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Symmetry and bifurcation of periodic solutions in Neumann boundary value problems

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Abstract. We study a vector-valued reaction-diffusion equation with Neumann boundary conditions ($u : [0, \pi] \to \mathbb{R}^2$). Unlike what is observed for scalar equations, where no heteroclinic connections involving periodic solutions occur, we find that steady-state/Hopf and Hopf/Hopf mode interactions produce heteroclinic solutions connecting at least one solution of standing wave type. This is achieved by restricting a problem with periodic boundary conditions and equivariant under O(2) symmetry to a suitable fixed-point space.

For completeness, we include a description of the solutions for Hopf bifurcation and mode interactions involving Hopf bifurcation, namely, steady-state/Hopf and Hopf/Hopf.

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1. Introduction

Consider a reaction-diffusion equation on $[-\pi, \pi]$ given by

$$\dot{u} = D\frac{\partial^2 u}{\partial \xi^2} + f(u, \lambda),$$

where $u = u(u_1(\xi), u_2(\xi))$, D is a 2×2 matrix, $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is a nonlinear mapping and λ is the bifurcation parameter as usual. We say that this problem has *periodic boundary conditions* (PBC) if $u(-\pi) = u(\pi)$ and $u'(-\pi) = u'(\pi)$. We define *Neumann boundary conditions* (NBC) on the smaller interval $[0, \pi]$ if $u'(0) = u'(\pi) = 0$.

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Crawford et al. [3] proved that a solution $u : [0, \pi] \to \mathbb{R}$ to a NBC problem can be reflected about the origin and extended periodically to produce an even solution $\tilde{u}$ to the corresponding PBC problem on $[-\pi, \pi]$. This extension is done by defining

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \geq 0, \\ u(-x) & \text{if } x < 0. \end{cases}$$

The natural domain for these solutions is the space of 2π-periodic functions. The converse is also trivially true, that is, the restriction of an even 2π-periodic solution u of a PBC from $[-\pi, \pi]$ to $[0, \pi]$ is a solution to the corresponding NBC (since u is smooth and even, NBC are automatically satisfied on $[0, \pi]$). Observe also that an even solution is a solution in $\mathrm{Fix}(\kappa)$ where $\kappa \cdot x = -x$, $x \in \mathbb{R}$. This rules out the existence of rotating waves, and all periodic solutions are standing waves. The natural symmetries for PBC are then those of the domain $[-\pi, \pi]$ which, when the ends are identified, is a circle. See, for instance, Golubitsky and Stewart [6]. Hence, the natural setting for extracting information from a problem with PBC to one with NBC is that of $\mathbf{O}(2)$ symmetry.

Several authors (see Gomes and Stewart [8], [9] and Crawford et al. [3]) have pursued the study of bifurcation problems with boundary conditions on rectangular domains. It was shown that hidden symmetries and change of genericity in solutions may occur. In Gomes and Stewart [8] the concern is about the solutions arising from Hopf bifurcation, including mode interactions. In this paper, we extend this study to Neumann boundary conditions and the case where the problem has $\mathbf{O}(2)$ symmetry. This setting can explain some patterns arising in the Taylor–Couette experiment, as remarked by Crawford et al. [3] and Golubitsky and Stewart [6].

In the next section, we use the solutions to the PBC problem with $\mathbf{O}(2)$ symmetry to swiftly obtain solutions to the NBC case. We consider single Hopf bifurcation as well as mode interactions of the types Hopf/steady-state and Hopf/Hopf, respectively, thus extending the study of reaction-diffusion equations with PBC to those with NBC. Our results are obtained by restricting the information concerning PBC to the appropriate fixed-point space and therefore proofs are omitted. The results obtained show how boundary conditions constrain the solutions. The use of amplitude-phase equations further simplifies the problem: the amplitude equations are equivariant under the action of a smaller group. Solutions to the original problem are obtained by restoring the phase. Even though the results are obtained through what can be considered an exercise, we believe that their precise statement provides a useful reference.

The final and main section concerns the existence of heteroclinic connections between periodic solutions. We prove the existence of connections involving at least one standing wave in steady-state/Hopf and Hopf/Hopf mode interactions.

This shows that the asymptotic behaviour of solutions to vector-valued reaction-diffusion equations with NBC is richer than that observed in the scalar case. In fact, as observed by Fiedler et al. [4], among others, in the scalar case heteroclinic orbits can only connect equilibria when NBC are present.

2. Symmetry and bifurcations

In this section we address the way symmetry affects NBC problems in what concerns several types of bifurcation. This completes studies by Gomes and Stewart [8] in that it addresses NBC, and by Armbruster and Dangelmayr [1], by considering Hopf bifurcation and mode interactions involving Hopf bifurcations. As remarked above, this is done by restricting previously established results by other authors. The tables below are obtained by restricting these results to the fixed-point space of even solutions.

The results concerning single Hopf bifurcation are obtained from chapter XVII in Golubitsky et al. [7]. The case of steady-state/Hopf bifurcation is obtained from chapter XX in [7] and Hill and Stewart [10]. Finally, Hopf/Hopf mode interaction is based again on chapter XX in [7] and Melbourne et al. [11].

We consider a system of differential equations given in normal form on an appropriate space, which we specify below in each instance,

$$\dot{x} + X(x, \lambda) = 0, \tag{1}$$

where X is smooth and group equivariant, for the suitable action, and λ is the bifurcation parameter.

It is well known (see [7], for example) that either a Lyapunov–Schmidt reduction or the calculation of normal forms on a centre manifold produce natural temporal symmetries where a Hopf bifurcation is concerned. Hence, the action we want to consider is that of $\mathrm{O}(2) \times S^1$ when one Hopf mode is involved, and that of $\mathrm{O}(2) \times \mathbb{T}^2$ if there are two Hopf modes. It will be clear from the normal form calculations below that the results by Chossat and Golubitsky [2] apply and either reduction method provides the same equation up to cubic order, which is enough for our purposes. More importantly, these results guarantee not only that the existence of equilibrium solutions of the reduced equation correspond to solutions of the original problem, but also that the stability and asymptotic dynamics are preserved by the reduction (see also [6]).

2.1. The group action. We consider the restriction of problems equivariant under the appropriate group action in each case. The occurrence of an $\mathrm{O}(2)$-equivariant Hopf bifurcation requires the group action not to be absolutely irreducible. Therefore, we consider, for each Hopf mode, an action on the 4-dimensional space $\mathbb{C}^2$.

In each case we indicate those fixed-point spaces and respective isotropy subgroups that are contained in the fixed-point space where even solutions to the PBC problem exist. These are obtained by restriction of the isotropy lattice for the PBC problem (see again the references above).

The actions and representations we use are the following.

Single Hopf. We choose coordinates on $\mathbb{C}^2$ so that the action is given by

$$\begin{aligned}\theta(z_1, z_2) &= (e^{i\theta} z_1, e^{i\theta} z_2), \qquad \theta \in S^1,\\ \phi(z_1, z_2) &= (e^{-i\phi} z_1, e^{i\phi} z_2), \qquad \phi \in \mathrm{SO}(2),\\ \kappa(z_1, z_2) &= (z_2, z_1), \qquad \kappa = \text{flip in } \mathrm{O}(2).\end{aligned}$$

Solutions to the NBC problem are found in $\mathrm{Fix}(\mathbb{Z}_2 \oplus \mathbb{Z}_2^c) = \{(z, z) : z \in \mathbb{C}\}$, where $\mathbb{Z}_2 \oplus \mathbb{Z}_2^c = \langle \kappa, (\pi, \pi) \rangle$. In this subspace, we have only two orbit types, namely, the trivial solution with full isotropy and periodic solutions of standing wave type.

Steady-state/Hopf mode interactions. In this case, the action is on $\mathbb{C}^3$ and all eigenvalues are double. We choose coordinates so that the action of $\mathrm{O}(2) \times S^1$ is generated by

$$\begin{aligned}\theta(z_0, z_1, z_3) &= (z_0, e^{i\theta} z_1, e^{i\theta} z_2), \qquad \theta \in S^1,\\ \phi(z_0, z_1, z_3) &= (e^{im\phi} z_0, e^{il\phi} z_1, e^{-il\phi} z_2), \qquad \phi \in \mathrm{SO}(2),\\ \kappa(z_0, z_1, z_3) &= (\overline{z_0}, z_2, z_1), \qquad \kappa = \text{flip in } \mathrm{O}(2).\end{aligned}$$

The integers l and m are the mode numbers (we follow the notation of Golubitsky et al. [7]; Hill and Stewart [10] interchange the mode numbers and name the generators for the group action differently). The isotropy subgroups depend on these mode numbers. We consider three cases: when both mode numbers are equal to one, when m is odd and when m is even. In the two last instances, we assume that l and m are coprime. The information in Table 1 is obtained from chapter XX, 2.3, in [7] and also from results in Hill and Stewart [10].

Solutions to NBC problems are found in $\{(x, z, z) : x \in \mathbb{R}, z \in \mathbb{C}\}$. This corresponds to $\mathrm{Fix}(\mathbb{Z}_2(\kappa))$ if $l = m = 1$ or if m is odd. If m is even then it corresponds to the fixed-point space of $\mathbb{Z}_2(\kappa) \times \mathbb{Z}_2(\pi, l\pi)$.

The isotropy subgroup $\mathbb{Z}_2(\kappa) \times S^1$ corresponds to a branch of steady-states and $\mathbb{Z}_2(\kappa) \oplus \mathbb{Z}_2^c$ to a branch of periodic solutions. As in the single mode case, these periodic solutions are standing waves. There may also be a mixed-mode branch with isotropy $\mathbb{Z}_2(\kappa)$.

Table 1. Isotropy subgroups and fixed-point spaces for the action of $O(2)\times S^1$ on $\mathbb{C}$. The isotropy subgroup Σ_1 is $\mathbb{Z}_2(\kappa)\times S^1$, $\mathbb{Z}_2(\kappa)\dot{+}\mathbb{Z}_2(\frac{2\pi}{m},\pi)$, or $\mathbb{Z}_2(\kappa)\dot{+}\mathbb{Z}_2(\frac{2\pi}{m},0)\times S^1\times\mathbb{Z}_2(\pi,l\pi)$ depending on whether $l=m=1$, m is odd or m is even, respectively. Analogously, Σ_2 is $\mathbb{Z}_2(\kappa)\oplus\mathbb{Z}_2^c$, $\mathbb{Z}_2(\kappa)\dot{+}\mathbb{Z}_2(\frac{\pi}{l},\pi)$ or $\mathbb{Z}_2(\kappa)\dot{+}\mathbb{Z}_2(\frac{\pi}{l},\pi)\times\mathbb{Z}_2(\pi,l\pi)$ and Σ_3 is $\mathbb{Z}_2(\kappa)$, $\mathbb{Z}_2(\kappa)$ or $\mathbb{Z}_2(\kappa)\times\mathbb{Z}_2(\pi,l\pi)$. We use $\dot{+}$ to represent the semi-direct product.

Isotropy subgroup Σ	$\mathrm{Fix}(\Sigma)$
$O(2)\times S^1$	$\{(0,0,0)\}$
Σ_1	$\{(x,0,0) : x\in\mathbb{R}\}$
Σ_2	$\{(0,z,z) : z\in\mathbb{C}\}$
Σ_3	$\{(x,z,z) : x\in\mathbb{R}, z\in\mathbb{C}\}$

Hopf/Hopf mode interactions. We consider two representations for the action of $O(2)\times\mathbb{T}^2$, one on $\mathbb{C}^3$ and the other on $\mathbb{C}^4$. In the latter representation both eigenvalues are double. Coordinates are chosen so that the action is given by

$$\begin{aligned}\phi(z_0,z_1,z_2)&=(z_0,e^{i\phi}z_1,e^{-i\phi}z_2),\qquad \phi\in SO(2),\\ \kappa(z_0,z_1,z_2)&=(z_0,z_2,z_1),\qquad \kappa=\text{flip in } O(2),\\ (\theta,\psi)(z_0,z_1,z_2)&=(e^{i\theta}z_0,e^{i\psi}z_1,e^{i\psi}z_2),\qquad (\theta,\psi)\in\mathbb{T}^2,\end{aligned}$$

in the 6-dimensional case. In the 8-dimensional case, we have

$$\begin{aligned}\phi(z_1,z_2,z_3,z_4)&=(e^{il\phi}z_1,e^{-il\phi}z_2,e^{im\phi}z_3,e^{-im\phi}z_4),\qquad \phi\in SO(2),\\ \kappa(z_1,z_2,z_3,z_4)&=(z_2,z_1,z_4,z_3),\qquad \kappa=\text{flip in } O(2),\\ (\theta,\psi)(z_1,z_2,z_3,z_4)&=(e^{i\theta}z_1,e^{i\theta}z_2,e^{i\psi}z_3,e^{i\psi}z_4),\qquad (\theta,\psi)\in\mathbb{T}^2.\end{aligned}$$

We factor out the kernel of the action of $O(2)$ on $\mathbb{C}^4$ and assume that l and m are coprime.

The 6-dimensional case. In this case the symmetry is such that a decomposition into amplitude-phase equations, from the start, very much simplifies the study of the mode interaction. Defining $r_j=|z_j|$, the amplitude equations produce an ordinary differential equation on $\mathbb{R}^3$ whose zeros are in one-to-one correspondence with the periodic solutions of the original mode interaction problem. This equation is defined by a $\mathbb{Z}_2\times\mathbb{D}_4$-equivariant map, where $\mathbb{Z}_2$ is generated by κ_0 and $\mathbb{D}_4$ is generated by three other elements of order two as follows:

Table 2. Isotropy subgroups and fixed-point spaces for each group action in Hopf/Hopf mode interaction. The first half concerns the symmetry group of the amplitude equations for the 6-dimensional representation. The second half describes the isotropy subgroups and their fixed-point spaces for the 6-dimensional representation.

Representation	Isotropy subgroup Σ	$\mathrm{Fix}(\Sigma)$
6-dimensional	$\mathbb{Z}_2 \times \mathbb{D}_4$	$\{(0,0,0)\}$
	$\mathbb{D}_4$	$\{(r_0,0,0) : r_0 \in \mathbb{R}\}$
	$\mathbb{Z}_2(\kappa_0) \times \mathbb{Z}_2(\kappa)$	$\{(0,r_1,r_1) : r_1 \in \mathbb{R}\}$
	$\mathbb{Z}_2(\kappa)$	$\{(r_0,r_1,r_1) : r_0, r_1 \in \mathbb{R}\}$
8-dimensional	$\mathrm{O}(2) \times \mathbb{T}^2$	$\{(0,0,0)\}$
	$S(0,0,1) \times \mathbb{Z}_2(\kappa) \times \mathbb{Z}(\pi/l, \pi, 0)$	$\{(z_1,z_1,0,0) : z_1 \in \mathbb{C}\}$
	$S(0,1,0) \times \mathbb{Z}_2(\kappa) \times \mathbb{Z}(\pi/m, 0, \pi)$	$\{(0,0,z_3,z_3) : z_3 \in \mathbb{C}\}$
	$\mathbb{Z}_2(\kappa) \times \mathbb{Z}(\pi, l\pi, m\pi)$	$\{(z_1,z_1,z_3,z_3) : z_1, z_3 \in \mathbb{C}\}$

$$\begin{aligned}
\kappa_0(r_0,r_1,r_2) &= (r_0,r_1,r_2),\\
\kappa_1(r_0,r_1,r_2) &= (-r_0,-r_1,-r_2),\\
\kappa_2(r_0,r_1,r_2) &= (r_0,r_1,-r_2),\\
\kappa(r_0,r_1,r_2) &= (r_0,r_2,r_1).
\end{aligned}$$

Solutions to the NBC problem belong to $\mathrm{Fix}\big(\mathbb{Z}_2(\kappa)\big) = \{(r_0,r_1,r_1) : r_0, r_1 \in \mathbb{R}\}$. Table 2 contains information about isotropy subgroups and fixed-point spaces obtained from data in [7], chapter XX.

The 8-dimensional case. We are interested in solutions in $\{(z_1,z_1,z_3,z_3) : z_1, z_3 \in \mathbb{C}\}$. We therefore restrict the information provided by Golubitsky et al. [7] and Melbourne et al. [11] to this space to obtain the corresponding information contained in Table 2. We use the notation of [7] for the isotropy subgroups, that is, $\mathbb{Z}(\phi,\theta,\psi) = \langle(\phi,\theta,\psi)\rangle \subset \mathrm{SO}(2) \times \mathbb{T}^2$ and $S(k,l,m) = \{(k\theta, l\theta, m\theta) : \theta \in S^1\}$.

2.2. Invariant theory and normal forms. We present a sequence of lemmas which provide the necessary information for the construction of normal forms for X in (1). In the mode interaction cases, we use λ to denote the bifurcation parameter and any necessary unfolding parameters. Hence, λ may be multi-dimensional. As before, these results are obtained by restricting results in Golubitsky et al. [7] and Hill and Stewart [10] to the appropriate space. Specific references are provided for each lemma.

Lemma 2.1 ([7], Proposition XVII, 2.1). *In the case of single mode Hopf bifurcation,*

(a) *every* O(2) $\times S^1$*-invariant germ* f *has the form* $f(z, z) = P(N)$*, where* $N = 2|z|^2$*, and*

(b) *every* O(2) $\times S^1$*-equivariant vector field* X *has the form*

$$X(z, z, \lambda) = (p + iq)(z, z)^T,$$

where p *and* q *are* O(2) $\times S^1$*-invariant germs, depending on* λ*, and the superscript* T *denotes the transpose.*

For the steady-state/Hopf mode interaction we have the following result which puts together, and restricts, several results in [10].

Lemma 2.2 (Hill and Stewart [10]). (a) *A basis for the* O(2) $\times S^1$*-invariant is given by* $N_0 = x^2$*,* $N_1 = 2|z|^2$ *and* $T = x^\alpha |z|^{2\beta}$*, where* $\alpha = 2l$ *and* $\beta = m$ *when* m *is odd, while* $\alpha = l$ *and* $\beta = m/2$ *when* m *is even.*

(b) *Every* O(2) $\times S^1$*-equivariant vector field has the form*

$$X(x, z, \lambda) = c_1(x, 0, 0) + c_3(x^{\alpha-1}|z|^{2\beta}) + (p_1 + iq_1)(0, z, z) + (p_3 + iq_3)(0, x^\alpha |z|^{2\beta}\bar{z}^{-1}, x^\alpha |z|^{2\beta}\bar{z}^{-1}),$$

where c_i*,* p_1 *and* q_i *depend on the invariants and the bifurcation parameters.*

For the 6-dimensional Hopf/Hopf mode interaction, we have

Lemma 2.3 ([7], Theorem XX, 3.1). *Vector fields commuting with* O(2) $\times \mathbb{T}^2$ *have the form*

$$X(z_0, z_1, z_1, \lambda) = (p_0 + iq_0)(z_0, 0, 0)^T + (p_1 + iq_1)(0, z_1, z_1)^T,$$

where p_i *and* q_i *are functions of the parameters and of* $\rho = |z_0|^2$ *and* $N = 2|z_1|^2$.

Finally, the invariant theory for the 8-dimensional Hopf/Hopf mode interaction is given in

Lemma 2.4 ([7], Theorem XX, 3.2). *Any* O(2) $\times \mathbb{T}^2$*-equivariant vector field* X *has normal form given by*

$$\begin{aligned} X(z_1, z_1, z_3, z_3, \lambda) = \big(&(p_1 + iq_1)z_1 + (r_1 + is_1)|z_1|^{2m}\overline{z_1}^{-1}|z_3|^{2l}, \\ &(p_1 + iq_1)z_1 + (r_1 + is_1)|z_1|^{2m}\overline{z_1}^{-1}|z_3|^{2l}, \\ &(p_3 + iq_3)z_3 + (r_3 + is_3)|z_1|^{2m}|z_3|^{2l}\overline{z_3}^{-1}, \\ &(p_3 + iq_3)z_3 + (r_3 + is_3)|z_1|^{2m}|z_3|^{2l}\overline{z_3}^{-1}\big), \end{aligned}$$

where p_i, q_i, r_i and s_i are functions of the parameters and of $N_i = 2|z_i|^2$ and $\beta = |z_1|^{2m}|z_3|^{2l}$.

2.3. Bifurcations. Using the Birkhoff normal form of the vector field X obtained in the previous subsection, we state conditions on the coefficients that guarantee that the required Hopf bifurcations occur. Similarly to the 6-dimensional Hopf/Hopf mode interaction, we shall use amplitude-phase equations to further simplify the normal forms. The amplitude equations are equivariant under the action of groups smaller than the original one. The study of the amplitude equations provides a lower-dimensional setting for the problem and reduces its bifurcation analysis, including bifurcation diagrams, to previously studied problems. Results for the original problem are obtained by restoring the phase as described below.

Single Hopf. We assume that $p(0) = 0$, $q(0) = 1$ and $p_\lambda(0) \neq 0$ to ensure genericity of the bifurcation. The branch of standing waves is given by

$$\lambda = -\frac{2p_N(0)}{p_\lambda(0)}a^2 + \text{higher-order terms},$$

where the subscripts indicate derivatives and $a \in \mathbb{R}$ refers to the orbit representative for standing-waves which is (a, a). Writing $z = xe^{i\psi}$, where $x \in \mathbb{R}$ is the amplitude and ψ the phase, we obtain the following amplitude-phase equations

$$\dot{x} + h(x, \lambda) = 0,$$
$$\dot{\psi} + q(x, \lambda) = 0,$$

with $h(x, \lambda) = p(x, \lambda)x$. Up to degree two, we have $dh = 2p_N(0)x^2$, meaning that the stability of the branch of standing waves is uniquely determined by the sign of $p_N(0)$ ($p_N(0) > 0$ corresponds to a supercritical stable branch and $p_N(0) < 0$ to a subcritical unstable one). This is all the information required to draw the bifurcation diagram in the non-degenerate case.

Furthermore, we can study the degenerate case by observing that the amplitude equation possesses symmetry $\mathbb{Z}_2$ and thus the solutions are those of the $\mathbb{Z}_2$-symmetric problems presented by Golubitsky and Schaeffer in [5], chapter VI. Note that the branches of solutions are to be interpreted as branches of periodic solutions, after having restored the phase.

Steady-state/Hopf mode interaction. The Birkhoff normal form, as remarked above, depends on the mode numbers l and m. We assume that $c_1(0) = 0$, $p_1(0) = 0$ and $q_1(0) = 1$. Nondegeneracy conditions can be found in Table XX, 2.6 of Golubitsky et al. [7]. We divide this section according to the mode numbers as before.

Mode numbers $l = m = 1$. If we write $x_0 = x$ and $x_1 e^{i\psi} = z$, where x_1 is the amplitude and ψ the phase, we obtain the following amplitude-phase equations

$$\begin{aligned} \dot{x}_0 + (c_1 + c_3 x_1^2)x_0 &= 0, \\ \dot{x}_1 + (p_1 + p_3 x_0^2)x_1 &= 0, \\ \dot{\psi} + q_1 + q_3 x_0^2 &= 0. \end{aligned}$$

The above amplitude equations have $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetry. The study of problems equivariant under this group is the content of chapter X in Golubitsky and Schaeffer [5]. When interpreting the solutions obtained by Golubitsky and Schaeffer, the phase needs to be restored. We remark that even the most generic normal form possesses modal parameters which condition the bifurcation diagrams. For all parameter values, both a branch of steady-state solutions and a branch of periodic, standing wave solutions are present. For certain regions in modal parameter space, mixed-mode branches can be found and a secondary Hopf bifurcation may take place along the mixed-mode branch.

Mode numbers m and l coprime. If m is odd then the amplitude equations again have $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetry. This case is therefore analogous to the previous one, with mode numbers both equal to unity. If m is even, the amplitude equations are

$$\begin{aligned} \dot{x} + p_1 x + p_3 x^{l-1} y^m &= 0, \\ \dot{y} + q_1 y + q_3 x^l y^{m-1} &= 0. \end{aligned}$$

These equations are $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetric with additional symmetry-breaking terms given by $p_3 x^{l-1} y^m$ and $q_3 x^l y^{m-1}$. Such problems have been studied by Armbruster and Dangelmayr in [1], with the mode numbers interchanged. Here, again, modal parameters appear in the least degenerate normal form producing a rich variety of bifurcation diagrams.

We finish what concerns steady-state/Hopf mode interactions with the observation, made by Hill and Stewart [10], that if the mode numbers are $(m, l) = (2, 1)$ there is a tertiary Hopf bifurcation from the branch of standing waves, for some values of the unfolding parameter. This is preserved under the restriction on the boundary.

Hopf/Hopf mode interactions. We divide this paragraph according to the dimension of the representation for the group action. We assume non-resonance of the eigenvalues $\pm i\omega_0$ and $\pm i\omega_1$, that is, we assume that ω_0/ω_1 is irrational.

The 6-dimensional case. In order to guarantee the occurrence of the Hopf/Hopf mode interaction, we assume $p_0(0) = 0$, $p_1(0) = 0$, $q_0(0) = \omega_0$ and $q_1(0) = \omega_1$.

Writing $z_0 = xe^{i\xi}$ and $z_1 = ye^{i\zeta}$, we obtain the following amplitude-phase equations

$$\begin{aligned} \dot{x} + p_0 x &= 0, \\ \dot{y} + p_1 y &= 0, \\ \dot{\xi} + q_0 &= 0, \\ \dot{\zeta} + q_1 &= 0. \end{aligned}$$

The amplitude equations possess $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetry and can be studied from Chapter X in [5]. Note that the mixed-mode solutions correspond to motion on a 2-torus.

The 8-dimensional case. Changing to amplitude-phase equations, by writing $z_1 = xe^{i\xi}$ and $z_3 = ye^{i\zeta}$, we obtain

$$\begin{aligned} \dot{x} + (p_1 + r_1 x^{2m-2} y^{2l})x &= 0, \\ \dot{y} + (p_3 + r_3 x^{2m} y^{2l-2})y &= 0, \\ \dot{\xi} + q_1 + s_1 x^{2m-2} y^{2l} &= 0, \\ \dot{\zeta} + q_3 + s_3 x^{2m} y^{2l-2} &= 0. \end{aligned}$$

These amplitude equations again have $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetry.

3. Heteroclinic connections

This section addresses the existence of heteroclinic connections between two periodic solutions or between a periodic solution and an equilibrium. We stress that neither type of heteroclinic connection is possible in scalar problems with NBC (see Fiedler et al. [4] who address this type of connection between rotating waves in PBC problems). We prove the existence of heteroclinic connections involving periodic solutions in steady-state/Hopf mode interactions with both mode numbers equal to unity and in Hopf/Hopf mode interactions. Recall that reducing to amplitude-phase equations leads to a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivariant bifurcation problem for the amplitude equations. We use the approach of Melbourne et al. [11].

Theorem 3.1. *For the NBC problems described above, there exists an open set of values for the coefficients in the normal form of X, such that the asymptotic behaviour of solutions is described by a heteroclinic connection between*

- *a standing wave and a steady-state in the case of a steady-state/Hopf mode interaction;*
- *two standing waves in the case of a Hopf/Hopf mode interaction.*

Proof. According to Theorem X, 2.4 in Golubitsky and Schaeffer [5], the non-degenerate normal form for a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivariant problem is

$$\dot{x} = (\varepsilon_2\lambda + my^2 + \varepsilon_1 x^2)x,$$
$$\dot{y} = (\varepsilon_4\lambda + \varepsilon_3 y^2 + nx^2)y,$$

where m and n are modal parameters, $\varepsilon_i = \pm 1$ and the mode interaction may be unfolded by adding $\varepsilon_4\sigma y$ to the second equation and small perturbations to the modal parameters. The equilibria for this normal form are the origin, one equilibrium on the horizontal axis, one on the vertical axis and, for certain parameter values, equilibria outside the axes corresponding to mixed-mode solutions.

In these coordinates, the fixed-point spaces of the two non-trivial isotropy subgroups of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ are the coordinate axes. For parameter values such that there are no mixed-mode branches, these axes are adjacent (i.e., there is no other invariant line in a wedge region defined by them; see Definition 2.2 in [11]) and there are no equilibria outside of them. Then, provided that the origin is unstable and solutions remain bounded inside the region defined by the fixed-point spaces, the Poincaré–Bendixson Theorem guarantees the existence of a connection from the equilibrium in one axis to the equilibrium in the other.

In order to prove boundedness of solutions, we use Proposition 2.6 in Melbourne et al. [11]. Note that, due to the different way in which differential equations are written by Golubitsky and Schaeffer [5] (using $\dot{x} + f(x,\lambda) = 0$) and Melbourne et al. [11] (using $\dot{x} = f(x,\lambda)$), the correspondence between the coefficients used in Proposition 2.6 in [11] and the normal form in [5] is as follows: $a_1 = -\varepsilon_2$, $b_1 = -\varepsilon_1$, $c_1 = -m$, $a_2 = -\varepsilon_4$, $b_2 = -\varepsilon_3$ and $c_2 = -n$. Thus, case A in [5] considers values for the coefficients so that Proposition 2.6 in [11] applies and solutions are bounded provided $m + n > -2$. This defines an open set of parameters in the modal plane in which a heteroclinic trajectory connects those solutions (saddle-sink connection) that correspond to the non-trivial equilibria on the coordinate axes. After restoring the phase, these solutions are steady-states on the horizontal axis and standing waves on the vertical axes, in the steady-state/Hopf mode interaction, and two standing waves in either instance of the Hopf/Hopf mode interaction.

The heteroclinic connections just described exist in the original PDE. In fact, they are structurally stable connections taking place inside a (invariant) fixed-point space and therefore they persist under transformations which preserve the symmetry. □

We do not address the existence of connections involving mixed-mode solutions or secondary branches. Also notice that the heteroclinic cycles found by Melbourne et al. [11] do not occur in NBC problems due to the simplicity of the isotropy lattice. Nevertheless, the theorem above shows how differently solutions to NBC problems behave when going from the scalar to the simplest vector-valued case.

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