-1}\right)^{1/p}\left(v(t) - (G(x) - G_{\min}) - 1\right)^{1/p} \\ &\le M\left(\frac{p}{p-1}\right)^{1/p}\left(v(t) + \alpha\right)^{1/p}, \end{aligned} \tag{2.2}$$

where α is a real positive number. Finally, integrating on the interval $[0, T]$, we deduce that

$$v(t) \le c_1 v_0 + c_2|t - t_0|^{p'},$$

where c_1 and c_2 are positive constants depending on M and p.

This last expression shows that $v(t) = V(x(t), y(t))$ is bounded for t in any bounded interval. Also, $V(x, y) > 0$ for each $(x, y) \in \mathbb{R}^2$ and $V(x, y) \to +\infty$ as $x^2 + y^2 \to +\infty$. Therefore, the general theory of Lyapunov functions for first order systems (see, e.g., [7]) shows that it is possible to extend any solution of (1.1) to the whole interval $[0, T]$, so that the global existence of solutions follows. □

We can also easily prove the following.

Lemma 2.2. *For any $R_1 > 0$ there is $R_2 \geq R_1$ such that for every solution $z(\cdot)$ of* (2.1), *we have*

$$|z(0)| \leq R_1 \implies |z(t)| \leq R_2 \quad \textit{for all } t \in [0,T],$$
$$|z(0)| \geq R_2 \implies |z(t)| \geq R_1 \quad \textit{for all } t \in [0,T],$$

where $|z(t)|$ denotes the usual Euclidean norm of $z(t) = \big(x(t), y(t)\big)$.

Proof. Integrating (2.2) we see that

$$v(t) \leq c(v_0 + \alpha) + T^{p'}.$$

Therefore, if $R_1 > 0$ is given, if we set for $w = (x, y) \in \mathbb{R}^2$:

$$\begin{aligned} c_1 &= 1 + \sup\{V(w) \mid |w| \leq R_1\}, \\ c_2 &= c(c_1 + \alpha) + T^{p'}, \\ R_2 &= 1 + \sup\{|w| \mid |V(w)| \leq c_2\}, \end{aligned}$$

the result of Lemma 2.2 follows. □

Let us define now $r(t)$ and $\theta(t)$ by

$$\begin{aligned} x(t) &= r(t)\cos\theta(t), \\ y(t) &= r(t)\sin\theta(t) \end{aligned}$$

for any solution of (1.1). A more or less straightforward computation gives

$$\begin{aligned} \theta'(t) = \frac{d\theta}{dt} &= -(p-1)|\sin\theta|^{p/(p-1)}|\cos\theta|^{(p-2)/(p-1)} \\ &\quad - \left(g(x) + \frac{n}{t}|x'|^{p'-2}x' + h(t)\right)\frac{\cos\theta}{r} \end{aligned}$$

for all $t \in [0,T]$.

Lemma 2.3. *There exists a constant d such that for any solution $z(t) = \big(x(t), y(t)\big)$ of* (2.1) *we have*

$$|z(0)| = r \geq d \implies \frac{d}{dt}\theta(t,z) < 0$$

for all $t \in [0,T]$.

Proof. The proof is standard; see, e.g., [4]. □

Let us define for each $r \geq d$ the following nonnegative integer

$$n_*(r) = \max\left\{ n \in \mathbb{Z}^+ \mid n \leq \inf \frac{|\theta(T,z) - \theta(0,z)|}{2\pi}, z(0) = r \right\},$$

where the infimum is taken over all solutions $z(\cdot)$ of (2.1). Therefore, $n_*(r)$ is the least "rotation number" of the solutions of the planar system around the origin, during the time $[0, T]$.

We can now state our main result.

Theorem 2.4. *Suppose that* (H1)–(H4) *are satisfied. Then equation* (2.1) *has at least one T-periodic solution.*

Proof. The proof is based on the following lemma which gives a fundamental property of the rotation number $n_*(r)$:

Lemma 2.5. *We have*

$$\lim_{r \to +\infty} n_*(r) = +\infty. \tag{2.3}$$

Proof. Let $z(t) = \big(x(t), y(t)\big)$ be any solution of (2.1) such that $|z(0)| = r \geq d > 0$ and fix $0 < \varepsilon < 1$. We will estimate $\frac{|\theta(T) - \theta(0)|}{2\pi}$ from below.

By assumptions (H2) and (H4) there exist $R_0 = R_0(\varepsilon)$ and $M > 0$ such that

$$\tau_G(s) = \left(\frac{p}{p-1}\right)^{1/p} \left| \int_0^s \frac{d\xi}{\big(G(s) - G(\xi)\big)^{1/p}} \right| < \varepsilon \left(\frac{p}{p-1}\right)^{1/p}, \qquad \text{for } |s| \geq R_0 \tag{2.4}$$

and

$$g(s)\,\mathrm{sign}(s) > pM > 0, \qquad \text{for } |s| \geq R_0. \tag{2.5}$$

Fix $R_1 = R_1(\varepsilon)$ such that relation (2.9) below is satisfied.

Consider $R_2(\varepsilon) > R_1 > R_0$ and assume that

$$|z(t)| \geq R_1$$

for all $t \in [0, T]$ if $|z(0)| = r > R_2$ (see Lemma 2.2). Our aim is to find a lower estimate for $\frac{|\theta(T) - \theta(0)|}{2\pi}$ if r is large.

Let

$$A = \{(a,b) \in \mathbb{R}^2 \mid |(a,b)| = \sqrt{a^2 + b^2} \geq R_1\}.$$

One can see that $A = \bigcup_{i=1}^{6} A_i$ where

$$A_1 = \{(a,b) \in A \mid |a| \le R_0, b > 0\},$$
$$A_2 = \{(a,b) \in A \mid a \ge R_0, b \ge 0\},$$
$$A_3 = \{(a,b) \in A \mid a \ge R_0, b \le 0\},$$
$$A_4 = \{(a,b) \in A \mid |a| \le R_0, b < 0\},$$
$$A_5 = \{(a,b) \in A \mid a \le -R_0, b \le 0\},$$
$$A_6 = \{(a,b) \in A \mid a \le -R_0, b \ge 0\}.$$

For $i = 1, \dots, 6$ define $I_i = \{t \in [0,T] \mid z(t) \in A_i\}$. Then

$$I_i = \Big(\bigcup_{i=1}^{n_i^*} J_j^{(i)}\Big) \cup P_i,$$

where $J_j^{(i)}$ are non-degenerate closed disjoints intervals contained in $[0,T]$ and maximal with respect to the property that $z(t) \in A_i$ for all $t \in J_j^{(i)}$. Moreover, for each i, j, k, l the intersection $J_j^{(i)} \cap J_l^{(k)}$ either is empty or consists of exactly one point, and P_i is a finite set which consists of at most two points.

Observe that

$$z(t) \in A = \bigcup_{i=1}^{6} A_i$$

for all $t \in [0,T]$ and that any final transition obeys the following "cycle rule"

$$A_1 \to A_2 \to A_3 \to A_4 \to A_5 \to A_6 \to A_1. \tag{2.6}$$

Thus $|n_i^* - n_k^*| \le 1$ for all $i \ne k$. It follows that

$$\frac{|\theta(T) - \theta(0)|}{2\pi} \ge \min\{n_i^* \mid 1 \le i \le 6\} - 1 \ge n_k^* - 2 \tag{2.7}$$

for all fixed k with $1 \le k \le 6$. On the other hand it is clear that

$$\sum_{i=1}^{6} \Big(\sum_{j=1}^{n_i^*} |J_j^{(i)}| \Big) = T, \tag{2.8}$$

where $|J_j^{(i)}| = \mathrm{meas}(J_j^{(i)})$.

Since $\theta'(t) < 0$ on $[0,T]$, we have $x' > 0$ on A_1, $x' < 0$ on A_4, and $y' < 0$ on $A_2 \cap A_3$, $y' > 0$ on $A_5 \cap A_6$. To guarantee that $z(t)$ completes any rotation we must derive an estimate of $\frac{|\theta(T)-\theta(0)|}{2\pi}$. We claim that

$$\mathrm{meas}(J_j^{(i)}) \le \varepsilon$$

for all $i = 1, \dots, 6$ and all $j = 1, \dots, n_i^*$. Indeed, we denote for concreteness and simplicity by

$$J = [\alpha_1, \alpha_2] \subseteq [0, T]$$

(with $\alpha_1 < \alpha_2$) a fixed generic interval among the $J_j^{(i)}$.

First suppose that

$$z(t) \in A_1$$

for $t \in J$ (the case $z(t) \in A_4$ is treated similarly). This yields that

$$|z(t)| = \sqrt{x^2(t) + y^2(t)} \geq R_1, \qquad |x(t)| \leq R_0$$

for all $t \in J$. Then from the first equation in (2.1) we have

$$|x'|^{p-1} = |x'|^{p-2}x' = y = |y(t)| \geq \sqrt{R_1^2 - R_0^2}.$$

In other words we have

$$x' \geq (R_1^2 - R_0^2)^{1/(2(p-1))} \geq \frac{2R_0}{\varepsilon}. \tag{2.9}$$

Here $R_1 = R_1(\varepsilon)$ is chosen such that it satisfies the latter inequality, that is, $(R_1^2 - R_0^2)^{1/(2(p-1))} \geq \frac{2R_0}{\varepsilon}$.

By integration on J we obtain that

$$2R_0 \geq x(\alpha_2) - x(\alpha_1) = \int_{\alpha_1}^{\alpha_2} x'(s)\, ds \geq \frac{2R_0}{\varepsilon}(\alpha_2 - \alpha_1).$$

Hence we deduce that

$$\operatorname{meas}(J) = \alpha_2 - \alpha_1 \leq \varepsilon.$$

Next we treat the case $z(t) \in A_2$ for $t \in J$. We have from the first equation of (2.1) that

$$y(t) = |x'|^{p-2}x' > 0.$$

Then $x'(t) > 0$. Hence, using the fact that $G(\cdot)$ is increasing on $[R_0, +\infty[$, we obtain that

$$x(t) \leq x(\alpha_2) \qquad \text{and} \qquad G\big(x(t)\big) \leq G\big(x(\alpha_2)\big)$$

for $t \in J$. We now introduce for $t \in J$ the function

$$w(t) = G\big(x(t)\big) - Mx(t) + \frac{1}{p'}|y|^{p'}.$$

Differentiating $w(\cdot)$ with respect to time, we get

$$\begin{aligned} w'(t) &= \big(g\big(x(t)\big) - M\big)x'(t) + |y|^{p'-2}yy' \\ &= -\big(M - g\big(x(t)\big)|y|^{p'-2}y + |y|^{p'-2}yy' \\ &= -|y|^{p'-2}y\big(M - g\big(x(t)\big) - y'\big) \\ &= -|y|^{p'-2}y\Big\{M + \frac{n}{t}y + h\Big\} \\ &\le -|y|^{p'-2}y\{M + h\} \le 0. \end{aligned}$$

Thus $w(\cdot)$ is non-increasing in J. Therefore we have

$$\begin{aligned} G\big(x(t)\big) - Mx(t) + \frac{1}{p'}|y|^{p'} &\ge G\big(x(\alpha_2)\big) - Mx(\alpha_2) + \frac{1}{p'}|y(\alpha_2)|^{p'} \\ &\ge G\big(x(\alpha_2)\big) - Mx(\alpha_2) \end{aligned}$$

for $t \in J = [\alpha_1, \alpha_2]$. By the mean value theorem this implies that

$$\begin{aligned} \frac{1}{p'}|y|^{p'} &\ge G\big(x(\alpha_2)\big) - G\big(x(t)\big) - M\big(x(\alpha_2) - x(t)\big) \\ &\ge \frac{1}{p'}G\big(x(\alpha_2)\big) - G\big(x(t)\big) + \frac{x(\alpha_2) - x(t)}{p}\left(\frac{G\big(x(\alpha_2)\big) - G\big(x(t)\big)}{x(\alpha_2) - x(t)} - pM\right) \\ &\ge \frac{1}{p'}\big\{G\big(x(\alpha_2)\big) - G\big(x(t)\big)\big\} + \frac{x(\alpha_2) - x(t)}{p}\big(g(\xi) - pM\big), \end{aligned}$$

where $x(t) \le \xi \le x(\alpha_2)$. It follows by (2.5) that

$$(x')^p = |y|^{p'} = |x'|^p \ge G\big(x(\alpha_2)\big) - G\big(x(t)\big).$$

Hence

$$x' \ge \big(G\big(x(\alpha_2)\big) - G\big(x(t)\big)\big)^{1/p}$$

for all $t \in [\alpha_1, \alpha_2[$. Thus

$$\frac{x'}{\big(G\big(x(\alpha_2)\big) - G\big(x(t)\big)\big)^{1/p}} \ge 1$$

for all $t \in [\alpha_1, \alpha_2[$. Using (2.4) and the inequality $x(\alpha_2) \geq R_0$ in A_2, we get by change of variable

$$\begin{aligned} J = \operatorname{meas}(J) = \alpha_2 - \alpha_1 &= \int_{\alpha_1}^{\alpha_2} 1\, ds \\ &\leq \int_{\alpha_1}^{\alpha_2} \frac{x'(s)\, ds}{\big(G(x(\alpha_2)) - G(x(s))\big)^{1/p}} \\ &= \int_{x(\alpha_1)}^{x(\alpha_2)} \frac{d\xi}{\big(G(x(\alpha_2)) - G(\xi)\big)^{1/p}} \\ &\leq \int_0^{x(\alpha_2)} \frac{d\xi}{\big(G(x(\alpha_2)) - G(\xi)\big)^{1/p}} = \frac{\tau_G\big(x(\alpha_2)\big)}{(p')^{\frac{1}{p}}} \leq \varepsilon. \end{aligned}$$

Similarly we can obtain the same estimate if $z(t) \in A_3, A_5, A_6$ for $t \in J$. Thus it follows from (2.8) that

$$T < \varepsilon \sum_{i=1}^{6} n_i^*,$$

which implies that

$$T < 6\varepsilon(n^* + 1).$$

Here $n^* = \min\{n_i^*, 1 \leq i \leq 6\}$. Consequently, we conclude from (2.7) that

$$\frac{|\theta(T, z) - \theta(0, z)|}{2\pi} > \frac{T}{6\varepsilon} - 3,$$

where $z(0) = r \geq R_2$. Finally, by the definition of $n_*(r)$, we have

$$n_*(r) \geq \left[\frac{T}{6\varepsilon} - 3\right],$$

where $[s]$ is the integer part of $s \in \mathbb{R}$. When $r \to +\infty$, we have $\varepsilon \to 0^+$, so we conclude that

$$\lim_{r \to +\infty} n_*(r) = +\infty. \qquad \square$$

To complete the proof one uses standard approximation and compactness arguments to obtain the existence of T-periodic solutions as fixed points of the Poincaré return map associated to the planar system (2.1). The argument is entirely similar to the proof of Corollary 1 in [3], the main hypothesis needed being condition (2.3) on the rotation number of the solutions of the planar system (2.1). □

Acknowledgement. The author was supported by FCT (The Portuguese Foundation for Science and Technology) with the fellowship SFRH/BPD/20934/2004.

References

[1] P. Amster and M. C. Mariani, A second order ODE with a nonlinear final condition. *Electron. J. Differential Equations* (2001), No. 75, 1–9. Zbl 0997.34015 MR 1872054

[2] T. R. Ding, R. Iannacci, and F. Zanolin, On periodic solutions of sublinear Duffing equations. *J. Math. Anal. Appl.* **158** (1991), 316–332. Zbl 0727.34030 MR 1117564

[3] T. R. Ding and F. Zanolin, Periodic solutions of Duffing's equations with superquadratic potential. *J. Differential Equations* **97** (1992), 328–378. Zbl 0763.34030 MR 1167538

[4] A. El Hachimi and F. de Thelin, Infinitely many radially symmetric solutions for a quasilinear elliptic problem in a ball. *J. Differential Equations* **128** (1996), 78–102. Zbl 0852.34021 MR 1392397

[5] A. El Hachimi and J.-P. Gossez, A note on a nonresonance condition for a quasilinear elliptic problem. *Nonlinear Anal.* **22** (1994), 229–236. Zbl 0816.35031 MR 1258959

[6] A. Fonda and F. Zanolin, On the use of time-maps for the solvability of nonlinear boundary value problems. *Arch. Math.* (*Basel*) **59** (1992), 245–259. Zbl 0766.34012 MR 1174402

[7] J. K. Hale, *Ordinary differential equations*. 2nd ed., Robert E. Krieger Publishing, Huntington, New York 1980. Zbl 0433.34003 MR 0587488

[8] R. Manásevich and F. Zanolin, Time-mappings and multiplicity of solutions for the one-dimensional p-Laplacian. *Nonlinear Anal.* **21** (1993), 269–291. Zbl 0792.34021 MR 1237587

[9] F. Nakajima, Even and periodic solutions of the equation $u'' + g(u) = e(t)$. *J. Differential Equations* **83** (1990), 277–299. Zbl 0693.34045 MR 1033189

Received July 25, 2006; revised September 25, 2007

M. R. Sidi Ammi, Departamento de Matemática, Universidade de Aveiro, Campus Universitário de Santiago, 3810-193 Aveiro, Portugal
E-mail: sidiammi@ua.pt