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Local solutions for a Timoshenko system in noncylindrical domains

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Abstract. In this paper we study a Timoshenko system modeling transverse vibrations of thin elastic beams in a moving boundary domain. Existence and uniqueness of a local solution is proved by using Faedo–Galerkin approximation.

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1. Introduction

Hyperbolic systems of the type

$$\left.\begin{aligned} \phi'' - M\Big(\int_\alpha^\beta \Big|\frac{\partial \phi}{\partial x}\Big|^2\,dx\Big)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial v}{\partial x} = \varphi, \\ v'' - \frac{\partial^2 v}{\partial x^2} + \frac{\partial \phi}{\partial x} + v = \psi, \end{aligned}\right\} \tag{1.1}$$

are usually called Timoshenko systems (see e.g. [10]) and describe transverse vibrations of a beam when rotatory inertia and shear forces are taken into account. In this model, the beam has a length $L = \beta - \alpha$ and its transverse vibrations at a time t are represented by $\phi = \phi(x,t)$, $\alpha \le x \le \beta$. The function $v = v(x,t)$ represents the slope of the deformation curve ϕ due to the rotatory effect and thus producing a shearing deformation.

The system (1.1) has been studied by several authors. For instance, the existence of local solutions was considered by Tucsnak in [9]. The problem of exact controllability, in the n-dimensional case, with $M(\lambda) = 1$, was studied by Medeiros [5]. A necessary and sufficient condition for exponential stability of (1.1), when $M(\lambda) = 1$, was established by Muñoz Rivera and Racke [7]. Other related results can be found, for instance, in [1], [3], [8].

The objective of this work is to extend the system (1.1) to a moving boundary context. This is done by assuming the ends α, β of the beam time dependent, and therefore defining the system in a noncylindrical domain $\hat{Q}$. More precisely, we have

$$\hat{Q} = \bigcup_{0<t<T}]\alpha(t), \beta(t)[\times \{t\}$$

and the lateral boundary

$$\hat{\Sigma} = \bigcup_{0<t<T} \{\alpha(t), \beta(t)\} \times \{t\}.$$

The noncylindrical domain $\hat{Q}$ means that the beam at rest configuration is modeled by the interval $[\alpha(0), \beta(0)]$ and its ends change in time according to the functions α and β. This situation may occur, for instance, due to the temperature variation.

Our problem is then defined by

$$\left.\begin{aligned} \phi'' - M\left(\int_{\alpha(t)}^{\beta(t)} \left|\frac{\partial \phi}{\partial x}\right|^2 dx\right) \frac{\partial^2 \phi}{\partial x^2} + |\phi|^\rho \phi + \frac{\partial v}{\partial x} = \varphi \\ v'' - \frac{\partial^2 v}{\partial x^2} + \frac{\partial \phi}{\partial x} = \psi \end{aligned} \quad \text{in } \hat{Q} \right\} \tag{1.2}$$

with initial conditions

$$\left.\begin{aligned} \phi(x,0) = \phi_0(x),\ \phi'(x,0) = \phi_1(x) \\ v(x,0) = v_0(x),\ v'(x,0) = v_1(x) \end{aligned} \quad \text{in } \Omega_0 =]\alpha(0), \beta(0)[\right\} \tag{1.3}$$

and boundary conditions

$$\phi = v = 0 \qquad \text{on } \hat{\Sigma}, \tag{1.4}$$

where $'$ denotes the derivative with respect to t.

We have added a nonlinear perturbation of the type $|\phi|^\rho \phi$, $\rho > 0$, in order to deal with a more general model. This kind of nonlinearity is a forcing term for the system and appears often in relativistic quantum mechanics. It has been considered by several authors in hyperbolic, parabolic and elliptic equations.

We study the existence and uniqueness of a local solution for the system (1.2)–(1.4). The method employed here is to transform the system (1.2)–(1.4) into an equivalent one defined in standard cylindrical domains, but with time variable coefficients. This is done, for instance, by following the arguments in Medeiros [6]. Then we use Faedo–Galerkin method and compactness arguments as in Lions [4].

The article is organized as follows: introduction; notations, assumptions and local results; approximation and estimates; proof of the theorems.

2. Notations, assumptions and local results

We obtain existence and uniqueness of solutions for the nonlinear system (1.2)–(1.4) assuming the following hypotheses on the functions M, α and β:

Assumptions.

(M_1) $M \in C^1([0,+\infty[;\mathbb{R})$ and there exists $m_0 \in \mathbb{R}^+$ such that $M(\lambda) \geq m_0$ for all $\lambda \geq 0$.

(H_1) $\alpha, \beta \in C^2([0,+\infty[;\mathbb{R})$ with $\alpha(t) < \beta(t)$, $\alpha'(t) < 0$, $\beta'(t) > 0$, and $\gamma'(t) \leq \left(\frac{m_0}{2}\right)^{1/2}$ for all $t \geq 0$, where $\gamma(t) = \beta(t) - \alpha(t)$;

(H_2) $|\alpha''(t) + y\gamma''(t)| \leq \frac{1}{\gamma(t)}|\alpha'(t) + y\gamma'(t)|^2$ for all $y \in [0,1]$ and $t \geq 0$;

(H_3) there exists $m_1 \in \mathbb{R}^+$ such that $m_1 < 1 - \left(\alpha'(t) + y\gamma'(t)\right)^2$ for all $y \in [0,1]$ and $t \geq 0$.

Remark 2.1. The assumptions $\alpha'(t) < 0$ and $\beta'(t) > 0$ mean that $\hat{Q}$ is strictly increasing in the sense that $\gamma(t) = \beta(t) - \alpha(t)$ is strictly increasing.

Remark 2.2. The assumption (H_1) implies that $|\alpha' + y\gamma'| < \gamma'$ for all $0 \leq y \leq 1$. In fact, from the signs of α', β' and the definition of γ, we have

$$\alpha' + y\gamma' < y\gamma' < \gamma' \qquad \text{if } 0 \leq y \leq 1,$$

and

$$\alpha' + y\gamma' = (\beta' + y\gamma') - \gamma' > -\gamma' \qquad \text{if } y \geq 0.$$

Assumption (H_1) also implies $|\alpha'| < \gamma'$ and $\beta' < \gamma'$. Using assumption (H_2) we conclude that

$$|\alpha''| \leq \frac{|\alpha'|^2}{\gamma} \qquad \text{and} \qquad |\beta''| \leq \frac{|\beta'|^2}{\gamma}.$$

Remark 2.3. To transform the noncylindrical domain $\hat{Q}$ into a cylindrical domain Q we use a suitable change of variables established by the following application:

$$\tau : \hat{Q} \to Q =]0,1[\times]0,T[, \qquad (x,t) \mapsto (y,t) = \left(\frac{x - \alpha(t)}{\gamma(t)}, t\right).$$

The application τ transforms $\Omega_t =]\alpha(t), \beta(t)[$ into $]0,1[$. The inverse $\tau^{-1} : Q \to \hat{Q}$ is given by $\tau^{-1}(y,t) = (x,t)$ with $x = \alpha(t) + \gamma(t)y$. The applications τ and τ^{-1} are C^2 by (H_1).

Using the new variables

$$u(y,t) = (\phi \circ \tau^{-1})(y,t) = \phi(x,t),$$
$$v(y,t) = (\nu \circ \tau^{-1})(y,t) = \nu(x,t),$$

we have the following identities:

$$\begin{aligned}
\phi'' &= u'' - \frac{1}{\gamma^2}\{2\gamma'(\alpha' + y\gamma') - \gamma(\alpha'' + y\gamma'')\}\frac{\partial u}{\partial y} - \frac{2}{\gamma}(\alpha' + y\gamma')\frac{\partial u'}{\partial y} \\
&\quad + \frac{1}{\gamma^2}(\alpha' + y\gamma')^2\frac{\partial^2 u}{\partial y^2}, \\
\nu'' &= v'' - \frac{1}{\gamma^2}\{2\gamma'(\alpha' + y\gamma') - \gamma(\alpha'' + y\gamma'')\}\frac{\partial v}{\partial y} - \frac{2}{\gamma}(\alpha' + y\gamma')\frac{\partial v'}{\partial y} \\
&\quad + \frac{1}{\gamma^2}(\alpha' + y\gamma')^2\frac{\partial^2 v}{\partial y^2}, \\
\frac{\partial^k \phi}{\partial x^k} &= \frac{1}{\gamma^k}\frac{\partial^k u}{\partial y^k}, \ \frac{\partial^k \nu}{\partial x^k} = \frac{1}{\gamma^k}\frac{\partial^k v}{\partial y^k}, \ |\phi|^\rho\phi = |u|^\rho u, \ \int_{\alpha(t)}^{\beta(t)}\left|\frac{\partial \phi}{\partial x}\right|^2 dx = \frac{1}{\gamma}\int_0^1\left|\frac{\partial u}{\partial y}\right|^2 dy.
\end{aligned}$$

Moreover, we define f, g, u_0, u_1, v_0, v_1 by

$$\begin{aligned}
&f(y,t) = (\varphi \circ \tau^{-1})(y,t) = \varphi(x,t), \\
&g(y,t) = (\psi \circ \tau^{-1})(y,t) = \psi(x,t), \\
&u_0(y) = \phi_0(\alpha_0 + \gamma_0 y), \qquad v_0(y) = \nu_0(\alpha_0 + \gamma_0 y), \\
&u_1(y) = \phi_1(\alpha_0 + \gamma_0 y) + (\alpha'(0) + \gamma'(0)y)\phi_{0x}(\alpha_0 + \gamma_0 y), \\
&v_1(y) = \nu_1(\alpha_0 + \gamma_0 y) + (\alpha'(0) + \gamma'(0)y)\nu_{0x}(\alpha_0 + \gamma_0 y),
\end{aligned}$$

where $\gamma_0 = \gamma(0)$, $\alpha_0 = \alpha(0)$ and $\beta_0 = \beta(0)$. Then problem (1.2)–(1.4) is transformed into a cylindrical problem with variable coefficients

$$\left.\begin{aligned}
&u'' - \frac{1}{\gamma^2}\overline{M}\left(\frac{1}{\gamma}\|u(t)\|^2\right)\frac{\partial^2 u}{\partial y^2} - \frac{\partial}{\partial y}\left(a(y,t)\frac{\partial u}{\partial y}\right) \\
&\qquad + b(y,t)\frac{\partial u'}{\partial y} + c(y,t)\frac{\partial u}{\partial y} + |u|^\rho u + \frac{1}{\gamma}\frac{\partial v}{\partial y} = f(y,t) \quad \text{in } Q, \\
&v'' - \frac{\partial}{\partial y}\left(d(y,t)\frac{\partial v}{\partial y}\right) + b(y,t)\frac{\partial v'}{\partial y} + c(y,t)\frac{\partial v}{\partial y} + \frac{1}{\gamma}\frac{\partial u}{\partial y} = g(y,t) \quad \text{in } Q,
\end{aligned}\right\} \tag{2.1}$$

with initial and boundary conditions

$$u(\cdot,0)=u_0,\ u'(\cdot,0)=u_1,\ v(\cdot,0)=v_0,\ v'(\cdot,0)=v_1 \quad \text{in }]0,1[\,, \tag{2.2}$$

$$u=v=0 \quad \text{on } \Sigma=\bigcup_{0<t<T}\{(0,t),(1,t)\}, \tag{2.3}$$

where the variable coefficients and function $\overline{M}$ are given as follows:

$$a(y,t)=\frac{m_0}{2\gamma^2}-\frac{1}{\gamma^2}(\alpha'+y\gamma')^2>0, \qquad b(y,t)=-\frac{2}{\gamma}(\alpha'+y\gamma'),$$

$$c(y,t)=-\frac{1}{\gamma}(\alpha''+y\gamma''), \qquad d(y,t)=\frac{1}{\gamma^2}-\frac{1}{\gamma^2}(\alpha'+y\gamma')^2>0,$$

$$\overline{M}(\lambda)=M(\lambda)-\frac{m_0}{2}\geq\frac{m_0}{2}>0.$$

We have the following results.

Theorem 2.1. *Suppose that M satisfies (M_1) and that α, β satisfy* (H_1)–(H_3). *Given $\phi_0, v_0 \in H_0^1(\Omega_0)\cap H^2(\Omega_0)$, $\phi_1, v_1 \in H_0^1(\Omega_0)$ and $\varphi,\psi \in L^\infty\big([0,T];H_0^1(\Omega_t)\big)$, there exist a positive constant $T_0<T$ and a unique pair of functions $\{\phi,v\}$ with $\phi:\hat{Q}_0\to\mathbb{R}$ and $v:\hat{Q}_0\to\mathbb{R}$, where*

$$\hat{Q}_0=\bigcup_{0<t<T_0}\Omega_t\times\{t\},$$

satisfying

$$\phi,v\in L^\infty\big(0,T_0;H_0^1(\Omega_t)\cap H^2(\Omega_t)\big),$$
$$\phi',v'\in L^\infty\big(0,T_0;H_0^1(\Omega_t)\big),$$
$$\phi'',v''\in L^\infty\big(0,T_0;L^2(\Omega_t)\big),$$

and it is a solution of (1.2)–(1.4) *in $\hat{Q}_0$.*

Theorem 2.2. *Suppose that the hypotheses of Theorem* 2.1 *hold and suppose that $f,g\in L^\infty\big([0,T];H_0^1(0,1)\big)$, $u_0,v_0\in H_0^1(0,1)\cap H^2(0,1)$ and $u_1,v_1\in H_0^1(0,1)$ are given. Then there exist a positive constant $T_0<T$ and a unique pair of functions $\{u,v\}$ with $u,v:Q_0\to\mathbb{R}$, where $Q_0=\,]0,1[\,\times\,]0,T_0[$, that satisfies the conditions*

$$u,v\in L^\infty\big(0,T_0;H_0^1(0,1)\cap H^2(0,1)\big),$$
$$u',v'\in L^\infty\big(0,T_0;H_0^1(0,1)\big),$$
$$u'',v''\in L^\infty\big(0,T_0;L^2(0,1)\big),$$

and is a solution of (2.1)–(2.3) *in Q_0.*

The proof of the theorems are given in the next two sections. We end this section with a few notations. The scalar product and norm in $L^2(0,1)$ are denoted by $(\cdot,\cdot)$ and $|\cdot|$ respectively. In $H_0^1(0,1)$, we use the corresponding notation $((\cdot,\cdot))$ and $\|\cdot\|$. We denote by $a(t,u,w)$ and $d(t,v,w)$ the bilinear forms

$$a(t,u,w) = \int_0^1 a(y,t)\frac{\partial u}{\partial y}\frac{\partial w}{\partial y}dy \quad \text{and} \quad d(t,v,w) = \int_0^1 d(y,t)\frac{\partial v}{\partial y}\frac{\partial w}{\partial y}dy,$$

which are continuous, symmetric and coercive in $H_0^1(0,1)$.

3. Approximations and estimates

Let $\{w_j\}_{j\in\mathbb{N}}$ be the complete orthonormal set of $L^2(0,1)$ given by the eigenfunctions of $-\frac{d^2}{dx^2}w_j = \lambda_j w_j$, $w(0) = w(1) = 0$, and define

$$V_m = \operatorname{span}\{w_1, w_2, \dots, w_m\}$$

the subspace of $H_0^1(0,1)\cap H^2(0,1)$ generated by the first m terms of $\{w_j\}$.

For each $m\in\mathbb{N}$ we construct a pair of functions u_m and v_m given by

$$\begin{cases} u_m(y,t) = \sum_{j=1}^m g_{j_m}(t)w_j(y), \\ v_m(y,t) = \sum_{j=1}^m h_{j_m}(t)w_j(y), \end{cases}$$

with $y\in\,]0,1[$ and $t\in\,]0,T_m[$. The pair $\{g_{j_m}(t), h_{j_m}(t)\}$ is a solution for the following system:

$$\left.\begin{aligned} &\left(u_m'' - \frac{1}{\gamma^2}\overline{M}\left(\frac{1}{\gamma}\|u_m(t)\|^2\right)\frac{\partial^2 u_m}{\partial y^2} - \frac{\partial}{\partial y}\left(a(y,t)\frac{\partial u_m}{\partial y}\right)\right.\\ &\qquad\left. + b(y,t)\frac{\partial u_m'}{\partial y}(t) + c(y,t)\frac{\partial u_m}{\partial y} + |u_m(t)|^\rho u_m(t) + \frac{1}{\gamma}\frac{\partial v_m}{\partial y}, w_j\right)(t) = (f(t), w_j),\\ &\left(v_m'' - \frac{\partial}{\partial y}\left(d(y,t)\frac{\partial v_m}{\partial y}\right) + b(y,t)\frac{\partial v_m'}{\partial y} + c(y,t)\frac{\partial v_m}{\partial y} + \frac{1}{\gamma}\frac{\partial u_m}{\partial y}, w_j\right)(t) = (g(t), w_j) \end{aligned}\right\} \tag{3.1}$$

for all $w_j\in V_m$ and with initial conditions

$$\begin{aligned} &u_m(0) = u_{0m}\to u_0, \quad v_m(0) = v_{0m}\to v_0 \quad \text{in } H_0^1(0,1)\cap H^2(0,1),\\ &u_m'(0) = u_{1m}\to u_1, \quad v_m'(0) = v_{1m}\to v_1 \quad \text{in } H_0^1(0,1), \end{aligned} \tag{3.2}$$

with $m\to\infty$ and $j = 1,2,\dots,m$.

First estimate. Substituting $w_j = u_m'$ and $w_j = v_m'$ into the equations $(3.1)_1$ and $(3.1)_2$, respectively, we obtain the following equations:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{|u_m'(t)|^2 + \frac{1}{\gamma}\hat{M}\left(\frac{1}{\gamma}\|u_m(t)\|^2\right) + a(t,u_m,u_m) + \frac{2}{p}\|u_m(t)\|_p^p\right\} \\
&\quad + \frac{\gamma'}{2\gamma^2}\hat{M}\left(\frac{1}{\gamma}\|u_m(t)\|^2\right) + \frac{\gamma'}{2\gamma^3}\overline{M}\left(\frac{1}{\gamma}\|u_m(t)\|^2\right)\|u_m(t)\|^2 \\
&\quad + \frac{\gamma'}{\gamma}a(t,u_m,u_m) + \frac{\gamma'}{\gamma}|u_m'(t)|^2 \\
&= -\int_0^1 c(y,t)\frac{\partial u_m}{\partial y}u_m'\,dy - \frac{1}{\gamma}\int_0^1 \frac{\partial v_m}{\partial y}u_m'\,dy \\
&\quad + \frac{1}{2}\int_0^1 b(y,t)h(y,t)\left(\frac{\partial u_m}{\partial y}\right)^2 dy + \int_0^1 f(t)u_m'(t)\,dy, \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{|v_m'(t)|^2 + d(t,v_m,v_m)\} + \frac{\gamma'}{\gamma}d(t,v_m,v_m) + \frac{\gamma'}{\gamma}|v_m'(t)|^2 \\
&= -\int_0^1 c(y,t)\frac{\partial v_m}{\partial y}v_m'\,dy - \frac{1}{\gamma}\int_0^1 \frac{\partial u_m}{\partial y}v_m'\,dy \\
&\quad + \frac{1}{2}\int_0^1 b(y,t)h(y,t)\left(\frac{\partial v_m}{\partial y}\right)^2 dy + \int_0^1 g(t)v_m'\,dy, \qquad (3.4)
\end{aligned}
$$

where $p = \rho + 2$ and the functions $\hat{M}(\lambda)$ and h are given by

$$
\hat{M}(\lambda) = \int_0^\lambda \overline{M}(s)\,ds \quad \text{and} \quad h(y,t) = \frac{1}{\gamma}(\alpha'' + y\gamma'') - \frac{\gamma'}{\gamma^2}(\alpha' + y\gamma').
$$

Summing up the equations in (3.3)–(3.4), we estimate the terms on the right-hand side using assumptions (H_1) and (H_2), Remark 2.2 and Young's inequality. Thus we have

$$
\begin{aligned}
\boldsymbol{I}_1 &= -\left(\int_0^1 c(y,t)\frac{\partial u_m}{\partial y}u_m'\,dy + \int_0^1 c(y,t)\frac{\partial v_m}{\partial y}v_m'\,dy\right) \\
&\le \int_0^1 \frac{1}{\gamma}|\alpha'' + y\gamma''|\left|\frac{\partial u_m}{\partial y}\right||u_m'|\,dy + \int_0^1 \frac{1}{\gamma}|\alpha'' + y\gamma''|\left|\frac{\partial v_m}{\partial y}\right||v_m'|\,dy \\
&\le \int_0^1 \frac{1}{\gamma^2}|\alpha' + y\gamma'|^2\left|\frac{\partial u_m}{\partial y}\right||u_m'|\,dy + \int_0^1 \frac{1}{\gamma^2}|\alpha' + y\gamma'|^2\left|\frac{\partial v_m}{\partial y}\right||v_m'|\,dy \\
&\le \frac{m_0}{4\gamma_0^2}\left(|u_m'(t)|^2 + |v_m'(t)|^2 + \|u_m(t)\|^2 + \|v_m(t)\|^2\right).
\end{aligned}
$$

By Young's inequality, we get

$$\begin{aligned} \boldsymbol{I_2} &= -\Big(\frac{1}{\gamma}\int_0^1 \frac{\partial v_m}{\partial y} u_m'\, dy + \frac{1}{\gamma}\int_0^1 \frac{\partial u_m}{\partial y} v_m'\, dy\Big) \\ &\leq \frac{1}{2\gamma_0}\big(|u_m'(t)|^2 + |v_m'(t)|^2 + \|u_m(t)\|^2 + \|v_m(t)\|^2\big). \end{aligned}$$

Using hypothesis (H_1) and (H_2), we obtain

$$\begin{aligned} \boldsymbol{I_3} &= \frac{1}{2}\int_0^1 b(y,t)h(y,t)\left(\frac{\partial u_m}{\partial y}\right)^2 dy \\ &\leq \int_0^1 \left|\frac{1}{\gamma^2}(\alpha' + y\gamma')(\alpha'' + y\gamma'') - \frac{\gamma'}{\gamma^3}(\alpha' + y\gamma')^2\right| \left|\frac{\partial u_m}{\partial y}\right|^2 dy \\ &\leq \int_0^1 \left[\frac{1}{\gamma^2}|\alpha' + y\gamma'|\,|\alpha'' + y\gamma''| + \frac{\gamma'}{\gamma^3}|\alpha' + y\gamma'|^2\right] \left|\frac{\partial u_m}{\partial y}\right|^2 dy \\ &\leq 2\left(\frac{\gamma'}{\gamma}\right)^3 \leq \frac{2}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{3/2} \|u_m(t)\|^2. \end{aligned}$$

Using similar arguments it follows that

$$\boldsymbol{I_4} = \frac{1}{2}\int_0^1 b(y,t)h(y,t)\left(\frac{\partial v_m}{\partial y}\right)^2 dy \leq \frac{2}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{3/2} \|v_m(t)\|^2.$$

By Young's inequality, we have

$$\begin{aligned} \boldsymbol{I_5} &= \int_0^1 f(t)u_m'(t)\, dy + \int_0^1 g(t)v_m'(t)\, dy \\ &\leq \frac{1}{2}\big(|f(t)|^2 + |g(t)|^2 + |u_m'(t)|^2 + |v_m'(t)|^2\big). \end{aligned}$$

By integration from 0 to t, $t \leq T_m$, and using estimates $\boldsymbol{I_1}, \ldots, \boldsymbol{I_5}$ we come to the inequality

$$\begin{aligned} &\frac{1}{2}\Bigg\{|u_m'(t)|^2 + |v_m'(t)|^2 + \frac{1}{\gamma}\hat{M}\left(\frac{1}{\gamma}\|u_m(t)\|^2\right) + a(t,u_m,u_m) \\ &\qquad + d(t,v_m,v_m) + \frac{2}{p}\|u_m(t)\|_p^p\Bigg\} \\ &+ \int_0^t \frac{\gamma'}{2\gamma^2}\hat{M}\left(\frac{1}{\gamma}\|u_m(s)\|^2\right) + \frac{\gamma'}{2\gamma^3}\overline{M}\left(\frac{1}{\gamma}\|u_m(s)\|^2\right)\|u_m(s)\|^2\, ds \end{aligned}$$

$$+\int_0^t \frac{\gamma'}{\gamma}\left(a(s,u_m,u_m)+d(s,v_m,v_m)+|u_m'(s)|^2+|v_m'(s)|^2\right)ds$$
$$\leq \frac{1}{2}\int_0^T |f(s)|^2\,ds+\frac{1}{2}\int_0^T |g(s)|^2\,ds+\frac{1}{2}\Bigg\{|u_{1m}|^2+|v_{1m}|^2$$
$$+\frac{1}{\gamma_0}\hat{M}\left(\frac{1}{\gamma}\|u_{0m}\|^2\right)+a(0,u_{0m},u_{0m})+d(0,v_{0m},v_{0m})\Bigg\}$$
$$+K_0\int_0^t \{|u'(s)|^2+|v'(s)|^2+\|u(s)\|^2+\|v(s)\|^2\}\,ds \tag{3.5}$$

where $K_0=\max\left\{\frac{m_0}{4\gamma_0^2},\frac{1}{2\gamma_0},\frac{2}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{3/2},\frac{1}{2}\right\}$.

Under the hypotheses on the functions f, g and convergences (3.2) we have

$$\frac{1}{2}\Bigg\{\int_0^T |f(s)|^2\,ds+\frac{1}{2}\int_0^T |g(s)|^2\,ds+|u_{1m}|^2+|v_{1m}|^2$$
$$+\frac{1}{\gamma_0}\hat{M}\left(\frac{1}{\gamma}\|u_{0m}\|^2\right)+at(0,u_{0m},u_{0m}t)+d(0,v_{0m},v_{0m})\Bigg\}\leq K_1.$$

Some terms in (3.5) are in fact non-negatives as we can see:

$$\int_0^t \frac{\gamma'}{2\gamma^2}\hat{M}\left(\frac{1}{\gamma}\|u_m(s)\|^2\right)ds+\frac{\gamma'}{2\gamma^3}\overline{M}\left(\frac{1}{\gamma}\|u_m(s)\|^2\right)\|u_m(s)\|^2\,ds\geq 0,$$
$$\int_0^t \frac{\gamma'}{\gamma}\left(a(s,u_m,u_m)+d(s,v_m,v_m)+|u_m'(s)|^2+|v_m'(s)|^2\right)ds\geq 0,$$
$$\hat{M}\left(\frac{1}{\gamma}\|u_m(t)\|^2\right)=\int_0^{(1/\gamma)\|u_m(t)\|^2}\overline{M}(s)\,ds\geq\frac{m_0}{2\gamma_1}\|u_m(t)\|^2,$$
$$d(t,v_m,v_m)=\int_0^1 d(y,t)\left|\frac{\partial v}{\partial y}\right|^2 dy\geq\frac{m_1}{\gamma_1^2}\|v_m(t)\|^2,$$
$$a(t,u_m,u_m)\geq 0,$$

where $\gamma_1=\sup_{0\leq t<T}\gamma(t)$.

Let $K_2=\min\left\{\frac{1}{2},\frac{m_0}{4\gamma_1^2},\frac{m_1}{2\gamma_1^2}\right\}$. In view of (3.5) we get

$$|u_m'(t)|^2+|v_m'(t)|^2+\|u_m(t)\|^2+\|v_m(t)\|^2$$
$$\leq\frac{K_1}{K_2}+\frac{K_0}{K_2}\int_0^t\left(|u_m'(s)|^2+|v_m'(s)|^2+\|u_m(s)\|^2+\|v_m(s)\|^2\right)ds, \tag{3.6}$$

where $\frac{K_0}{K_2}$, $\frac{K_1}{K_2}$ are constants which not depend on m.

By (3.6) and Gronwall's lemma, we obtain that

$$|u_m'(t)|^2 + |v_m'(t)|^2 + \|u_m(t)\|^2 + \|v_m(t)\|^2 \le K_3 \qquad \text{on } [0,T], \tag{3.7}$$

where $K_3 = \frac{K_1}{K_2} e^{(K_0/K_2)T}$.

Second estimate. In the first approximate equation of (3.1) we take $w_j = -\frac{\partial^2 u_m'}{\partial y^2}$. This gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_m'(t)\|^2 + \frac{1}{2\gamma^2}\overline{M}\left(\frac{1}{\gamma}\|u_m(t)\|^2\right)\frac{d}{dt}\left|\frac{\partial^2 u_m}{\partial y^2}\right|^2 + a\left(t, u_m, -\frac{\partial^2 u_m'}{\partial y^2}\right)\\
&\quad + \left(b(y,t)\frac{\partial u_m'}{\partial y}, -\frac{\partial^2 u_m'}{\partial y^2}\right) + \left(c(y,t)\frac{\partial u_m}{\partial y}, -\frac{\partial^2 u_m'}{\partial y^2}\right) + \left(|u_m|^\rho u_m, -\frac{\partial^2 u_m'}{\partial y^2}\right)\\
&\quad + \left(\frac{1}{\gamma}\frac{\partial v_m}{\partial y}, -\frac{\partial^2 u_m'}{\partial y^2}\right) = \left(f, -\frac{\partial^2 u_m'}{\partial y^2}\right).
\end{aligned}\tag{3.8}$$

In the identity (3.8), we have

$$\begin{aligned}
\frac{1}{2\gamma^2}\overline{M}\left(\frac{1}{\gamma}\|u_m(t)\|^2\right)\frac{d}{dt}\left|\frac{\partial^2 u_m}{\partial y^2}\right|^2 &= \frac{d}{dt}\left[\frac{1}{2\gamma^2}\overline{M}\left(\frac{1}{\gamma}\|u_m(t)\|^2\right)\left|\frac{\partial^2 u_m}{\partial y^2}\right|^2\right]\\
&\quad - \left[\frac{1}{2\gamma^2}\overline{M}\left(\frac{1}{\gamma}\|u_m(t)\|^2\right)\right]'\left|\frac{\partial^2 u_m}{\partial y}\right|^2
\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}
&a\left(t, u_m, -\frac{\partial^2 u_m'}{\partial y^2}\right)\\
&\quad = \frac{1}{2}\frac{d}{dt}\int_0^1 a(y,t)\left(\frac{\partial^2 u_m}{\partial y^2}\right)^2 dy - \frac{1}{2}\int_0^1 a'(y,t)\left(\frac{\partial^2 u_m}{\partial y^2}\right)^2 dy\\
&\qquad - \int_0^1 \frac{\partial}{\partial y}\left[\frac{\partial a}{\partial y}\frac{\partial u_m}{\partial y}\right]\frac{\partial u_m'}{\partial y}\,dy + \frac{\partial a}{\partial y}\frac{\partial u_m}{\partial y}\frac{\partial u_m'}{\partial y}\bigg|_0^1.
\end{aligned}\tag{3.10}$$

Integrating by parts and using that $b(y,t) = -\frac{2}{\gamma}(\alpha' + y\gamma')$, we get

$$\begin{aligned}
\left(b(y,t)\frac{\partial u_m'}{\partial y}, -\frac{\partial^2 u_m'}{\partial y^2}\right) &= \frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}\left(\frac{\partial u_m'}{\partial y}\right)^2 dy - \frac{1}{2}b(y,t)\left(\frac{\partial u_m'}{\partial y}\right)^2\bigg|_0^1\\
&= \frac{\beta'}{\gamma}\left(\frac{\partial u_m'}{\partial y}(1,t)\right)^2 - \frac{\alpha'}{\gamma}\left(\frac{\partial u_m'}{\partial y}(0,t)\right)^2,
\end{aligned}\tag{3.11}$$

which is positive by the hypothesis $\alpha' < 0$ and $\beta' > 0$ on the increasing boundary.

We rewrite the other terms of equation (3.8) as

$$\left(c(y,t)\frac{\partial u_m}{\partial y}, -\frac{\partial^2 u_m'}{\partial y^2}\right) = \int_0^1 \frac{\partial}{\partial y}\left[c(y,t)\frac{\partial u_m}{\partial y}\right]\frac{\partial u_m'}{\partial y}\,dy - c(y,t)\frac{\partial u_m}{\partial y}\frac{\partial u_m'}{\partial y}\Big|_0^1, \quad (3.12)$$

$$\left(|u_m|^\rho u_m, -\frac{\partial^2 u_m'}{\partial y^2}\right) = \int_0^1 \frac{\partial}{\partial y}[|u_m|^\rho u_m]\frac{\partial u_m'}{\partial y}\,dy, \quad (3.13)$$

$$\left(\frac{1}{\gamma}\frac{\partial v_m}{\partial y}, -\frac{\partial^2 u_m'}{\partial y^2}\right) = -\frac{1}{\gamma}\frac{\partial v_m}{\partial y}\frac{\partial u_m'}{\partial y}\Big|_0^1 + \frac{1}{\gamma}\int_0^1 \frac{\partial^2 v_m}{\partial y^2}\frac{\partial u_m'}{\partial y}\,dy, \quad (3.14)$$

$$\left(f, -\frac{\partial^2 u_m'}{\partial y^2}\right) = \int_0^1 \frac{\partial f}{\partial y}\frac{\partial u_m'}{\partial y}\,dy. \quad (3.15)$$

Substituting (3.9)–(3.15) into (3.8), we obtain the following identity:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_m'(t)\|^2 + \frac{1}{2}\frac{d}{dt}\left[\frac{1}{\gamma^2}\overline{M}\left(\frac{1}{\gamma}\|u_m(t)\|^2\right)\left|\frac{\partial^2 u_m}{\partial y^2}\right|^2\right] \\
&\quad + \frac{1}{2}\frac{d}{dt}\int_0^1 a(y,t)\left|\frac{\partial^2 u_m}{\partial y^2}\right|^2 dy + \frac{\beta'}{\gamma}\left(\frac{\partial u_m'}{\partial y}(1,t)\right)^2 - \frac{\alpha'}{\gamma}\left(\frac{\partial u_m'}{\partial y}(0,t)\right)^2 \\
&= \left[c(y,t) - \frac{\partial a}{\partial y}\right]\frac{\partial u_m}{\partial y}\frac{\partial u_m'}{\partial y}\Big|_0^1 + \frac{1}{\gamma}\frac{\partial v_m}{\partial y}\frac{\partial u_m'}{\partial y}\Big|_0^1 \\
&\quad + \left[\frac{1}{2\gamma^2}\overline{M}\left(\frac{1}{\gamma}\|u_m(t)\|^2\right)\right]'\left|\frac{\partial^2 u_m}{\partial y}\right|^2 \\
&\quad - \frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}\left(\frac{\partial u_m'}{\partial y}\right)^2 dy + \frac{1}{2}\int_0^1 a'(y,t)\left|\frac{\partial^2 u_m}{\partial y^2}\right|^2 dy \\
&\quad - \int_0^1 \frac{\partial}{\partial y}\left[\left(c(y,t) - \frac{\partial a}{\partial y}\right)\frac{\partial u_m}{\partial y}\right]\frac{\partial u_m'}{\partial y}\,dy - \frac{1}{\gamma}\int_0^1 \frac{\partial^2 v_m}{\partial y^2}\frac{\partial u_m'}{\partial y}\,dy \\
&\quad - \int_0^1 \frac{\partial}{\partial y}(|u_m|^\rho u_m)\frac{\partial u_m'}{\partial y}\,dy + \int_0^1 \frac{\partial f}{\partial y}\frac{\partial u_m'}{\partial y}\,dy. \quad (3.16)
\end{aligned}$$

Remark 3.1. Using the identity

$$\frac{\partial u_m(0)}{\partial y} = \int_0^1 \frac{\partial}{\partial y}\left((1-y)\frac{\partial u_m(y)}{\partial y}\right) dy = \int_0^1 (1-y)\frac{\partial^2 u_m(y)}{\partial y^2}\,dy - \int_0^1 \frac{\partial u_m(y)}{\partial y}\,dy$$

we have

$$\left|\frac{\partial u_m(0)}{\partial y}\right|_{\mathbb{R}} \le \left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|_{L^2(0,1)} + \left|\frac{\partial u_m(t)}{\partial y}\right|_{L^2(0,1)}.$$

By the Poincaré inequality, since $\lambda_1 = \pi^2$, we obtain that

$$\left|\frac{\partial u_m(0)}{\partial y}\right|_{\mathbb{R}} \le \left(\frac{1+\pi}{\pi}\right)\left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|_{L^2(0,1)}.$$

In the same way we use

$$\frac{\partial u_m(1)}{\partial y} = \int_0^1 \frac{\partial}{\partial y}\left(y\frac{\partial u_m(y)}{\partial y}\right)dy = \int_0^1 y\frac{\partial^2 u_m(y)}{\partial y^2}dy + \int_0^1 \frac{\partial u_m(y)}{\partial y}dy$$

to obtain

$$\left|\frac{\partial u_m(1)}{\partial y}\right|_{\mathbb{R}} \le \left(\frac{1+\pi}{\pi}\right)\left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|_{L^2(0,1)}.$$

Note that the first term on the right-hand side of the equation (3.16) can be written as

$$\begin{aligned}
&\left[c(y,t) - \frac{\partial a}{\partial y}\right]\frac{\partial u_m}{\partial y}\frac{\partial u_m'}{\partial y}\bigg|_0^1 \\
&\quad = \left[-\frac{1}{\gamma}(\alpha'' + y\gamma'') + \frac{2\gamma'}{\gamma^2}(\alpha' + y\gamma')\right]\frac{\partial u_m}{\partial y}\frac{\partial u_m'}{\partial y}\bigg|_0^1 \\
&\quad = \left[-\frac{\beta''}{\gamma} + \frac{2\gamma'\beta'}{\gamma^2}\right]\frac{\partial u_m}{\partial y}(1,t)\frac{\partial u_m'}{\partial y}(1,t) \\
&\qquad + \left[\frac{\alpha''}{\gamma} - \frac{2\gamma'\alpha'}{\gamma^2}\right]\frac{\partial u_m}{\partial y}(0,t)\frac{\partial u_m'}{\partial y}(0,t).
\end{aligned} \tag{3.17}$$

Using assumptions (H_1) and (H_2), we have

$$\begin{aligned}
\left|-\frac{\beta''}{\gamma} + \frac{2\gamma'\beta'}{\gamma^2}\right| &\le \frac{|\beta''|}{\gamma} + \frac{2\gamma'|\beta'|}{\gamma^2} \le \left[\frac{|\beta'|}{\gamma}\right]^{1/2}\left[\frac{|\beta'|^{3/2}}{\gamma^{3/2}} + \frac{2\gamma'|\beta'|^{1/2}}{\gamma^{3/2}}\right] \\
&\le \left[\frac{|\beta'|}{\gamma}\right]^{1/2}\left[\frac{3}{\gamma_0^{3/2}}\left(\frac{m_0}{2}\right)^{3/4}\right].
\end{aligned}$$

Thus by Young's inequality we get

$$\begin{aligned}
&\left|\left[-\frac{\beta''}{\gamma}+\frac{2\gamma'\beta'}{\gamma^2}\right]\frac{\partial u_m}{\partial y}(1,t)\frac{\partial u_m'}{\partial y}(1,t)\right| \\
&\quad\le \left[\frac{|\beta'|}{\gamma}\right]^{1/2}\left[\frac{3}{\gamma_0^{3/2}}\left(\frac{m_0}{2}\right)^{3/4}\right]\left|\frac{\partial u_m}{\partial y}(1,t)\right|\left|\frac{\partial u_m'}{\partial y}(1,t)\right| \\
&\quad=\left\{\left[\frac{3(2^{1/2})}{\gamma_0^{3/2}}\left(\frac{m_0}{2}\right)^{3/4}\right]\left|\frac{\partial u_m}{\partial y}(1,t)\right|\right\}\left\{\left[\frac{|\beta'|}{2\gamma}\right]^{1/2}\left|\frac{\partial u_m'}{\partial y}(1,t)\right|\right\} \\
&\quad\le \frac{9}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{3/2}\left|\frac{\partial u_m}{\partial y}(1,t)\right|^2+\frac{1}{4}\frac{|\beta'|}{\gamma}\left|\frac{\partial u_m'}{\partial y}(1,t)\right|^2. \qquad (3.18)
\end{aligned}$$

The term $\frac{9}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{3/2}\left|\frac{\partial u_m}{\partial y}(1,t)\right|^2$ can be estimated as in Remark 3.1:

$$\frac{9}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{3/2}\left|\frac{\partial u_m}{\partial y}(1,t)\right|^2 \le \frac{9}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{3/2}\left(\frac{\pi}{\pi+1}\right)^2\left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|^2_{L^2(0,1)}.$$

Using the same arguments on $\left[\frac{\alpha''}{\gamma}-\frac{2\gamma'\alpha'}{\gamma^2}\right]\frac{\partial u_m}{\partial y}(0,t)\frac{\partial u_m'}{\partial y}(0,t)$ we obtain that

$$\begin{aligned}
&\left|\left[\frac{\alpha''}{\gamma}-\frac{2\gamma'\alpha'}{\gamma^2}\right]\frac{\partial u_m}{\partial y}(0,t)\frac{\partial u_m'}{\partial y}(0,t)\right| \\
&\qquad\le \frac{9}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{3/2}\left(\frac{\pi}{\pi+1}\right)^2\left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|^2+\frac{1}{4}\frac{|\alpha'|}{\gamma}\left|\frac{\partial u_m'}{\partial y}(0,t)\right|^2. \qquad (3.19)
\end{aligned}$$

Therefore, substituting (3.18), (3.19) into (3.17), we can estimate the first term on the right-hand side of the equation (3.16) as follows:

$$\begin{aligned}
\boldsymbol{J_1} &= \left[c(y,t)-\frac{\partial a}{\partial y}\right]\frac{\partial u_m}{\partial y}\frac{\partial u_m'}{\partial y}\Bigg|_0^1 \\
&\le \frac{1}{4}\left[\frac{\beta'}{\gamma}\left|\frac{\partial u_m'}{\partial y}(1,t)\right|^2-\frac{\alpha'}{\gamma}\left|\frac{\partial u_m'}{\partial y}(0,t)\right|^2\right]+\frac{18}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{3/2}\left(\frac{\pi}{\pi+1}\right)^2\left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|^2.
\end{aligned}$$

The term $\frac{1}{4}\left[\frac{\beta'}{\gamma}\left|\frac{\partial u_m'}{\partial y}(1,t)\right|^2-\frac{\alpha'}{\gamma}\left|\frac{\partial u_m}{\partial y}(0,1)\right|^2\right]$ on the left-hand side of equation (3.16) is compensated with $\frac{\beta'}{\gamma}\left(\frac{\partial u_m'}{\partial y}(1,t)\right)^2-\frac{\alpha'}{\gamma}\left(\frac{\partial u_m}{\partial y}(0,t)\right)^2$, which gives a positive contribution to the left side of equation (3.16).

Remark 3.2. Using that $\alpha'(t)<0$ and $\beta'(t)>0$ for all $t\ge 0$, it follows that there are constants ε and δ with $0<\varepsilon<\beta'(t)$ and $0<\delta<-\alpha'(t)$ for all $t\in[0,T]$.

Then second term on the right-hand side of the equation (3.16) can be written as follows:

$$\frac{1}{\gamma}\frac{\partial v_m}{\partial y}\frac{\partial u_m'}{\partial y}\bigg|_0^1 = \frac{1}{\gamma}\left(\frac{\partial v_m}{\partial y}\frac{\partial u_m'}{\partial y}(1,t) - \frac{\partial v_m}{\partial y}\frac{\partial u_m'}{\partial y}(0,t)\right).$$

By Young's inequality, we have

$$\begin{aligned}\left|\frac{1}{\gamma}\frac{\partial v_m}{\partial y}\frac{\partial u_m'}{\partial y}(1,t)\right| &\le \frac{1}{\gamma}\left|\frac{\partial v_m}{\partial y}(1,t)\right|\left|\frac{\partial u_m'}{\partial y}(1,t)\right| \\ &= \left\{\left(\frac{2}{\varepsilon\gamma}\right)^{1/2}\left|\frac{\partial v_m}{\partial y}(1,t)\right|\right\}\left\{\left(\frac{\varepsilon}{2\gamma}\right)^{1/2}\left|\frac{\partial u_m'}{\partial y}(1,t)\right|\right\} \\ &\le \frac{1}{\varepsilon\gamma}\left|\frac{\partial v_m}{\partial y}(1,t)\right|^2 + \frac{1}{4}\frac{\varepsilon}{\gamma}\left|\frac{\partial u_m'}{\partial y}(1,t)\right|^2. \end{aligned} \tag{3.20}$$

Using Remarks 3.1 and 3.2 we obtain from 3.20 that

$$\left|\frac{1}{\gamma}\frac{\partial v_m}{\partial y}\frac{\partial u_m'}{\partial y}(1,t)\right| \le \frac{1}{\varepsilon\gamma_0}\left(\frac{\pi}{\pi+1}\right)^2\left|\frac{\partial^2 v_m(t)}{\partial y^2}\right|^2 + \frac{1}{4}\frac{\beta'}{\gamma}\left|\frac{\partial u_m'}{\partial y}(1,t)\right|^2. \tag{3.21}$$

Proceeding analogously, we have

$$\left|\frac{1}{\gamma}\frac{\partial v_m}{\partial y}\frac{\partial u_m'}{\partial y}(0,t)\right| \le \frac{1}{\delta\gamma_0}\left(\frac{\pi}{\pi+1}\right)^2\left|\frac{\partial^2 v_m(t)}{\partial y^2}\right|^2 + \frac{1}{4}\frac{(-\alpha')}{\gamma}\left|\frac{\partial u_m'}{\partial y}(0,t)\right|^2, \tag{3.22}$$

where $\delta > 0$ is given by Remark 3.2.

From inequalities (3.21) and (3.22) we have the following estimate for the second term on the right-hand side of (3.16):

$$\begin{aligned}\boldsymbol{J_2} &= \frac{1}{\gamma}\frac{\partial v_m}{\partial y}\frac{\partial u_m'}{\partial y}\bigg|_0^1 \\ &\le \frac{1}{4}\left[\frac{\beta'}{\gamma}\left|\frac{\partial u_m'}{\partial y}(1,t)\right|^2 - \frac{\alpha'}{\gamma}\left|\frac{\partial u_m'}{\partial y}(0,t)\right|^2\right] + \left[\frac{1}{\varepsilon}+\frac{1}{\delta}\right]\frac{1}{\gamma_0}\left(\frac{\pi}{\pi+1}\right)^2\left|\frac{\partial^2 v_m(t)}{\partial y^2}\right|^2.\end{aligned}$$

From the first estimate, hypothesis H_3 and the equality

$$\begin{aligned}\left[\frac{1}{2\gamma^2}\overline{M}\left(\frac{1}{\gamma}\|u_m(t)\|^2\right)\right]' &= \frac{1}{\gamma^3}\overline{M}'\left(\frac{1}{\gamma}\|u_m(t)\|^2\right)((u_m', u_m)) \\ &\quad - \frac{\gamma'}{2\gamma^4}\overline{M}'\left(\frac{1}{\gamma}\|u_m(t)\|^2\right)\|u_m(t)\|^2 - \frac{\gamma'}{\gamma^3}\overline{M}\left(\frac{1}{\gamma}\|u_m(t)\|^2\right),\end{aligned}$$

the third term on the right-hand side of equation (3.16) can be estimated as follows:

$$\begin{aligned}\boldsymbol{J_3} &= \frac{1}{2}\int_0^1 \left[\frac{1}{\gamma^2}\overline{M}\left(\frac{1}{\gamma}\|u_m(t)\|^2\right)\right]'\left|\frac{\partial^2 u_m}{\partial y^2}\right|^2 dy \\ &\leq \left[\frac{1}{2\gamma_0^4}\left(\frac{m_0}{2}\right)^{1/2} M_1 K_3 + \frac{1}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{1/2} M_0\right]\left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|^2 \\ &\quad + \frac{1}{\gamma_0^3} M_1 K_3^{1/2}\|u_m'(t)\|\left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|^2,\end{aligned}$$

where $M_0 = \max_{0\leq\lambda\leq K_3/\gamma_0}\{|\overline{M}(\lambda)|\}$ and $M_1 = \max_{0\leq\lambda\leq K_3/\gamma_0}\{|M'(\lambda)|\}$.

From assumption H_1 we obtain as estimate for the fourth term on the right-hand side of the equation (3.16)

$$\boldsymbol{J_4} = \frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}\left|\frac{\partial u_m'}{\partial y}\right|^2 dy = -\frac{\gamma'}{\gamma}\int_0^1\left|\frac{\partial u_m'}{\partial y}\right|^2 dy \leq \left(\frac{m_0}{2}\right)^{1/2}\frac{1}{\gamma_0}\|u_m'(t)\|^2.$$

Using hypotheses (H_1) and (H_2), the first estimate and Young's inequality, we obtain for the fifth term of (3.16)

$$\begin{aligned}\boldsymbol{J_5} &= -\int_0^1 \frac{\partial}{\partial y}\left[\left(c(y,t) - \frac{\partial a}{\partial y}\right)\frac{\partial u_m}{\partial y}\right]\frac{\partial u_m'}{\partial y}dy \\ &= \int_0^1\left[-2\left(\frac{\gamma'}{\gamma}\right)^2 + \frac{\gamma''}{\gamma}\right]\frac{\partial u_m}{\partial y}\frac{\partial u_m'}{\partial y}dy \\ &\quad + \int_0^1\left[-\frac{2\gamma'}{\gamma^2}(\alpha' + y\gamma') + \frac{1}{\gamma}(\alpha'' + y\gamma'')\right]\frac{\partial^2 u_m}{\partial y^2}\frac{\partial u_m'}{\partial y}dy \\ &\leq 3\int_0^1\left(\frac{\gamma'}{\gamma}\right)^2\left|\frac{\partial u_m}{\partial y}\right|\left|\frac{\partial u_m'}{\partial y}\right|dy + \int_0^1\left[2\left(\frac{\gamma'}{\gamma}\right)^2 + \left(\frac{\gamma'}{\gamma}\right)^2\right]\left|\frac{\partial^2 u_m}{\partial y^2}\right|\left|\frac{\partial u_m'}{\partial y}\right|dy \\ &\leq \frac{3m_0 K_3^{1/2}}{2\gamma_0^2}\|u_m'(t)\| + \frac{3m_0}{2\gamma_0^2}\left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|\|u_m'(t)\| \\ &\leq \frac{9m_0^2 K_3}{8\gamma_0^4} + \|u_m'(t)\|^2 + \frac{9m_0^2}{8\gamma_0^4}\left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|^2.\end{aligned}$$

An estimate for the sixth term on the right-hand side of the equation (3.16) follows from hypotheses (H_1) and (H_2):

$$\boldsymbol{J_6} = \frac{1}{2}\int_0^1 a'(y,t)\left|\frac{\partial^2 u_m}{\partial y^2}\right|^2 dy$$

$$
\begin{aligned}
&= -\frac{m_0\gamma'}{2\gamma^3}\int_0^1\left|\frac{\partial^2 u_m}{\partial y^2}\right|^2 dy \\
&\quad + \int_0^1 \frac{1}{\gamma}(\alpha' + y\gamma')\left(\frac{2\gamma'}{\gamma^2}(\alpha' + y\gamma') - \frac{1}{\gamma}(\alpha'' + y\gamma'')\right)\left|\frac{\partial^2 u_m}{\partial y^2}\right|^2 dy \\
&\leq \frac{1}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{3/2}\left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|^2 + \int_0^1\left[2\left(\frac{\gamma'}{\gamma}\right)^3 + \left(\frac{\gamma'}{\gamma}\right)^3\right]\left|\frac{\partial^2 u_m}{\partial y^2}\right|^2 dy \\
&\leq \frac{4}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{3/2}\left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|^2.
\end{aligned}
$$

Using Hölder's inequality and the continuous embedding $H^1(0,1) \hookrightarrow L^p(0,1)$, $1 \leq p < \infty$, we have

$$
\begin{aligned}
\boldsymbol{J_7} = \int_0^1 \frac{\partial}{\partial y}(|u_m|^\rho u_m)\frac{\partial u_m'}{\partial y}\,dy &= (\rho+1)\int_0^1 |u_m|^\rho \frac{\partial u_m}{\partial y}\frac{\partial u_m'}{\partial y}\,dy \\
&\leq (\rho+1)|u_m(t)|_{\rho q}^\rho \left|\frac{\partial u_m(t)}{\partial y}\right|_r \|u_m'(t)\| \\
&\leq (\rho+1)\|u_m(t)\|^\rho \left|\frac{\partial u_m(t)}{\partial y}\right|_r \|u_m'(t)\| \\
&\leq (\rho+1)K_3^{\rho/2}\left|\frac{\partial u_m(t)}{\partial y}\right|_r \|u_m'(t)\|.
\end{aligned}
$$

Sobolev's inequality gives

$$
\left|\frac{\partial u_m(t)}{\partial y}\right|_r \leq C_1\|u_m(t)\|_{H^2}
$$

and the regularity theory for elliptic equations ensures that

$$
\|u_m(t)\|_{H^2} \leq C_2\left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|_{L^2}.
$$

Therefore, from Young's inequality we obtain the following estimate for the seventh term on the right-hand side of (3.16):

$$
\begin{aligned}
\boldsymbol{J_7} &\leq (\rho+1)K_3^{\rho/2}C_1C_2\left|\frac{\partial^2 u_m(t)}{\partial y^2}\right| \|u_m'(t)\| \\
&\leq \frac{1}{2}\left((\rho+1)K_3^{\rho/2}C_1C_2\right)\left[\|u_m'(t)\|^2 + \left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|^2\right].
\end{aligned}
$$

For the remaining terms on the right-hand side of (3.16) we obtain

$$J_8 = \frac{1}{\gamma}\int_0^1 \frac{\partial^2 v_m}{\partial y^2}\frac{\partial u_m'}{\partial y}dy \leq \frac{1}{2\gamma_0}\left|\frac{\partial^2 v_m(t)}{\partial y^2}\right|^2 + \frac{1}{2\gamma_0}\|u_m'(t)\|^2,$$

$$J_9 = \int_0^1 \frac{\partial f}{\partial y}\frac{\partial u_m'}{\partial y}dy \leq \frac{1}{2}\|f(t)\|^2 + \frac{1}{2}\|u_m'(t)\|^2.$$

Now, taking $w_j = -\frac{\partial^2 v_m'}{\partial y^2}$ in second equation of (3.1) and using the same arguments as in (3.8), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_m'(t)\|^2 + \frac{1}{2}\frac{d}{dt}\left[\int_0^1 d(y,t)\left(\frac{\partial^2 v_m}{\partial y^2}\right)^2\right] - \frac{1}{2}b(y,t)\left|\frac{\partial v_m'}{\partial y}\right|^2\Bigg|_0^1 \\
&\quad = \frac{1}{2}\int_0^1 d'(y,t)\left(\frac{\partial^2 v_m}{\partial y^2}\right)^2 dy - \frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}\left(\frac{\partial v_m'}{\partial y}\right)^2 dy + \left[c(y,t) - \frac{\partial d}{\partial y}\right]\frac{\partial v_m}{\partial y}\frac{\partial v_m'}{\partial y}\Bigg|_0^1 \\
&\quad - \int_0^1 \frac{\partial}{\partial y}\left[\left(c(y,t) - \frac{\partial d}{\partial y}\right)\frac{\partial v_m}{\partial y}\right]\frac{\partial v_m'}{\partial y}dy - \frac{1}{\gamma}\frac{\partial u_m}{\partial y}\frac{\partial v_m'}{\partial y}\Bigg|_0^1 \\
&\quad - \frac{1}{\gamma}\int_0^1 \frac{\partial^2 u_m}{\partial y^2}\frac{\partial v_m'}{\partial y}dy + \int_0^1 \frac{\partial g}{\partial y}\frac{\partial v_m'}{\partial y}dy. \qquad (3.23)
\end{aligned}$$

To estimate (3.23) we proceed analogously:

$$J_{10} = \frac{1}{2}\int_0^1 d'(y,t)\left|\frac{\partial^2 v_m}{\partial y^2}\right|^2 dy \leq \left[\frac{1}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{1/2} + \frac{3}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{3/2}\right]\left|\frac{\partial^2 v_m(t)}{\partial y^2}\right|^2,$$

$$J_{11} = \frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}\left|\frac{\partial v_m'}{\partial y}\right|^2 dy \leq \left(\frac{m_0}{2}\right)^{1/2}\frac{1}{\gamma_0}\|v_m'(t)\|^2,$$

$$\begin{aligned}
J_{12} &= \left[c(y,t) - \frac{\partial d}{\partial y}\right]\frac{\partial v_m}{\partial y}\frac{\partial v_m'}{\partial y}\Bigg|_0^1 \leq \frac{1}{4}\left[\frac{\beta'}{\gamma}\left|\frac{\partial v_m'}{\partial y}(1,t)\right|^2 - \frac{\alpha'}{\gamma}\left|\frac{\partial v_m'}{\partial y}(0,t)\right|^2\right] \\
&\quad + \frac{18}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{3/2}\left(\frac{\pi}{\pi+1}\right)^2\left|\frac{\partial^2 v_m(t)}{\partial y^2}\right|^2,
\end{aligned}$$

$$\begin{aligned}
J_{13} &= -\int_0^1 \frac{\partial}{\partial y}\left[\left(c(y,t) - \frac{\partial d}{\partial y}\right)\frac{\partial v_m}{\partial y}\right]\frac{\partial v_m'}{\partial y}dy \leq \frac{9m_0^2 K_3}{8\gamma_0^4} + \|v_m'(t)\|^2 \\
&\quad + \frac{9m_0^2}{8\gamma_0^4}\left|\frac{\partial^2 v_m(t)}{\partial y^2}\right|^2,
\end{aligned}$$

$$\boldsymbol{J_{14}} = \frac{1}{\gamma}\frac{\partial u_m}{\partial y}\frac{\partial v_m'}{\partial y}\bigg|_0^1 \leq \frac{1}{4}\left[\frac{\beta'}{\gamma}\left|\frac{\partial v_m'}{\partial y}(1,t)\right|^2 - \frac{\alpha'}{\gamma}\left|\frac{\partial v_m'}{\partial y}(0,t)\right|^2\right]$$

$$+ \left[\frac{1}{\varepsilon}+\frac{1}{\delta}\right]\frac{1}{\gamma_0}\left(\frac{\pi}{\pi+1}\right)^2\left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|^2,$$

$$\boldsymbol{J_{15}} = \frac{1}{\gamma}\int_0^1 \frac{\partial^2 u_m}{\partial y^2}\frac{\partial v_m'}{\partial y}dy \leq \frac{1}{2\gamma_0}\left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|^2 + \frac{1}{2\gamma_0}\|v_m'(t)\|^2,$$

$$\boldsymbol{J_{16}} = \int_0^1 \frac{\partial g}{\partial y}\frac{\partial v_m'}{\partial y}dy \leq \frac{1}{2}\|g(t)\|^2 + \frac{1}{2}\|v_m'(t)\|^2.$$

Let us define the functional

$$F\big(u_m(t), v_m(t)\big) = \frac{1}{2}\Bigg\{\|u_m'(t)\|^2 + \|v_m'(t)\|^2 + \frac{1}{\gamma^2}\overline{M}\left(\frac{1}{\gamma}\|u_m(t)\|^2\right)\left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|^2$$

$$+ \int_0^1 a(y,t)\left(\frac{\partial^2 u_m}{\partial y^2}\right)^2 dy + \int_0^1 d(y,t)\left(\frac{\partial^2 v_m}{\partial y^2}\right)^2 dy\Bigg\}.$$

Summing up the equations (3.16) and (3.23) and using estimates $\boldsymbol{J_1}, \ldots, \boldsymbol{J_{16}}$, we obtain

$$\frac{1}{2}\frac{d}{dt}F\big(u_m(t), v_m(t)\big) + \frac{1}{2}\left[\frac{\beta'}{\gamma}\left(\frac{\partial u_m'}{\partial y}(1,t)\right)^2 - \frac{\alpha'}{\gamma}\left(\frac{\partial u'}{\partial y}(0,t)\right)^2\right]$$

$$+ \frac{1}{2}\left[\frac{\beta'}{\gamma}\left(\frac{\partial v_m'}{\partial y}(1,t)\right)^2 - \frac{\alpha'}{\gamma}\left(\frac{\partial v'}{\partial y}(0,t)\right)^2\right]$$

$$\leq K_4 + \frac{1}{2}[\|f(t)\|^2 + \|g(t)\|^2] + K_5\Bigg\{\|u_m'(t)\|^2 + \|v_m'(t)\|^2$$

$$+ \left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|^2 + \left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|^2\|u_m'(t)\| + \left|\frac{\partial^2 v_m(t)}{\partial y^2}\right|^2\Bigg\}. \tag{3.24}$$

where K_4, K_5 are constants which do not depend on m.

Since

$$\frac{\beta'}{\gamma}\left(\frac{\partial u_m'}{\partial y}(1,t)\right)^2 - \frac{\alpha'}{\gamma}\left(\frac{\partial u'}{\partial y}(0,t)\right)^2 > 0,$$

$$\frac{\beta'}{\gamma}\left(\frac{\partial v_m'}{\partial y}(1,t)\right)^2 - \frac{\alpha'}{\gamma}\left(\frac{\partial v'}{\partial y}(0,t)\right)^2 > 0,$$

we get from (3.24)

$$\frac{d}{dt}F\big(u_m(t), v_m(t)\big) \le 2K_4 + \|f(t)\|^2 + \|g(t)\|^2 + K_6\big\{F\big(u_m(t), v_m(t)\big)^{3/2} + F\big(u_m(t), v_m(t)\big)\big\}. \tag{3.25}$$

The following Gronwall type lemma is essential for estimating the functional $F(u_m, v_m)$.

Lemma 3.1. *Let μ be a positive and differentiable function such that*

$$\mu'(t) \le \theta(t) + \kappa\mu(t) + \sigma\mu^{\lambda}(t)$$

where $\theta(t)$ is a positive function, $\theta \in L^1(0, T)$, κ, σ and λ are positive constants and $\lambda > 1$. Then there exists $T_0 \in (0, T)$ such that μ is bounded on $[0, T_0]$.

Proof. See [2]. □

Using 3.25 and Lemma 3.1, there exists $T_0 > 0$ such that

$$F\big(u_m(t), v_m(t)\big) \le K_7 \qquad \text{for all } 0 \le t \le T_0.$$

Therefore

$$\|u_m'(t)\|^2 + \|v_m'(t)\|^2 + \left|\frac{\partial^2 u_m(t)}{\partial y^2}\right|^2 + \left|\frac{\partial^2 v_m(t)}{\partial y^2}\right|^2 \le K_8 \qquad \text{for all } t \in [0, T_0]. \tag{3.26}$$

Third estimate. Substituting $w_j = u_m''(t)$ and $w_j = v_m''(t)$ into first and second equations of system (3.1), respectively, we obtain the following equalities:

$$\begin{aligned} |u_m''(t)|^2 = {} & \frac{1}{\gamma^2}\overline{M}\left(\frac{1}{\gamma}\|u_m(t)\|^2\right)\int_0^1 \frac{\partial^2 u_m}{\partial y^2}u_m''\,dy + \int_0^1 \frac{\partial}{\partial y}\left(a(y,t)\frac{\partial u_m}{\partial y}\right)u_m''\,dy \\ & - \int_0^1 b(y,t)\frac{\partial u_m'}{\partial y}u_m''\,dy - \int_0^1 c(y,t)\frac{\partial u_m}{\partial y}u_m''\,dy - \int_0^1 |u_m|^{\rho}u_m u_m''\,dy \\ & - \frac{1}{\gamma}\int_0^1 \frac{\partial v_m}{\partial y}u_m''\,dy + \int_0^1 f u_m''\,dy, \end{aligned} \tag{3.27}$$

$$\begin{aligned} |v_m''(t)|^2 = {} & \int_0^1 \frac{\partial}{\partial y}\left(d(y,t)\frac{\partial v_m}{\partial y}\right)v_m''\,dy - \int_0^1 b(y,t)\frac{\partial v_m'}{\partial y}v_m''\,dy \\ & - \int_0^1 c(y,t)\frac{\partial v_m}{\partial y}v_m''\,dy - \frac{1}{\gamma}\int_0^1 \frac{\partial u_m}{\partial y}v_m''\,dy + \int_0^1 g v_m''\,dy. \end{aligned} \tag{3.28}$$

Taking into account the hypotheses (M_1) and (H_1)–(H_3), we obtain the following estimates:

$$
\begin{aligned}
|u_m''(t)|^2 \leq{} & \frac{M_0}{\gamma_0^2}\left|\frac{\partial^2 u_m(t)}{\partial y^2}\right| |u_m''(t)| + \frac{m_0}{\gamma_0^2}\|u_m(t)\|\,|u_m''(t)| \\
& + \frac{m_0}{\gamma_0^2}\left|\frac{\partial^2 u_m(t)}{\partial y^2}\right| |u_m''(t)| + \frac{2}{\gamma_0}\left(\frac{m_0}{2}\right)^{1/2}\|u_m'(t)\|\,|u_m''(t)| \\
& + \frac{m_0}{2\gamma_0^2}\|u_m(t)\|\,|u_m''(t)| + |u_m(t)|_{2(\rho+1)}^{\rho+1}|u''(t)| \\
& + \frac{1}{\gamma_0}\|v(t)\|\,|u''(t)| + |f(t)|\,|u''(t)|, \qquad (3.29)
\end{aligned}
$$

$$
\begin{aligned}
|v_m''(t)|^2 \leq{} & \frac{m_0}{\gamma_0^2}\|v_m(t)\|\,|v_m''(t)| + \frac{1}{\gamma_0^2}\left[1 + \left(\frac{m_0}{2}\right)\right]\left|\frac{\partial^2 v_m(t)}{\partial y^2}\right| |v_m''(t)| \\
& + \frac{2}{\gamma_0}\left(\frac{m_0}{2}\right)^{1/2}\|v_m'(t)\|\,|v_m''(t)| + \frac{m_0}{2\gamma_0^2}\|v_m(t)\|\,|v_m''(t)| \\
& + \frac{1}{\gamma_0}\|u_m(t)\|\,|v_m''(t)| + |g(t)|\,|v_m''(t)|. \qquad (3.30)
\end{aligned}
$$

By Sobolev's embedding theorem, we know that $H_0^1(0,1) \hookrightarrow L^{2(\rho+1)}(0,1)$. Therefore, using the first and second estimates we obtain from (3.29) and (3.30) that

$$
|u_m''(t)| \leq K_9 \quad \text{and} \quad |v_m''(t)| \leq K_{10} \quad \text{on } [0, T_0]. \qquad (3.31)
$$

4. Proof of theorems

Proof of Theorem 2.2. Estimates (3.7), (3.26) and (3.31) assure the existence of a subsequence $\{u_k, v_k\}_{k\in\mathbb{N}}$ of sequence $\{u_m, v_m\}_{m\in\mathbb{N}}$ and a pair $\{u, v\}$ such that the following convergences:

$$
\begin{aligned}
& u_k \rightharpoonup u, && v_k \rightharpoonup v && \text{weak star in } L^\infty\big(0, T_0; H_0^1(0,1) \cap H^2(0,1)\big), && (4.1)\\
& u_k' \rightharpoonup u', && v_k' \rightharpoonup v' && \text{weak star in } L^\infty\big(0, T_0; H_0^1(0,1)\big), && (4.2)\\
& u_k'' \rightharpoonup u'', && v_k'' \rightharpoonup v'' && \text{weak star in } L^\infty\big(0, T_0; L^2(0,1)\big). && (4.3)
\end{aligned}
$$

Using compact embedding from $H_0^1(0,1) \cap H^2(0,1) \hookrightarrow H_0^1(0,1)$ and $H_0^1(0,1) \hookrightarrow L^2(0,1)$, by the compactness theorem of Aubin–Lions [4] we can conclude that exists a subsequence, still denoted by $\{u_k, v_k\}_{k\in\mathbb{N}}$, such that

$$u_k \to u \qquad \text{strong in } L^2\big(0, T_0; H_0^1(0,1)\big), \tag{4.4}$$

$$u_k' \to u' \qquad \text{strong in } L^2\big(0, T_0; L^2(0,1)\big). \tag{4.5}$$

From (4.3) we have for all $w \in L^2(0,1)$

$$(u_k'', w) \rightharpoonup (u'', w) \qquad \text{weak star in } L^\infty(0, T_0), \tag{4.6}$$

which implies weak convergence in $L^2(0, T_0)$.

Analogously, we obtain

$$(v_k'', w) \rightharpoonup (v'', w) \qquad \text{weak in } L^2(0, T_0). \tag{4.7}$$

From the second estimate we have

$$\frac{\partial^2 u_k}{\partial y^2} \rightharpoonup \frac{\partial^2 u}{\partial y^2} \qquad \text{weak star in } L^\infty\big(0, T_0; L^2(0,1)\big).$$

Then

$$\left(\frac{\partial^2 u_k}{\partial y^2}, w\right) \rightharpoonup \left(\frac{\partial^2 u}{\partial y^2}, w\right) \qquad \text{weak star in } L^\infty(0, T_0). \tag{4.8}$$

Let us denote $\eta_k(t) = \frac{1}{\gamma^2}\overline{M}\left(\frac{1}{\gamma}\|u_k(t)\|^2\right)$ and $\eta(t) = \frac{1}{\gamma^2}\overline{M}\left(\frac{1}{\gamma}\|u(t)\|^2\right)$. Then, taking into account the convergence (4.4) and hypothesis (H_2) on $M(\lambda)$, we obtain

$$\begin{aligned}\int_0^{T_0} |\eta_k(t) - \eta(t)|\,dt &\le K_{11}\int_0^{T_0}\Big|\,\|u_k(t)\|^2 - \|u(t)\|^2\Big|\,dt \\ &\le K_{11}\int_0^{T_0}\|u_k(t) - u(t)\|[\|u_k(t)\| + \|u(t)\|]\,dt.\end{aligned}$$

Using convergence (4.4), since $[\|u_k(t)\| + \|u(t)\|]$ is bounded, we get

$$\eta_k(t) \to \eta(t) \qquad \text{in } L^2(0, T_0).$$

From weak convergences $a(t, u_k, w) \rightharpoonup a(t, u, w)$, $d(t, v_k, w) \rightharpoonup d(t, v, w)$ in $L^2(0, T_0)$ for all $w \in H_0^1(0,1)$, and integration by parts we obtain that

$$-\left(\frac{\partial}{\partial y}\left(a(y,t)\frac{\partial u_k}{\partial y}\right), w\right) \rightharpoonup -\left(\frac{\partial}{\partial y}\left(a(y,t)\frac{\partial u}{\partial y}\right), w\right) \qquad \text{weak in } L^2(0, T_0).$$

Analogously, we have

$$-\left(\frac{\partial}{\partial y}\left(d(y,t)\frac{\partial v_k}{\partial y}\right), w\right) \rightharpoonup -\left(\frac{\partial}{\partial y}\left(d(y,t)\frac{\partial v}{\partial y}\right), w\right) \quad \text{weak in } L^2(0,T_0).$$

Using the same arguments as above, we have for all $w \in L^2(0,1)$

$$\begin{aligned}
\left(b(y,t)\frac{\partial u_k'}{\partial y}, w\right) &\rightharpoonup \left(b(y,t)\frac{\partial u'}{\partial y}, w\right) && \text{weak in } L^2(0,T_0),\\
\left(b(y,t)\frac{\partial v_k'}{\partial y}, w\right) &\rightharpoonup \left(b(y,t)\frac{\partial v'}{\partial y}, w\right) && \text{weak in } L^2(0,T_0),\\
\left(c(y,t)\frac{\partial u_k}{\partial y}, w\right) &\rightharpoonup \left(c(y,t)\frac{\partial u}{\partial y}, w\right) && \text{weak in } L^2(0,T_0),\\
\left(c(y,t)\frac{\partial v_k}{\partial y}, w\right) &\rightharpoonup \left(c(y,t)\frac{\partial v}{\partial y}, w\right) && \text{weak in } L^2(0,T_0),\\
\left(\frac{1}{\gamma}\frac{\partial u_k}{\partial y}, w\right) &\rightharpoonup \left(\frac{1}{\gamma}\frac{\partial u}{\partial y}, w\right) && \text{weak in } L^2(0,T_0),\\
\left(\frac{1}{\gamma}\frac{\partial v_k}{\partial y}, w\right) &\rightharpoonup \left(\frac{1}{\gamma}\frac{\partial v}{\partial y}, w\right) && \text{weak in } L^2(0,T_0).
\end{aligned}$$

Convergence (4.4) assures that there exists a subsequence, still denoted by $(u_k)_{k\in\mathbb{N}}$, such that

$$|u_k|^\rho u_k \to |u|^\rho u \quad \text{in }]0,1[\times]0,T_0[\text{ almost everywhere.} \tag{4.9}$$

Using the embedding $H_1(0,1) \hookrightarrow L^{2\rho+2}(0,1)$ and the first estimate, we obtain

$$\int_0^1 \big|\,|u_k|^\rho u_k\big|^2\,dy = \int_0^1 |u_k|^{2\rho+2}\,dy = |u_k|_{2\rho+2}^{2\rho+2} \le c\|u_k\|_{H_0^1(0,1)}^{2\rho+2} \le cK_3^{\rho+1}. \tag{4.10}$$

Thus

$$|u_k|^\rho u_k \text{ is bounded in } L^\infty\big(0,T_0;L^2(0,1)\big).$$

From (4.9) and (4.10) and Lions' lemma [4] we obtain

$$|u_k|^\rho u_k \rightharpoonup |u|^\rho u \quad \text{in } L^2\big(0,T_0;L^2(0,1)\big).$$

Therefore, we have for all $w \in L^2(0,1)$

$$(|u_k|^\rho u_k, w)\,dt \rightharpoonup (|u|^\rho u, w) \quad \text{weak in } L^2(0,T_0).$$

Using these convergences we can pass to the limit in the approximated system (3.1) for the subsequence $\{u_k, v_k\}_{k\in\mathbb{N}}$, which proves the existence of a solution $\{u, v\}$ to the cylindrical problem (2.1) in $L^2(0, T_0; L^2(0,1))$.

From (4.1), (4.2) and (4.3), we have

$$u(0) = u_0, \qquad v(0) = v_0, \qquad u'(0) = u_1 \qquad \text{and} \qquad v'(0) = v_1.$$

Uniqueness. Let $\{u, v\}$ and $\{\check{u}, \check{v}\}$ be solutions of (3.1) under the conditions of Theorem 2.2. Let $w = u - \check{u}$ and $z = v - \check{v}$. Then $\{w, z\}$ satisfies

$$\begin{aligned} w'' &- \frac{1}{\gamma^2}\overline{M}(\gamma|u(t)|^2)\frac{\partial^2 w}{\partial y^2} - \frac{1}{\gamma^2}\left[\overline{M}\left(\frac{1}{\gamma}\|u(t)\|^2\right) - \overline{M}\left(\frac{1}{\gamma}\|\check{u}(t)\|^2\right)\right]\frac{\partial^2 \check{u}}{\partial y^2} \\ &- \frac{\partial}{\partial y}\left(a(y,t)\frac{\partial w}{\partial y}\right) + b(y,t)\frac{\partial w'}{\partial y} + c(y,t)\frac{\partial w}{\partial y} + |u|^\rho u - |\check{u}|^\rho \check{u} + \frac{1}{\gamma}\frac{\partial z}{\partial y} = 0, \end{aligned} \tag{4.11}$$

$$z'' - \frac{\partial}{\partial y}\left(d(y,t)\frac{\partial z}{\partial y}\right) + b(y,t)\frac{\partial z'}{\partial y} + c(y,t)\frac{\partial z}{\partial y} + \frac{1}{\gamma}\frac{\partial w}{\partial y} = 0, \tag{4.12}$$

with initial data and boundary conditions

$$w(0) = z(0) = w'(0) = z'(0) = 0, \qquad w(0,t) = w(1,t) = z(0,t) = z(1,t) = 0$$

for all $t \in]0, T_0[$.

Multiplying (4.11) and (4.12) by w' and z', respectively, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}&\left\{|w'(t)|^2 + \frac{1}{\gamma^2}\overline{M}\left(\frac{1}{\gamma}\|u(t)\|^2\right)\|w(t)\|^2 + a(t,w,w)\right\} + \frac{\gamma'}{\gamma}a(t,w,w) + \frac{\gamma'}{\gamma}|w'(t)|^2 \\ &= \left[\frac{1}{\gamma^2}\overline{M}(\gamma\|u(t)\|^2) - \frac{1}{\gamma^2}\overline{M}(\gamma\|\check{u}(t)\|^2)\right]\int_0^1 \frac{\partial^2 \check{u}}{\partial y^2}w'\,dy \\ &\quad + \frac{1}{\gamma^3}\overline{M}'\left(\frac{1}{\gamma}\|u(t)\|^2\right)\left[((u',u)) - \frac{\gamma'}{\gamma}\|u(t)\|^2\right]\|w(t)\|^2 \\ &\quad - \frac{\gamma'}{\gamma^3}\overline{M}\left(\frac{1}{\gamma}\|u(t)\|^2\right)\|w(t)\|^2 + \int_0^1 b(y,t)h(y,t)\left(\frac{\partial w}{\partial y}\right)^2 dy \\ &\quad - \int_0^1 c(y,t)\frac{\partial w}{\partial y}w'\,dy - \int_0^1 [|u|^\rho u - |\check{u}|^\rho \check{u}]w'\,dy - \frac{1}{\gamma}\int_0^1 \frac{\partial z}{\partial y}w'\,dy, \end{aligned} \tag{4.13}$$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}&\{|z'(t)|^2 + d(t,z,z)\} + \frac{\gamma'}{\gamma}d(t,z,z) + \frac{\gamma'}{\gamma}|z'(t)|^2 \\ &= \frac{1}{2}\int_0^1 b(y,t)h(y,t)\left(\frac{\partial z}{\partial y}\right)^2 dy - \int_0^1 c(y,t)\frac{\partial z}{\partial y}z'\,dy - \frac{1}{\gamma}\int_0^1 \frac{\partial w}{\partial y}z'\,dy. \end{aligned} \tag{4.14}$$

We estimate (4.13) step by step

$$\begin{aligned}
\boldsymbol{T_1} &= -\frac{1}{\gamma^2}\left[\overline{M}\left(\gamma\|u(t)\|^2\right) - \overline{M}\left(\gamma\|\breve{u}(t)\|^2\right)\right]\int_0^1 \frac{\partial^2 \breve{u}}{\partial y^2} w'\,dy \\
&\le \frac{1}{\gamma_0} C\|w(t)\|[\|u(t)\| + \|\breve{u}(t)\|]\left|\frac{\partial^2 \breve{u}(t)}{\partial y^2}\right| |w'(t)| \\
&\le \frac{1}{\gamma_0} C K_3^{1/2} K_8^{1/2}[\|w(t)\|^2 + |w'(t)|^2], \\
\boldsymbol{T_2} &= \frac{1}{\gamma^3}\overline{M}'\left(\frac{1}{\gamma}\|u(t)\|^2\right)\left[((u',u)) - \frac{\gamma'}{\gamma}\|u(t)\|^2\right]\|w(t)\|^2 \\
&\le \frac{1}{\gamma_0^3} M_1 K_3^{1/2}\left[K_8^{1/2} + \frac{1}{\gamma_0}\left(\frac{m_0}{2}\right)^{1/2} K_3^{1/2}\right]\|w(t)\|^2, \\
\boldsymbol{T_3} &= \frac{\gamma'}{\gamma}\overline{M}\left(\frac{1}{\gamma}\|u(t)\|^2\right)\|w(t)\|^2 \le \frac{1}{\gamma_0} M_0\left(\frac{m_0}{2}\right)^{1/2}\|w(t)\|^2.
\end{aligned}$$

Let $F:\mathbb{R}\to\mathbb{R}$, $F\in C^1(\mathbb{R})$, given by $F(\lambda)=|\lambda|^\rho\lambda$. It is clear that $F'(\lambda) = (\rho+1)|\lambda|^\rho$. Therefore given $\xi_1,\xi_2\in\mathbb{R}$, by the mean-value theorem, there exists $\xi=\xi_1+\theta(\xi_2-\xi_1)$, $\theta\in\,]0,1[$, such that

$$\begin{aligned}
&|F(\xi_2)-F(\xi_1)| \le |F'(\xi)|\,|\xi_2-\xi_1| \implies \big||\xi_2|^\rho\xi_2 - |\xi_1|^\rho\xi_1\big| \le (\rho+1)|\xi|^\rho|\xi_2-\xi_1| \\
&\qquad \implies \big||\xi_2|^\rho\xi_2 - |\xi_1|^\rho\xi_1\big| \le (\rho+1)|\xi_1+(\xi_2-\xi_1)\theta|^\rho|\xi_2-\xi_1|.
\end{aligned}$$

In particular, if $\xi_1(y,t)=u(y,t)$ and $\xi_2(y,t)=\breve{u}(y,t)$, we have

$$\begin{aligned}
\big||u|^\rho u - |\breve{u}|^\rho\breve{u}\big| &\le (\rho+1)|u+(\breve{u}-u)\theta|^\rho|\breve{u}-u| \\
&\le (\rho+1)\{|u|+|\breve{u}|+|u|\}^\rho|w| \le (\rho+1)\{2|u|+2|\breve{u}|\}^\rho|w| \\
&\le (\rho+1)2^\rho\{|u|+|\breve{u}|\}^\rho|w| \le (\rho+1)2^\rho 2^\rho\{|u|^\rho+|\breve{u}|^\rho\}|w|.
\end{aligned}$$

Using Hölder's inequality and Sobolev's embedding theorem we obtain that

$$\begin{aligned}
\boldsymbol{T_4} &= \int_0^1 [|u|^\rho u - |\breve{u}|^\rho v]w'\,dy \\
&\le (\rho+1)2^{2\rho}\int_0^1 \{|u|^\rho + |\breve{u}|^\rho\}|w|\,|w'|\,dy \\
&\le (\rho+1)2^{2\rho}\{|u(t)|_{r\rho}^\rho + |\breve{u}(t)|_{r\rho}^\rho\}|w(t)|_q|w'(t)|_2 \\
&\le K_{12}[\|w(t)\|^2 + |w'(t)|^2].
\end{aligned}$$

The other terms on the right-hand side of (4.13) are estimated using the hypotheses (H_1) and (H_2) as follows:

$$\boldsymbol{T_5} = \frac{1}{2}\int_0^1 b(y,t)h(y,t)\left|\frac{\partial w}{\partial y}\right|^2 dy \le \frac{2}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{3/2}\|w(t)\|^2.$$

From the Young inequality we get

$$\boldsymbol{T_6} = -\int_0^1 c(y,t)\frac{\partial w}{\partial y}w'\,dy \le \frac{m_0}{4\gamma_0^2}[\|w(t)\|^2 + |w'(t)|^2],$$

$$\boldsymbol{T_7} = -\frac{1}{\gamma}\int_0^1 \frac{\partial z}{\partial y}w'\,dy \le \frac{1}{2\gamma_0}[\|z(t)\|^2 + |w'(t)|^2].$$

In the analogous way we estimate the terms on the right-hand side of equation (4.14). Summing up the equations (4.13) and (4.14), integrating from 0 to t we obtain that

$$\begin{aligned}\frac{1}{2}\bigg\{&|w'(t)|^2 + |z'(t)|^2 + \|w(t)\|^2 + \frac{1}{\gamma^2}\overline{M}\left(\frac{1}{\gamma}\|u(t)\|^2\right)\|w(t)\|^2 + a(t,w,w)\\ &+ d(t,z,z)\bigg\} + \int_0^t \frac{\gamma'(s)}{\gamma(s)}[|w'(s)|^2 + |z'(s)|^2 + a(s,w,w) + d(s,z,z)]\,ds\\ &\le K_{12}\int_0^t [|w'(s)|^2 + |z'(s)|^2 + \|w(s)\|^2 + \|z(s)\|^2]\,ds. \end{aligned} \tag{4.15}$$

From the next inequalities

$$\int_0^t \frac{\gamma'(s)}{\gamma(s)}[|w'(s)|^2 + |z'(s)|^2 + a(s,w,w) + d(s,z,z)]\,ds \ge 0,$$

$$\frac{1}{\gamma^2}\overline{M}\left(\frac{1}{\gamma}\|u(t)\|^2\right)\|w(t)\|^2 \ge \frac{m_0}{2\gamma_1^2}\|w(t)\|^2,$$

$$a(t,w,w) \ge 0 \quad \text{and} \quad d(t,z,z) \ge \frac{m_1}{\gamma_1}\|z(t)\|^2,$$

turning back to (4.15) we obtain that

$$\begin{aligned}&|w'(t)|^2 + |z'(t)|^2 + \|w(t)\|^2 + \|z(t)\|^2\\ &\quad \le K_{13}\int_0^t [|w'(s)|^2 + |z'(s)|^2 + \|w(s)\|^2 + \|z(s)\|^2]\,ds.\end{aligned}$$

Finally Gronwall's lemma assures that $w(t) = z(t) = 0$ in $H_0^1(0,1)$ for all $t \in [0, T_0]$. This proves Theorem 2.2.

Proof of Theorem 2.1. Let $\{u, v\}$ be a solution of Theorem 2.2. We consider the functions $\phi(x,t) = u(y,t)$ and $\psi(x,t) = v(y,t)$, where $x = \alpha(t) + y\gamma(t)$ with initial data is given by the functions

$$u_0 = \phi(x,0) = \phi_0\big(\alpha(0) + \gamma(0)y\big), \qquad v_0 = v(x,0) = v_0\big(\alpha(0) + \gamma(0)y\big),$$
$$u_1 = \phi'(x,0) = \phi_1\big(\alpha(0) + \gamma(0)y\big) + \big(\alpha'(0) + \gamma'(0)y\big)\phi_{0x}\big(\alpha(0) + \gamma(0)y\big),$$
$$v_1 = v'(x,0) = v_1\big(\alpha(0) + \gamma(0)y\big) + \big(\alpha'(0) + \gamma'(0)y\big)v_{0x}\big(\alpha(0) + \gamma(0)y\big),$$
$$\gamma_0 = \gamma(0), \qquad \alpha_0 = \alpha(0) \qquad \text{and} \qquad \beta_0 = \beta(0),$$

and

$$f(y,t) = \varphi(\alpha + y\gamma, t), \qquad g(y,t) = \psi(\alpha + y\gamma, t).$$

To verify that $\phi(x,t)$ and $v(x,t)$ are solutions of Theorem 1.1, it is sufficient to observe that the mapping $\tau : (x,t) \mapsto \left(\frac{x-\alpha}{\gamma}, t\right)$ of the domain $\hat{Q}_0$ into $Q_0 =]0,1[\times]0,T_0[$ is of class C^2 and from (2.1)–(2.3) we also have that $\{\phi, v\}$ satisfies the (1.2)–(1.4). The regularity of $\{u, v\}$ given by Theorem 2.2 implies the regularity of $\{\phi, v\}$ in Theorem 2.1. Thus the uniqueness of the solution of (1.2)–(1.4) is a direct consequence of the uniqueness of (2.1)–(2.3).

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