

Sharp estimates for periodic solutions to the Euler–Poisson–Darboux equation

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Abstract. We establish sharp estimates for distributional solutions to the Euler–Poisson–Darboux equation posed in a periodic domain. These equations are highly singular, and setting the Cauchy problem requires a precise understanding of the nature of the singularities that may arise in weak solutions. We consider initial data in a space of functions with fractional derivatives such that weak solutions are solely integrable, and we derive sharp continuous dependence estimates for solutions to the initial-value problem. Our results strongly depend on a key parameter arising in the Euler–Poisson–Darboux equation.

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1. Introduction

1.1. Aim of this paper. In this paper we establish sharp regularity estimates for periodic solutions to the (highly singular) Cauchy problem associated with the Euler–Poisson–Darboux (EPD) equation

$$u_{tt}(t, \theta) + \frac{2\omega}{t}u_t(t, \theta) - u_{\theta\theta}(t, \theta) = 0, \quad (1)$$

$$u_t(0, \theta) = u_1(\theta), \quad u(0, \theta) = u_2(\theta), \quad (2)$$

where ω is a real parameter, and $(t, \theta) \in (0, +\infty) \times (0, 2\pi)$. Since this is a singular initial-value problem, it is not surprising that the solutions of this equation are singular (in some sense) as one approaches the singularity $t = 0$. In this perspective, the initial data u_1, u_2 may be regarded as the coefficients of a singular asymptotic expansion of the solution as $t \rightarrow 0$. Solving the above Cauchy problem is equivalent to validating such an asymptotic expansion. Note that, at this stage, (2) is

only defined formally and, as we will see, the parameter ω should be involved in a rigorous formulation of the initial data.

One central question of interest in the present paper concerns the choice of appropriate spaces of initial data u_1, u_2 ; we are especially interested in ensuring that solutions to the above Cauchy problem belong to the space L^1 . To this end, we introduce an appropriate class of function spaces, denoted below by $W_{\text{per}}^{\omega,1}(0, 2\pi)$, which are variants of Sobolev spaces and yield us the desired optimal regularity statement, i.e., the solution operator associated with (1)–(2) maps $W_{\text{per}}^{\omega,1}(0, 2\pi)$ onto a subset of L^1 . We recall that EPD equations provide a typical example of singular Cauchy problem, and have served as a paradigm for the theory of singular and degenerate Cauchy problems. For various results on such equations, see the book [4] as well as [1], [2], [5].

1.2. Main result of this paper. To begin with let us assume that $u_1 = 0$ and start with the observation that the function

$$t^{1-2\omega}(t^2 - \theta^2)_+^{\omega-1}$$

is a solution to the EPD equation (1) in the classical sense, at least away from the singular lines $t = |\theta|$. Here, f_+ denotes the positive part of f . Let u_2 be any sufficiently smooth, 2π -periodic function. Since the equation under consideration is linear, the convolution

$$U_2^\omega(t, \theta) := \int_{\mathbb{R}} t^{1-2\omega}(t^2 - (\theta - \theta')^2)_+^{\omega-1} u_2(\theta') d\theta' \tag{3}$$

is still a solution to (1), at least for those values of ω for which this integral is well-defined in a classical sense. Next, by a change of variables in (3), we obtain

$$U_2^\omega(t, \theta) = \int_{-1}^1 (1 - x^2)_+^{\omega-1} u_2(\theta + tx) dx \tag{4}$$

and, moreover,

$$C_{\omega-1} U_2^\omega(0, \theta) = u_2(\theta)$$

with $C_{\omega-1} = (\int_{-1}^1 (1 - x^2)_+^{\omega-1} dx)^{-1}$, as can easily be seen by letting t tend to zero.

At this stage, a natural question arises whether one can still give an appropriate meaning to the expression (4) when the function $(1 - x^2)_+^{\omega-1}$ is not integrable and, if we can do so, whether this expression still provides a solution (in a suitable sense) to the equation (1). As we show here, the answer is closely related to properly choosing the regularity space for the data u_1 . With this in mind, our objectives are describing such an optimal function space, deriving regularity estimates

for general solutions, and rigorously validating the corresponding asymptotic expansion.

The key point to observe is that the solution given by (4), for instance, takes the form of a scaled convolution in which the convolution kernel $(1 - x^2)_+^{\omega-1}$ has singularities of the type $x_+^{\omega-1}$. It is well known [6] that convolving a distribution u with a (suitably normalized) kernel of the form $x_+^{\omega-1}$ amounts to taking a fractional derivative (or integral) of order $-\omega$ of u , which we denote here by $D^{-\omega}u$.

In view of this fact, when trying to determine the optimal space $E = E(\omega)$ for the data u_2 and when imposing that the formal solution (4) remains in $L^1(0, 2\pi)$ for $t > 0$, one should use the close relation of these kernels with fractional derivatives and fractional integrals. That is, we need a rigorous version of the formal argument

$$U_2^\omega(t) \sim D^{-\omega}u_2 \in L^1(0, 2\pi) \iff u_2 \in E(\omega),$$

which suggests that $E(\omega)$ should be a suitable generalization of the Sobolev spaces $W^{k,1}$ of distributions u such that $D^k u$ is an integrable function.

This leads us here to define the spaces $W_{\text{per}}^{\omega,1}(0, 2\pi)$ which are suitable variants of the usual Sobolev spaces for periodic functions. They allow us, on one hand, to validate an asymptotic expansion for solutions with non-smooth initial data, and on the other hand, to determine the space of initial data for which solutions remain integrable for all positive times.

Observe next that the function

$$(t^2 - \theta^2)_+^{-\omega},$$

and, similarly, the convolution

$$U_1^\omega(t, \theta) = \int_{-1}^1 t^{1-2\omega} (1 - x^2)_+^{-\omega} u_1(\theta + tx) dx, \tag{5}$$

are also, formally at least, solutions of the EPD equation. Moreover, we have

$$\begin{aligned} C_{-\omega} t^{2\omega-1} U_1^\omega(t, \theta) &\rightarrow u_1(\theta), \\ t^{2\omega} (1 - 2\omega)^{-1} C_{-\omega} U_{1,t}^\omega(t, \theta) &\rightarrow u_1(\theta), \end{aligned}$$

provided these expressions make sense. This shows that for *smooth* data u_1, u_2 and for $\omega \in (0, 1)$, the function

$$u(t, \theta) := C_{-\omega} U_1^\omega(t, \theta) + C_{\omega-1} U_2^\omega(t, \theta) \tag{6}$$

is a solution to the EPD equation (2), and the asymptotic expansions

$$\begin{aligned} u(t, \theta) - t^{1-2\omega} u_1(\theta) - u_2(\theta) &= o(1), & t \rightarrow 0, \\ t^{2\omega} (1 - 2\omega)^{-1} u_t(t, \theta) - u_1(\theta) &= o(1), & t \rightarrow 0, \end{aligned} \tag{7}$$

hold pointwise. In this paper we generalize this result to *non-smooth* initial data and general exponents ω .

1.3. Outline of this paper. In Section 2 below, we define the singular distributions required for defining the fundamental kernels to the equation (1) and the fractional regularity spaces $W_{\text{per}}^{\lambda,1}(0, 2\pi)$. First we consider the classical fractional derivative kernel $\Phi_\lambda = x_+^{\lambda-1}/\Gamma(\lambda)$, whose main properties we recall. Next, we consider the truncated distribution $\gamma\Phi_\lambda$, where γ is a suitable cut-off function. This distribution allows us to introduce fractional derivatives of periodic functions which, due to support restrictions, cannot otherwise be convolved with the traditional fractional derivative kernel. Several properties of these distributions are described in Lemma 2.2 below. We then consider the distribution Ψ_λ , which is a normalized version of the singular distribution $(1 - x^2)_+^{\lambda-1}$. This distribution essentially represents the explicit solutions to the EPD equation, as shown in (4) and (5) above.

Next, in Section 3 we provide the definition of the spaces $W_{\text{per}}^{\lambda,1}(0, 2\pi)$, which relies on the truncated distribution $\gamma\Phi_{-\lambda}$. We study basic properties of these spaces in Lemmas 3.1 and 3.2 and then derive key estimates relating the distribution $\Psi_{-\lambda}$ with the spaces $W_{\text{per}}^{\lambda,1}(0, 2\pi)$. These estimates take the form

$$\|\Psi_{-\lambda} * T\|_{L^1(0, 2\pi)} \lesssim \|T\|_{\lambda,1}, \tag{8}$$

where T is a periodic distribution. Here and in what follows the constant implied in the notation \lesssim is independent of T . This estimate provides a continuous embedding of the space $W_{\text{per}}^{\lambda,1}(0, 2\pi)$ into the space of periodic distributions T for which $\Psi_{-\lambda} * T \in L^1(0, 2\pi)$. Since the solutions of the EPD equation are closely related to the convolution appearing in this estimate, it is this estimate which ultimately yields the optimal regularity result of interest.

Finally, in Section 4 we are in a position to handle the EPD equation (1) and we rigorously define its solutions using the singular distributions Ψ_ω . The formula here takes the form

$$Q^\omega = t^{1-2\omega} \sigma_{1/t}(\Psi_{1-\omega} * \sigma_t u_1) + \sigma_{1/t}(\Psi_\omega * \sigma_t u_2),$$

where σ is a scaling operator. This formula provides a rigorous meaning to the formal expression (6) for arbitrary parameter values ω and non-smooth initial data u_1, u_2 . We then derive our key estimates, in the spirit of (8). All these results come together in our main result, Theorem 4.4, where for all values of ω outside a

discrete exceptional set \mathcal{E} , we validate asymptotic expansions for solutions to the EPD equation.

The analysis outlined above gives only partial results for the case $\omega = 1/2$, and does not allow us to consider the case $\omega \in \mathcal{E}$, since the distribution Ψ_ω is not defined for these values. Therefore, the final two sections of this paper deal with these exceptional cases; see the discussion in Section 5 and Theorem 6.1.

2. A class of singular distributions

2.1. The distributions Φ_λ . In this section, we present standard material about singular distributions in one space variable. We refer to Gelfand–Shilov [6] and Hörmander [7] for further details. We denote by $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}'(\mathbb{R})$ the space of C^∞ functions with compact support and the space of distributions on \mathbb{R} , respectively. In particular, $\delta^{(k)}$ stands for the k -th derivative of the Dirac distribution, that is, $\langle \delta^{(k)}, \varphi \rangle = (-1)^k \varphi^{(k)}(0)$. We also set

$$\mathbb{N} := \{1, 2, 3, \dots\}, \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad \mathbb{Z} := \{\dots, -1, 0, 1, \dots\},$$

and $[\mu]$ is the integer part of μ satisfying by definition $[\mu] \leq \mu < [\mu] + 1$.

We recall some properties of the gamma and beta functions which will be useful throughout. The gamma function is defined by $\Gamma(\lambda) = \int_0^{+\infty} e^{-\xi} \xi^{\lambda-1} d\xi$ for $\lambda > 0$ and using that

$$\Gamma(\lambda + 1) = \lambda \Gamma(\lambda), \quad -\lambda \in \mathbb{R} \setminus \mathbb{N}_0,$$

(which implies $\Gamma(k + 1) = k!$ for $k = 0, 1, 2, \dots$) this function can be extended by analytic continuation to all $\lambda \in \mathbb{R} \setminus -\mathbb{N}_0$. The Gamma function blows up at every non-positive integer $-k$ and

$$\lim_{\lambda \rightarrow -k} (\lambda + k) \Gamma(\lambda) = (-1)^k / k!, \quad k \in \mathbb{N}_0, \tag{9}$$

and satisfies the duplication formula

$$\Gamma(\lambda) \Gamma(\lambda + 1/2) = 2^{1-2\lambda} \sqrt{\pi} \Gamma(2\lambda). \tag{10}$$

Furthermore, if a, b and $a + b$ are not negative integers we define the beta function by

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)},$$

which is also be given by the integral

$$B(a, b) = \int_0^1 \xi^{a-1} (1 - \xi)^{b-1} d\xi, \quad a, b > 0.$$

By taking $a = b$ and using the duplication formula (10) we find

$$B(a, a) = 2^{1-2a} B(a, 1/2). \tag{11}$$

Recall that the convolution of two distributions f, g satisfying certain assumptions on their support (either one of the supports is bounded or they are both bounded on the same side) is the distribution $f * g$ defined by

$$\langle f * g, \varphi \rangle := \langle f \otimes g, \varphi(x + y) \rangle = \langle f, \langle g, \varphi(x + y) \rangle \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}). \tag{12}$$

Denote by $f_+ = \max(f, 0)$ the positive part of a function f . We want to define the “function” $\xi_+^{\lambda-1}$ as a distribution normalized to be of unit mass, that is formally:

$$\Phi_\lambda = \begin{cases} \frac{\xi_+^{\lambda-1}}{\Gamma(\lambda)}, & -\lambda \notin \mathbb{N}_0, \\ \delta^{(k)}, & -\lambda = k \in \mathbb{N}_0. \end{cases} \tag{13}$$

The following proposition provides a rigorous definition.

Proposition 2.1 (Definition and properties of the distributions Φ_λ). *The following formula defines a one-parameter family of distributions supported on the half-line $[0, \infty)$:*

$$\Phi_\lambda := \begin{cases} \frac{(-1)^k}{\Gamma(\lambda + k)} \int_{\mathbb{R}} \xi_+^{\lambda-1+k} \varphi^{(k)}(\xi) d\xi, & -\lambda \notin \mathbb{N}_0, k := [-\lambda + 1]_+, \\ (-1)^n \varphi^{(n)}(0), & -\lambda = n \in \mathbb{N}_0, \end{cases}$$

for $\varphi \in \mathcal{D}(\mathbb{R})$. Moreover, they satisfy the normalization $\langle \Phi_\lambda, e^{-\xi} \rangle = 1$ and, provided the convergence, derivative, and convolution are understood in the sense of distributions, the following properties hold for all $\lambda, \lambda' \in \mathbb{R}$:

- (1) Φ_λ depends continuously upon λ , that is, $\lim_{\lambda' \rightarrow \lambda} \Phi_{\lambda'} = \Phi_\lambda$,
- (2) $\Phi_\lambda * \Phi_{\lambda'} = \Phi_{\lambda+\lambda'}$,
- (3) $\frac{d}{d\xi} \Phi_\lambda = \Phi_{\lambda-1}$.

Proof. Step 1. Defining $\xi_+^{\lambda-1}$ as a finite part. First of all, when $\lambda > 0$ the function $\xi_+^{\lambda-1}$ is locally integrable and, therefore, determines a distribution on \mathbb{R} . For $\lambda \leq 0$ however, the function $\xi_+^{\lambda-1}$ is not integrable at the origin. For values $-\lambda \in \mathbb{R} \setminus \mathbb{N}_0$

with $k := [-\lambda + 1]_+$ (so that $\lambda - 1 + k \in (-1, 0)$) we can introduce a distribution still denoted by $\xi_+^{\lambda-1}$ by

$$\langle \xi_+^{\lambda-1}, \varphi \rangle := \frac{(-1)^k}{(\lambda - 1 + k) \dots \lambda} \int_0^{+\infty} \xi^{\lambda-1+k} \varphi^{(k)}(\xi) d\xi \tag{14}$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$. By integration by parts an equivalent definition of the distribution $\xi_+^{\lambda-1}$ is found:

$$\langle \xi_+^{\lambda-1}, \varphi \rangle = \int_0^{+\infty} \xi^{\lambda-1} \left(\varphi(\xi) - \varphi(0) - \xi \varphi'(0) - \dots - \frac{\xi^{k-1}}{(k-1)!} \varphi^{(k-1)}(0) \right) d\xi.$$

To motivate the above definition, for $-\lambda \in \mathbb{R} \setminus \mathbb{N}_0$ with $k := [-\lambda + 1]_+$ we can compute

$$\langle \xi_+^{\lambda-1+k}, \varphi^{(k)} \rangle = \int_{\mathbb{R}} \xi_+^{\lambda-1+k} \varphi^{(k)}(\xi) d\xi = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{+\infty} \xi^{\lambda-1+k} \varphi^{(k)}(\xi) d\xi$$

and, by integrating by parts,

$$\begin{aligned} & \int_{\varepsilon}^{+\infty} \xi^{\lambda-1+k} \varphi^{(k)}(\xi) d\xi \\ &= -(\lambda + k - 1) \int_{\varepsilon}^{+\infty} \xi^{\lambda+k-2} \varphi^{(k-1)}(\xi) d\xi - \varepsilon^{\lambda+k-1} \varphi^{(k-1)}(\varepsilon) \\ &= (\lambda + k - 1)(\lambda + k - 2) \int_{\varepsilon}^{+\infty} \xi^{\lambda+k-3} \varphi^{(k-2)}(\xi) d\xi - \varepsilon^{\lambda+k-1} \varphi^{(k-1)}(\varepsilon) \\ &\quad + (\lambda + k - 1) \varepsilon^{\lambda+k-2} \varphi^{(k-2)}(\varepsilon), \end{aligned}$$

and so on, until

$$\begin{aligned} & \int_{\varepsilon}^{+\infty} \xi^{\lambda-1+k} \varphi^{(k)}(\xi) d\xi \\ &= (-1)^k (\lambda + k - 1)(\lambda + k - 2) \dots \lambda \int_{\varepsilon}^{+\infty} \xi^{\lambda-1} \varphi(\xi) d\xi \\ &\quad - \varepsilon^{\lambda+k-1} \varphi^{(k-1)}(\varepsilon) + (\lambda + k - 1) \varepsilon^{\lambda+k-2} \varphi^{(k-2)}(\varepsilon) + \dots \\ &\quad + (-1)^{k-1} (\lambda + k - 1) \dots (\lambda + 1) \varepsilon^{\lambda} \varphi(\varepsilon). \end{aligned}$$

In the above identity, we observe that the first singular term (for instance) takes the form

$$\begin{aligned} \varepsilon^{\lambda+k-1} \varphi^{(k-1)}(\varepsilon) &= \varepsilon^{\lambda+k-1} (\varphi^{(k-1)}(\varepsilon) - \varphi^{(k-1)}(0)) + \varepsilon^{\lambda+k-1} \varphi^{(k-1)}(0) \\ &= C_0 \varepsilon^{\lambda+k-1} + \mathcal{O}(\varepsilon^{\lambda+k}), \end{aligned}$$

where C_i (depends on φ and) denote constants that may change at each occurrence and can always be computed explicitly. Similarly, each singular term in $\varepsilon^{\lambda-1+j}$ ($j = 1, \dots, k$) takes the form

$$C_j \varepsilon^{\lambda+j-1} + \dots + C_k \varepsilon^{\lambda+k-1} + \mathcal{O}(\varepsilon^{\lambda+k}),$$

so that clearly the most singular term is of the order ε^λ , as expected.

Therefore, we may write

$$\int_\varepsilon^{+\infty} \xi^{\lambda-1} \varphi(\xi) d\xi = C_1 \varepsilon^\lambda + \dots + C_k \varepsilon^{\lambda+k-1} + \frac{(-1)^k}{(\lambda+k-1)\dots\lambda} \int_\varepsilon^{+\infty} \xi^{\lambda+k-1} \varphi^{(k)}(\xi) d\xi + \mathcal{O}(\varepsilon^{\lambda+k}).$$

Note that the first k terms contain singular powers of ε , while the other ones tend to zero with ε (since $\lambda+k > 0$). This leads us to define the distribution $\xi_+^{\lambda-1}$, for $\lambda < 0$, $-\lambda \in \mathbb{R} \setminus \mathbb{N}$, as the coefficient of the finite term in the above expansion of the integral and precisely leads us to (14). (The above derivation also justifies the terminology “finite part” of the divergent integral $\int_0^{+\infty} \xi^{\lambda-1} \varphi(\xi) d\xi$).

Step 2. Defining the distribution Φ_λ . Observing that the expression (14) is singular when $-\lambda \in \mathbb{N}$, it is convenient to normalize the distribution $\xi_+^{\lambda-1}$ with the factor $1/\Gamma(\lambda)$. This leads us precisely to the definition (13) where the second line in (13) will now be justified as we check the properties of Φ_λ stated in the proposition. Note that the definition (13) may be restated as

$$\langle \Phi_\lambda, \varphi \rangle := (-1)^k \langle \Phi_{\lambda+k}, \phi^{(k)} \rangle, \tag{15}$$

where k is such that $\lambda+k \geq 0$.

If λ is not a negative integer or zero, the first result is clear from (14) and the continuity of the gamma function. If $-\lambda = n \in \mathbb{N}_0$, the function $\lambda \mapsto \langle \xi_+^{\lambda-1}, \varphi \rangle$ has a simple pole at each such value of λ . At each such value, the residue is easily computed from (14) to be

$$\lim_{\lambda' \rightarrow -n} (\lambda' + n) \langle \xi_+^{\lambda'-1}, \varphi \rangle = \frac{\varphi^{(n)}(0)}{n!} = \frac{(-1)^n}{n!} \langle \delta^{(n)}, \varphi \rangle.$$

Therefore, using (9), in the sense of distributions we find for $-\lambda = n$

$$\begin{aligned} \lim_{\lambda' \rightarrow \lambda} \Phi_{\lambda'} &= \lim_{\lambda' \rightarrow \lambda} \frac{1}{\Gamma(\lambda')} \xi_+^{\lambda'-1} \\ &= \frac{(-1)^n}{n!} \delta^{(n)} \lim_{\lambda' \rightarrow \lambda} \frac{1}{\Gamma(\lambda')(\lambda' + n)} = \delta^{(n)} = \Phi_\lambda, \end{aligned}$$

which establishes the item (1) of the proposition.

The second claim is contained in Lemma 2.2 below. For a more direct proof, see [6]. Finally, the third claim is actually a particular case of the second, since

$$\frac{d}{d\xi} \Phi_\lambda = \delta' * \Phi_\lambda = \Phi_{-1} * \Phi_\lambda = \Phi_{\lambda-1}.$$

Alternatively, we may also compute

$$\frac{d}{d\xi} \Phi_\lambda = \frac{d}{d\xi} \frac{\xi^{\lambda-1}}{\Gamma(\lambda)} = \frac{(\lambda-1)\xi^{\lambda-2}}{\Gamma(\lambda)} = \frac{\xi^{\lambda-2}}{\Gamma(\lambda-1)} = \Phi_{\lambda-1},$$

which is easily justified in the sense of distributions by relying on the expression (14) if $-\lambda \notin \mathbb{N}_0$, or on (13) otherwise. This completes the proof of Proposition 2.1. \square

2.2. The truncated distribution $\Phi_\lambda \gamma$. In this section we introduce a variant of the distribution Φ_λ , which consists of multiplying it by a regular cut-off function. The aim is to obtain distributions with the same regularity, but with compact support.

For definiteness, we choose the cut-off functions to be regularizations of the characteristic function of the interval $(-\infty, 1)$, $\chi_{(-\infty, 1)}$. Let ρ_ε denote the standard mollifier function, and set

$$\gamma := \rho_a * \chi_{(-\infty, 1)}, \tag{16}$$

for some fixed $a \in (0, 1)$. Thus, we consider the distributions $\Phi_\lambda \gamma$, which are simply the product of Φ_λ by the smooth function γ .

The group property with respect to the convolution will be lost, so we determine the resulting error term in the following lemma.

Lemma 2.2. *For all $\mu > \lambda \geq 0$ one has the (semi-group) property*

$$\Phi_{-\lambda} \gamma * \Phi_\mu \gamma = \Phi_{\mu-\lambda} \gamma_{\mu, -\lambda}, \tag{17}$$

and for $\mu = \lambda$,

$$\Phi_{-\lambda} \gamma * \Phi_\lambda \gamma = \delta + \gamma_\lambda, \tag{18}$$

where $\gamma_\lambda, \gamma_{\mu, -\lambda}$ are smooth functions with compact support which vanish for $x > 2 + 2a$. Moreover, $\text{supp } \gamma_\lambda \subset (1 - a, 2 + 2a)$, and if $-\lambda + k \in (0, 1]$ one has

$$\|\gamma_\lambda\|_{L^1(\mathbb{R})} \lesssim a^{-k}, \tag{19}$$

where a is given in (16).

Proof. Suppose first that $\lambda \in (0, 1)$. and take $\mu > \lambda$. For all $\varphi \in \mathcal{D}(\mathbb{R})$, according to (12) and (15), we have

$$\begin{aligned} \langle \Phi_{-\lambda}\gamma * \Phi_{\mu}\gamma, \varphi \rangle &= \langle \Phi_{-\lambda}(x) \otimes \Phi_{\mu}(y), \gamma(x)\gamma(y)\varphi(x + y) \rangle \\ &= -\langle \Phi_{-\lambda+1}(x) \otimes \Phi_{\mu}(y), \gamma'(x)\gamma(y)\varphi(x + y) + \gamma(x)\gamma(y)\varphi'(x + y) \rangle, \end{aligned}$$

and since $-\lambda + 1 > 0$, this expression is an actual integral:

$$-\frac{1}{\Gamma(1 - \lambda)\Gamma(\mu)} \int_{\mathbb{R}} \int_{\mathbb{R}} x_+^{-\lambda} y_+^{\mu-1} (\gamma'(x)\gamma(y)\varphi(x + y) + \gamma(x)\gamma(y)\varphi'(x + y)) \, dx \, dy.$$

Next, performing the changes of variables $x + y = s$ and $r = y/s$ leads to

$$\begin{aligned} -\frac{1}{\Gamma(1 - \lambda)\Gamma(\mu)} \int_0^{\infty} s^{\mu-\lambda} \int_0^1 (1 - r)^{-\lambda} r^{\mu-1} (\gamma'(s(1 - r))\gamma(sr)\varphi(s) \\ + \gamma(s(1 - r))\gamma(sr)\varphi'(s)) \, dr \, ds. \end{aligned}$$

Integration by parts in the second term and straightforward calculation yield

$$\begin{aligned} \langle \Phi_{-\lambda}\gamma * \Phi_{\mu}\gamma, \varphi \rangle &= \frac{\Gamma(\mu - \lambda + 1)}{\Gamma(1 - \lambda)\Gamma(\mu)} \left(\int_0^{\infty} \Phi_{\mu-\lambda+1}(s)\varphi(s) \int_0^1 (1 - r)^{-\lambda} r^{\mu} [\gamma'(sr)\gamma(s(1 - r)) \right. \\ &\quad \left. - \gamma(sr)\gamma'(s(1 - r))] \, dr \, ds \right. \\ &\quad \left. + \int_0^{\infty} \Phi_{\mu-\lambda}(s)\varphi(s) \int_0^1 (1 - r)^{-\lambda} r^{\mu-1} \gamma(sr)\gamma(s(1 - r)) \, dr \, ds \right). \end{aligned} \tag{20}$$

Thus, we find

$$\Phi_{-\lambda}\gamma * \Phi_{\mu}\gamma = \Phi_{\mu-\lambda}\gamma_{\mu, -\lambda},$$

with

$$\begin{aligned} \gamma_{\mu, -\lambda}(s) &= \frac{\Gamma(\mu - \lambda + 1)}{\Gamma(1 - \lambda)\Gamma(\mu)} \left(\frac{s}{\mu - \lambda} \int_0^1 (1 - r)^{-\lambda} r^{\mu} (\gamma'(sr)\gamma(s(1 - r)) \right. \\ &\quad \left. - \gamma(sr)\gamma'(s(1 - r))) \, dr \right. \\ &\quad \left. + \int_0^1 (1 - r)^{-\lambda} r^{\mu-1} \gamma(sr)\gamma(s(1 - r)) \, dr \right). \end{aligned}$$

Since $\gamma(s) = 0$ for $s > 1 + a$ and $\gamma'(s)$ is concentrated on $(1 - a, 1 + a)$, one checks immediately that the (smooth) function $\gamma_{\mu, -\lambda}$ vanishes for $s > 2 + 2a$. Moreover,

if we put $\gamma \equiv 1$ (so that $\Phi_\lambda \gamma = \Phi_\lambda$), we would find $\gamma_{\mu, -\lambda} \equiv 1$, which is consistent with the group property for Φ_λ in Proposition 2.1. This completes the proof of (17).

To derive (18) we return to (20) and pass to the limit $\mu \rightarrow \lambda$. Using the continuity (with respect to λ) of the distribution Φ_λ , the first double integral converges to

$$\langle \gamma_\lambda, \varphi \rangle := \left\langle \Phi_1 \int_0^1 \frac{(1-r)^{-\lambda} r^\lambda}{\Gamma(1-\lambda)\Gamma(\lambda)} (\gamma'(sr)\gamma(s(1-r)) - \gamma(sr)\gamma'(s(1-r))) dr, \varphi \right\rangle,$$

and this distribution is actually a smooth function supported in $(1-a, 2+2a)$. Deriving $\|\gamma_\lambda\|_{L^1(\mathbb{R})} \lesssim 1/a$ from this expression is immediate, using the properties of the beta function and the bound $|\gamma'| \leq C/a$.

For the second double integral in (20), observe, on the one hand, that for any $\lambda \in (0, 1)$, the function

$$\frac{\Gamma(\mu - \lambda + 1)}{\Gamma(1 - \lambda)\Gamma(\mu)} \int_0^1 (1-r)^{-\lambda} r^{\mu-1} \gamma(sr)\gamma(s(1-r)) dr$$

is unity for $s < 1 - a$, since in that range $\gamma(sr)\gamma(s(1-r)) = 1$ for all $r \in (0, 1)$ and from the properties of the beta function. Since, on the other hand, $\Phi_{\mu-\lambda} \rightarrow \Phi_0 = \delta$, passing to the limit gives (18). This completes the proof of the lemma for $\lambda \in (0, 1)$.

Extending the result to all $\lambda > 0$ is done by performing similar calculations for $\lambda \in (k, k + 1)$, successively, using the relation (15). The functions appearing instead of $\gamma_{\mu, -\lambda}$ have correspondingly more complex expressions, involving γ and its derivatives up to the order $k + 1$, but similar support and smoothness properties, inherited in the same way from the properties of the function γ and its derivatives. For completeness, we provide the expression of γ_λ for $\lambda \in (1, 2)$:

$$\begin{aligned} \gamma_\lambda(s) = s \int_0^1 \frac{(1-r)^{1-\lambda} r^{\lambda+1}}{\Gamma(2-\lambda)\Gamma(\lambda)} & (\gamma''(s(1-r))\gamma(sr) \\ & - 2\gamma'(s(1-r))\gamma'(sr) + \gamma(s(1-r))\gamma''(sr)) dr \\ + \int_0^1 \frac{(1-r)^{1-\lambda} r^\lambda}{\Gamma(2-\lambda)\Gamma(\lambda)} & 2(\gamma(s(1-r))\gamma'(sr) - \gamma'(s(1-r))\gamma(sr)) dr. \end{aligned} \tag{21}$$

As in the case $\lambda \in (0, 1)$, the bound (19) follows from the properties of the beta function and from $|\gamma''| \leq Ca^{-2}$. This completes the proof of Lemma 2.2. \square

Remark 2.3. Only minor changes to the proof of the previous lemma would lead to the following generalization of (18): for all smooth function α

$$(\Phi_{-\lambda}\gamma) * (\Phi_\lambda\alpha\gamma) = \alpha(0)\delta + \Phi_1 A_\lambda,$$

where the (smooth) function $\Phi_1 A_\lambda$ is now supported in $(0, 2 + 2a)$.

2.3. The distribution Ψ_λ . Given two reals $a, b \in \mathbb{R}$ we define the “scaling-translation” operator $\phi \in \mathcal{D}(\mathbb{R}) \mapsto \tau_{a,b}\phi \in \mathcal{D}(\mathbb{R})$ by

$$(\tau_{a,b}\phi)(\xi) = \phi(a\xi + b), \quad \xi \in \mathbb{R},$$

and, by duality, we define the operator $T \in \mathcal{D}'(\mathbb{R}) \mapsto \tau_{a,b}T \in \mathcal{D}'(\mathbb{R})$ by

$$\langle \tau_{a,b}T, \phi \rangle := \left\langle T, \frac{1}{a}\tau_{1/a, -b/a}\phi \right\rangle, \quad \phi \in \mathcal{D}(\mathbb{R}).$$

For $b = 0$, we have the scaling operator

$$\sigma_a\varphi(\xi) := \tau_{a,0}\varphi(\xi) = \varphi(a\xi).$$

Further, we denote by χ_1 the characteristic function $\chi_{(-\infty, 1)}$, and we define the set of exceptional values as

$$\mathcal{E} := \{-1/2, -3/2, \dots\}.$$

Proceeding as in the proof of Proposition 2.1 one can view the formal expressions $(1 - \xi^2)_+^{\lambda-1}$ as singular distributions. (See (24)–(25) below for the explicit formula.) Then, after normalization, we arrive at the following one-parameter family of distributions supported on the interval $[-1, 1]$:

$$\Psi_\lambda := \begin{cases} C_{\lambda-1}(1 - \xi^2)_+^{\lambda-1}, & -\lambda \in \mathbb{R} \setminus \mathbb{N}_0, \lambda \notin \mathcal{E}, \\ \pi^{-1/2}\Gamma(-n + 1/2)(1 + |\xi|)^{-n-1}(\delta_{\xi=-1}^{(n)} + (-1)^n\delta_{\xi=1}^{(n)}), & -\lambda = n \in \mathbb{N}_0, \end{cases} \tag{22}$$

which are defined for all values except $\lambda \in \mathcal{E}$. Here, we have set

$$C_{\lambda-1} := \frac{\Gamma(\lambda + 1/2)}{\Gamma(\lambda)\Gamma(1/2)}.$$

Proposition 2.4 (Properties of the distributions Ψ_λ). *In the sense of distributions and for all $\lambda \in \mathbb{R} \setminus \mathcal{E}$ the following properties hold:*

- (1) $\Psi_\lambda = \pi^{-1/2}\Gamma(\lambda + 1/2)(1 + |\xi|)^{\lambda-1}(\tau_{1,1}(\chi_1(\xi)\Phi_\lambda) + \tau_{-1,1}(\chi_1(\xi)\Phi_\lambda))$.
- (2) Ψ_λ depends continuously upon λ , that is, $\lim_{\lambda' \rightarrow \lambda} \Psi_{\lambda'} = \Psi_\lambda$.
- (3) For $\lambda = 1/2$, one may define the distribution $\tilde{\Psi}_{1/2} := \Psi_{1/2} \ln(1 - \xi^2)$ as the distributional limit

$$\lim_{\lambda' \rightarrow 1/2} \frac{\Psi_{\lambda'} - \Psi_{1-\lambda'}}{1 - 2\lambda'} = \Psi_{1/2} \ln(K_{1/2}(1 - \xi^2)), \tag{23}$$

where the constant $K_{1/2}$ is defined by

$$\ln K_{1/2} := -\langle \Psi_{1/2} \ln(1 - \xi^2), 1 \rangle.$$

$$(4) \quad \frac{d}{d\xi} \Psi_\lambda = (1 - 2\lambda)\xi \Psi_{\lambda-1}.$$

Remark 2.5. As shown in the proof given below, the (unnormalized) distributions $(1 - \xi^2)_+^{\lambda-1}$ are actually defined for $\lambda \in \mathcal{E}$. However, our normalization constants $C_{\lambda-1}$ blow up for exactly these values, so that the distributions Ψ_λ remain undefined for these values. In fact, it is not possible to provide a normalization ensuring continuity for all values of the parameter λ .

Proof. Step 1. Definition of $(1 - \xi^2)_+^{\lambda-1}$. We begin with the observation that for $\lambda \notin \{0, -1\}$ the function $\xi \mapsto (1 - \xi^2)_+^{\lambda-1}$ satisfies the algebraic equation

$$(1 - \xi^2)_+^{\lambda-1} = \frac{2\lambda + 1}{2\lambda} (1 - \xi^2)_+^\lambda + (4\lambda(\lambda + 1))^{-1} \frac{d^2}{d\xi^2} (1 - \xi^2)_+^{\lambda+1}.$$

This elementary fact can be used to define the distribution associated with the function $(1 - \xi^2)_+^{\lambda-1}$ whenever $\lambda < 0$ and therefore the functions are not locally integrable, as follows.

Suppose first that $\lambda > 0$. Multiplying the above identity by a test function $\varphi \in \mathcal{D}(\mathbb{R})$, integrating over \mathbb{R} and using integration by parts twice in the last term, we obtain

$$\begin{aligned} \int_{\mathbb{R}} (1 - \xi^2)_+^{\lambda-1} \varphi(\xi) d\xi &= \frac{2\lambda + 1}{2\lambda} \int_{\mathbb{R}} (1 - \xi^2)_+^\lambda \varphi(\xi) d\xi \\ &\quad + (4\lambda(\lambda + 1))^{-1} \int_{\mathbb{R}} (1 - \xi^2)_+^{\lambda+1} \varphi''(\xi) d\xi. \end{aligned}$$

Observe that all of these integrals exist in a classical sense since $\lambda > 0$.

Suppose next that $\lambda \in (-1, 0)$. We can no longer integrate as above, but we may nevertheless set

$$\begin{aligned} \langle (1 - \xi^2)_+^{\lambda-1}, \varphi \rangle &:= \frac{2\lambda + 1}{2\lambda} \langle (1 - \xi^2)_+^\lambda, \varphi \rangle \\ &\quad + (4\lambda(\lambda + 1))^{-1} \langle (1 - \xi^2)_+^{\lambda+1}, \varphi'' \rangle, \end{aligned} \tag{24}$$

which defines $(1 - \xi^2)_+^{\lambda-1}$ as a distribution.

It is now clear that (24) may be used as a recursive formula to define the distributions $(1 - \xi^2)_+^{\lambda-1}$ for any $\lambda < 0$, $-\lambda \notin \mathbb{N}_0$. Indeed, let $k := \lceil -\lambda - 1 \rceil$ be such that $\lambda - 1 + k \in (-1, 0)$, and let us iterate the equation (24) on each term in the right-hand side to obtain

$$\begin{aligned} \langle (1 - \xi^2)_+^{\lambda-1}, \varphi \rangle &= \left\langle (1 - \xi^2)_+^{\lambda-1+k}, \sum_{j=0}^{k+1} a_j(\lambda) \varphi^{(j)} \right\rangle \\ &\quad + \left\langle (1 - \xi^2)_+^{\lambda+k}, \sum_{j=0}^{k+1} b_j(\lambda) \varphi^{(j)} \right\rangle, \end{aligned} \tag{25}$$

for some reals $a_j(\lambda), b_j(\lambda)$. Clearly, in view of (24), these constants clearly blow up as one approaches $-\lambda \in \mathbb{N}_0$. The above expansion is not unique; for instance, in (24) one could integrate by parts once more the right-hand side and obtain equivalent expressions involving higher derivatives of φ .

Step 2. Normalizing the distribution $(1 - \xi^2)_+^{\lambda-1}$. We impose on the one hand that the normalized distribution, when applied to a function constant on $(-1, 1)$, returns that same constant, and on the other hand, that the singularities generated by the $a_j(\lambda), b_j(\lambda)$ when $\lambda \in \mathbb{N}_0$, are eliminated (see (25) above). To this end, we define the normalization constants

$$C_{\lambda-1} = \left(\int_{-1}^1 (1 - \xi^2)_+^{\lambda-1} d\xi \right)^{-1}.$$

For those values of λ for which this integral converges it is easy to see that $C_{\lambda-1} = 2^{1-2\lambda} B(\lambda, \lambda)^{-1}$. Using the definition of the beta function in terms of the gamma function and the formula (11), we find

$$C_{\lambda-1} = B(\lambda, 1/2)^{-1} = \frac{\Gamma(\lambda + 1/2)}{\Gamma(\lambda)\Gamma(1/2)}. \tag{26}$$

This expression may then be considered for any λ for which the right-hand side above is defined, that is, for $\lambda \notin \mathcal{E}$. Note that $C_{\lambda-1}$ blows up for these values of λ . Observe also that $C_{-k} = 0$ if k is a positive integer, and that if $\lambda \notin \mathcal{E}$,

$$\frac{C_\lambda}{C_{\lambda-1}} = \frac{2\lambda + 1}{2\lambda}. \tag{27}$$

After an easy computation, (24) becomes

$$\Psi_\lambda = \Psi_{\lambda+1} + \alpha(\lambda)\Psi''_{\lambda+2},$$

or, for $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\langle \Psi_\lambda, \varphi \rangle = \langle \Psi_{\lambda+1}, \varphi \rangle + \alpha(\lambda) \langle \Psi_{\lambda+2}, \varphi'' \rangle,$$

with

$$\alpha(\lambda) := ((2\lambda + 1)(2\lambda + 3))^{-1}.$$

In analogy with (25), we may iterate the formula above and obtain an induction relation used to define Ψ_λ when $\lambda + k \in (0, 1)$,

$$\Psi_\lambda = \sum_{j=0}^k \beta_j(\lambda) \Psi_{\lambda+k}^{(j)} + \sum_{j=2}^{k+1} \eta_j(\lambda) \Psi_{\lambda+k+1}^{(j)} \tag{28}$$

or, for $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\langle \Psi_\lambda, \varphi \rangle = \left\langle \Psi_{\lambda+k}, \sum_{j=0}^k \beta_j(\lambda) \varphi^{(j)} \right\rangle + \left\langle \Psi_{\lambda+k+1}, \sum_{j=2}^{k+1} \eta_j(\lambda) \varphi^{(j)} \right\rangle.$$

Here, the coefficients $\beta_j(\lambda)$, $\eta_j(\lambda)$, which clearly blow up for $\lambda \in \mathcal{E}$, satisfy

$$\beta_j(\lambda), \eta_j(\lambda) = 0 \quad \text{if } j \text{ is odd.}$$

Step 3. Proof of the proposition. To show claim (1) of the proposition, we begin by expressing the distribution $(1 - \xi^2)_+^{\lambda-1}$ in terms of the distribution $\xi_+^{\lambda-1}$. For all $\varphi \in \mathcal{D}(\mathbb{R})$ (recall the notation $\chi_1 = \chi_{(-\infty, 1)}$),

$$\begin{aligned} \langle (1 - \xi^2)_+^{\lambda-1}, \varphi \rangle &= \langle \xi_+^{\lambda-1}, (2 - \xi)^{\lambda-1} \chi_1(\xi) (\varphi(\xi - 1) + \varphi(1 - \xi)) \rangle \\ &= \langle (1 + |\xi|)^{\lambda-1} (\tau_{1,1}(\xi_+^{\lambda-1} \chi_1(\xi)) + \tau_{-1,1}(\xi_+^{\lambda-1} \chi_1(\xi))), \varphi \rangle. \end{aligned} \tag{29}$$

To show this, we rely on the uniqueness of analytic continuation. First, consider the function of a complex variable

$$\mathbb{C} \ni \lambda \mapsto \langle (1 - \xi^2)_+^{\lambda-1}, \varphi \rangle.$$

Observe now that if $\Re(\lambda) > 0$ (here \Re denotes the real part), the above function is analytic, and the equation (29) is valid (by linear changes of variables in the integral expressions). Therefore, by uniqueness of analytic continuation, the equation (29) holds for $-\Re(\lambda) \notin \mathbb{N}$. This completes the derivation of claim (1).

In view of (13), (29) and $\Gamma(1/2) = \sqrt{\pi}$, we immediately find

$$\begin{aligned} \lim_{\lambda \rightarrow -n} \langle \Psi^\lambda, \varphi \rangle &= \frac{\Gamma(-n + 1/2)}{\Gamma(1/2)} \langle (1 + |\xi|)_+^{-n-1} (\tau_{1,1} \delta^{(n)} + \tau_{-1,1} \delta^{(n)}), \varphi \rangle \\ &= \pi^{-1/2} \Gamma(-n + 1/2) \langle (1 + |\xi|)_+^{-n-1} (\delta_{\xi=-1}^{(n)} + (-1)^n \delta_{\xi=1}^{(n)}), \varphi \rangle. \end{aligned} \tag{30}$$

Claim (2) is thus established.

Next, we note that we can define the distribution $(1 - \xi^2)^{\lambda-1} \ln(1 - \xi^2)$, at least for $-\lambda \notin \mathbb{N}_0$. This may be done simply by observing that for such values of λ , and for any $\varphi \in \mathcal{D}(\mathbb{R})$, the function $\lambda \mapsto \langle (1 - \xi^2)^{\lambda-1}, \varphi \rangle$ is analytic and

$$\langle (1 - \xi^2)^{\lambda-1} \ln(1 - \xi^2), \varphi \rangle = \frac{d}{d\lambda} \langle (1 - \xi^2)^{\lambda-1}, \varphi \rangle.$$

We denote these distributions (multiplied by the normalization constant $C_{\lambda-1}$) by $\tilde{\Psi}_\lambda$. Let us now show the third claim of the lemma, (23). For this, simply note that the quotient indicated is equal to $\frac{d}{d\lambda} \Psi_\lambda /_{\lambda=1/2}$. The result follows by computing this derivative:

$$\begin{aligned} \frac{d}{d\lambda} \Psi_\lambda &= \Psi_\lambda \ln(1 - \xi^2) + (1 - \xi^2)^{\lambda-1} \frac{d}{d\lambda} C_{\lambda-1} \\ &= \Psi_\lambda \ln(1 - \xi^2) - \langle \ln(1 - \xi^2) \Psi_\lambda, 1 \rangle \Psi_\lambda, \end{aligned}$$

which, for the value $\lambda = 1/2$, gives (23) (in this computation we have omitted for simplicity the test function φ).

Finally, consider claim (4). First, note that for $-\lambda \in \mathbb{N}_0$ and $\lambda = 1$, this may be checked directly from (22). Otherwise, then the claim will follow if we show that, in the sense of distributions,

$$\frac{d}{d\xi} (1 - \xi^2)^{\lambda-1} = -2(\lambda - 1)\xi(1 - \xi^2)^{\lambda-2}. \tag{31}$$

In that case, for $\lambda \notin \mathcal{E}$, and using also (27), we find

$$\begin{aligned} \frac{d}{d\xi} \Psi_\lambda &= \frac{d}{d\xi} C_{\lambda-1} (1 - \xi^2)^{\lambda-1} \\ &= (1 - 2\lambda) C_{\lambda-2} \xi (1 - \xi^2)^{\lambda-2} = (1 - 2\lambda) \xi \Psi_{\lambda-1}. \end{aligned}$$

Now, (31) is clearly true if $\lambda > 1$, since the distributions in this case reduce to regular functions. Suppose next that $\lambda \in (0, 1)$. Then, a straightforward computation using the relation (24) shows (31). Clearly, one may now proceed similarly for $\lambda \in (-1, 0)$, and so on, for all $\lambda \notin \mathcal{E}$. This completes the proof of Proposition 2.4. □

3. Estimates in fractional Sobolev spaces

3.1. Notation and definition. Following Gelfand–Shilov [6], the derivative of order λ of a distribution T supported on the half-line \mathbb{R}^+ is defined by convolution with the kernel $\Phi_{-\lambda}$ (given in (13) above):

$$D^\lambda T := \Phi_{-\lambda} * T.$$

For $\lambda > 0$ we also use the notation

$$I^\lambda T := D^{-\lambda} T,$$

and refer to $I^\lambda T$ as the integral of order λ of the distribution T . We also use the short-hand notation $T^{(\lambda)}$ instead of $D^\lambda T$.

Using this notion of fractional derivative, one cannot define the derivative of a periodic distribution T , since $\text{supp } T \not\subset \mathbb{R}^+$ (apart from the trivial case $T = 0$). One way to extend the notion of fractional derivative to periodic distributions is to replace the convolution kernel Φ_λ with a new kernel having the same type of singularity at zero, but having compact support. This is done simply by multiplying Φ_λ by a cut-off function. This procedure has the advantage that the convolution of T with the new kernel still is a periodic distribution and, since Φ_{-k} has compact support if $k \in \mathbb{N}_0$, this notion of fractional derivative is consistent with usual (integer-order) derivative.

Recall from (16) the definition of the cut-off functions γ ,

$$\gamma := \rho_a * \chi_{(-\infty, 1)}.$$

Given $\lambda \geq 0$, the periodic fractional Sobolev space of order λ , denoted by $W_{\text{per}}^{\lambda, 1}(0, 2\pi)$, consists of all periodic distributions $T \in \mathcal{D}'_{\text{per}}(0, 2\pi)$ such that $(\gamma\Phi_{-\lambda}) * T$ belongs to the Lebesgue space $L^1(0, 2\pi)$, that is,

$$W_{\text{per}}^{\lambda, 1}(0, 2\pi) := \{T \in \mathcal{D}'_{\text{per}}(0, 2\pi) : (\gamma\Phi_{-\lambda}) * T \in L^1(0, 2\pi)\}.$$

Similarly, we define (for $\lambda \geq 0$) the negative Sobolev spaces

$$W_{\text{per}}^{-\lambda, 1}(0, 2\pi) := \{T \in \mathcal{D}'_{\text{per}}(0, 2\pi) : (\gamma\Phi_\lambda) * T \in L^1(0, 2\pi)\}.$$

Now we define the norms associated with these spaces, for all $\lambda > 0$,

$$\|T\|_{\lambda, 1} := \|(\gamma\Phi_{-\lambda}) * T\|_{L^1(0, 2\pi)} + \|T\|_{L^1(0, 2\pi)}, \tag{32}$$

$$\|T\|_{-\lambda, 1} := \|(\gamma\Phi_\lambda) * T\|_{L^1(0, 2\pi)} + \|\gamma_\lambda * T\|_{L^1(0, 2\pi)}, \tag{33}$$

where γ_λ is the function given by (18) in Lemma 2.2. Also, define the semi-norms

$$|T|_{\pm\lambda, 1} := \|(\gamma\Phi_{\mp\lambda}) * T\|_{L^1(0, 2\pi)}.$$

Note that for all periodic functions $T \in L^1(0, 2\pi)$ and all integrable functions g with compact support, one has

$$\|g * T\|_{L^1(0, 2\pi)} \leq \|g\|_{L^1(\mathbb{R})} \|T\|_{L^1(0, 2\pi)}.$$

The above definitions are justified by the following result.

Lemma 3.1. *Let $k \in \mathbb{N}_0$.*

- (1) *The space $W_{\text{per}}^{k,1}(0, 2\pi)$ coincides with the usual Sobolev space $W^{k,1}(S^1)$ of functions on the sphere S^1 whose k -th distributional derivative is in $L^1(S^1)$.*
- (2) *For every $\lambda \geq 0$, the space $W_{\text{per}}^{-\lambda,1}(0, 2\pi)$ coincides with the space given by*

$$\{T \in \mathcal{D}'_{\text{per}}(0, 2\pi) : f, g \in L^1(0, 2\pi), T = (\gamma\Phi_{-\lambda}) * f - g\}.$$

Moreover, one may take

$$f = (\gamma\Phi_{\lambda}) * T, \quad g = \gamma_{\lambda} * T,$$

with γ_{λ} given in Lemma 2.2.

- (3) *For every real $\lambda \leq \mu$, one has the continuous embedding*

$$W_{\text{per}}^{\mu,1} \subset W_{\text{per}}^{\lambda,1}.$$

Proof. The first claim is simply a consequence of the fact that, since $\gamma(0) = 1$, $\Phi_{-k} = \Phi_{-k}\gamma$.

To deal with the second claim, suppose that $T \in W_{\text{per}}^{-\lambda,1}(0, 2\pi)$. Set $f := (\Phi_{\lambda}\gamma) * T \in L^1(0, 2\pi)$, $g := \gamma_{\lambda} * T$ (cf. (18)). Using Lemma 2.2, we find

$$(\gamma\Phi_{-\lambda}) * f = (\gamma\Phi_{-\lambda}) * (\Phi_{\lambda}\gamma) * T = \delta * T + \gamma_{\lambda} * T = T + g.$$

which establishes one inclusion in (2). For the other inclusion, suppose that $T = (\gamma\Phi_{-\lambda}) * f - g$, with $f, g \in L^1(0, 2\pi)$. Then, from Lemma 2.2, we find

$$(\gamma\Phi_{\lambda}) * T = \delta * f + \gamma_{\lambda} * f - (\gamma\Phi_{\lambda}) * g \in L^1(0, 2\pi).$$

This shows the second claim of the lemma.

We now turn to the proof of the lemma's last claim. Let $\lambda, \mu \geq 0$, and let us first check that $W_{\text{per}}^{-\lambda,1}(0, 2\pi) \subset W_{\text{per}}^{-\mu,1}(0, 2\pi)$, with continuous embedding, if $\lambda < \mu$. Let $T \in W_{\text{per}}^{-\mu,1}(0, 2\pi)$. Using Lemma 2.2 we find

$$\begin{aligned} \|T\|_{-\mu,1} &= \|(\gamma\Phi_{\mu}) * T\|_{L^1(0,2\pi)} + \|\gamma_{\mu} * T\|_{L^1(0,2\pi)} \\ &\leq \|(\gamma\Phi_{-\lambda}) * (\gamma\Phi_{\mu}) * (\gamma\Phi_{\lambda}) * T\|_{L^1(0,2\pi)} + \|\gamma_{\lambda} * (\gamma\Phi_{\mu}) * T\|_{L^1(0,2\pi)} \\ &\quad + \|\gamma_{\mu} * T\|_{L^1(0,2\pi)}, \end{aligned}$$

thus

$$\begin{aligned} \|T\|_{-\mu,1} &\leq \|\Phi_{\mu-\lambda}\gamma_{\mu,-\lambda}\|_{L^1(\mathbb{R})} \|(\gamma\Phi_{\lambda}) * T\|_{L^1(0,2\pi)} + \|\Phi_{\mu}\gamma\|_{L^1(\mathbb{R})} \|\gamma_{\lambda} * T\|_{L^1(0,2\pi)} \\ &\quad + \|(\gamma\Phi_{-\lambda}) * \gamma_{\mu} * (\gamma\Phi_{\lambda}) * T\|_{L^1(0,2\pi)} + \|\gamma_{\lambda} * \gamma_{\mu} * T\|_{L^1(0,2\pi)} \end{aligned}$$

and so

$$\begin{aligned} \|T\|_{-\mu,1} &\leq \|\Phi_{\mu-\lambda}\gamma_{\mu,-\lambda}\|_{L^1(\mathbb{R})} \|(\gamma\Phi_\lambda) * T\|_{L^1(0,2\pi)} + \|\Phi_\mu\gamma\|_{L^1(\mathbb{R})} \|\gamma_\lambda * T\|_{L^1(0,2\pi)} \\ &\quad + \|(\gamma\Phi_{-\lambda}) * \gamma_\mu\|_{L^1(\mathbb{R})} \|(\gamma\Phi_\lambda) * T\|_{L^1(0,2\pi)} + \|\gamma_\mu\|_{L^1(\mathbb{R})} \|\gamma_\lambda * T\|_{L^1(0,2\pi)} \\ &\lesssim \|(\gamma\Phi_\lambda) * T\|_{L^1(0,2\pi)} + \|\gamma_\lambda * T\|_{L^1(0,2\pi)} = \|T\|_{-\lambda,1}. \end{aligned}$$

This shows the desired result for negative Sobolev spaces. Now, suppose $T \in W_{\text{per}}^{\mu,1}(0,2\pi)$. Again using Lemma 2.2, we find

$$\begin{aligned} \|T\|_{\lambda,1} &= \|(\gamma\Phi_{-\lambda}) * T\|_{L^1(0,2\pi)} + \|T\|_{L^1(0,2\pi)} \\ &\leq \|(\gamma\Phi_\mu) * (\gamma\Phi_{-\lambda})\|_{L^1(\mathbb{R})} \|(\gamma\Phi_{-\mu}) * T\|_{L^1(0,2\pi)} \\ &\quad + \|\gamma_\mu * (\gamma\Phi_{-\lambda})\|_{L^1(\mathbb{R})} \|T\|_{L^1(0,2\pi)} + \|T\|_{L^1(0,2\pi)} \end{aligned}$$

and thus

$$\|T\|_{\lambda,1} \lesssim \|(\gamma\Phi_{-\mu}) * T\|_{L^1(0,2\pi)} + \|T\|_{L^1(0,2\pi)} = \|T\|_{\mu,1}.$$

This completes the proof of Lemma 3.1. □

We now establish an alternative characterization of the spaces $W_{\text{per}}^{\lambda,1}(0,2\pi)$, for $\lambda \geq 0$. In particular, we show that $T \in W_{\text{per}}^{\lambda,1}(0,2\pi)$ if and only if the (classical) fractional derivative of $T\psi$ is integrable for every $\psi \in \mathcal{D}(\mathbb{R})$.

Lemma 3.2. *Supposing that $\lambda \geq 0$, one has*

$$W_{\text{per}}^{\lambda,1}(0,2\pi) = \{T \in \mathcal{D}'_{\text{per}}(0,2\pi) : \Phi_{-\lambda} * (T\psi) \in L^1(0,2\pi), \psi \in \mathcal{D}(\mathbb{R})\}.$$

Proof. Suppose that $T \in W_{\text{per}}^{\lambda,1}(0,2\pi)$. This is equivalent to saying that $T \in L^1(0,2\pi)$ and $(\gamma\Phi_{-\lambda}) * T \in L^1(0,2\pi)$ (cf. (32)). We want to show that

$$\|\Phi_{-\lambda} * (T\psi)\|_{L^1(0,2\pi)} \lesssim \|T\|_{\lambda,1}.$$

We have

$$\|\Phi_{-\lambda} * (T\psi)\|_{L^1(0,2\pi)} \leq \|((1-\gamma)\Phi_{-\lambda}) * (T\psi)\|_{L^1(0,2\pi)} + \|(\gamma\Phi_{-\lambda}) * (T\psi)\|_{L^1(0,2\pi)}.$$

Now, the first term in the right-hand side is bounded by $C(\psi)\|T\|_{L^1(0,2\pi)}$, since $T\psi \in L^1(\mathbb{R})$ and $(1-\gamma)\Phi_{-\lambda} \in L^1_{\text{loc}}$. Next, consider the second term. Observe that when $\lambda \in \mathbb{N}$, the result is obvious since in that case $\Phi_\lambda\gamma = \delta^{(k)}$, and $T \in W_{\text{per}}^{k,1}(0,2\pi)$. Suppose, then, that $\lambda \in (0,2) \setminus \{1\}$ (the general case follows similarly). For $\varphi \in \mathcal{D}(\mathbb{R})$ we have

$$\begin{aligned} \langle (\gamma\Phi_{-\lambda}) * (T\psi), \varphi \rangle &= \langle \Phi_{-\lambda} \otimes T, \psi(y)\gamma(x)\varphi(x+y) \rangle \\ &= \langle \Phi_{-\lambda} \otimes T, (\psi(x+y) - x\psi'(x+y) \\ &\quad + x^2\psi''(\xi))\gamma(x)\varphi(x+y) \rangle \end{aligned}$$

thus

$$\begin{aligned} \langle (\gamma\Phi_{-\lambda}) * (T\psi), \varphi \rangle &= \langle \Phi_{-\lambda} \otimes T, \psi(x+y)\gamma(x)\varphi(x+y) \rangle \\ &\quad - (1-\lambda)\langle \Phi_{-\lambda+1} \otimes T, \psi'(x+y)\gamma(x)\varphi(x+y) \rangle \\ &\quad + (1-\lambda)(2-\lambda)\langle \Phi_{-\lambda+2} \otimes T, \psi''(\xi)\gamma(x)\varphi(x+y) \rangle, \end{aligned}$$

for some ξ depending on x, y . This gives after a change of variable is the last term,

$$\begin{aligned} \langle (\gamma\Phi_{-\lambda}) * (T\psi), \varphi \rangle &= \langle \psi((\gamma\Phi_{-\lambda}) * T), \varphi \rangle - (1-\lambda)\langle \psi'((\gamma\Phi_{-\lambda+1}) * T), \varphi \rangle \\ &\quad + (1-\lambda)(2-\lambda) \int_{\mathbb{R}} \varphi(s) \int_{\mathbb{R}} (\gamma\Phi_{-\lambda+2})(s-y)T(y)\psi(\xi) dy ds. \end{aligned}$$

Therefore, we find

$$\begin{aligned} &\|(\gamma\Phi_{-\lambda}) * (T\psi)\|_{L^1(0,2\pi)} \\ &\leq \| \psi((\gamma\Phi_{-\lambda}) * T) \|_{L^1(0,2\pi)} + |1-\lambda| \| \psi'((\gamma\Phi_{-\lambda+1}) * T) \|_{L^1(0,2\pi)} \\ &\quad + |1-\lambda|(2-\lambda) \int_0^{2\pi} \left| \int_{\mathbb{R}} (\gamma\Phi_{-\lambda+2})(\theta-y)T(y)\psi''(\xi) dy \right| d\theta. \end{aligned}$$

Thus

$$\begin{aligned} &\|(\gamma\Phi_{-\lambda}) * (T\psi)\|_{L^1(0,2\pi)} \\ &\leq \| \psi \|_{L^\infty} \|(\gamma\Phi_{-\lambda}) * T\|_{L^1(0,2\pi)} + |1-\lambda| \| \psi' \|_{L^\infty} \|(\gamma\Phi_{-\lambda+1}) * T\|_{L^1(0,2\pi)} \\ &\quad + |1-\lambda|(2-\lambda) \| \psi'' \|_{L^\infty} \|(\gamma\Phi_{-\lambda+2}) * |T| \|_{L^1(0,2\pi)}. \end{aligned}$$

For the second term, using Lemma 3.1 we find

$$\|(\gamma\Phi_{-\lambda+1}) * T\|_{L^1(0,2\pi)} \leq \|T\|_{\lambda-1,1} \lesssim \|T\|_{\lambda,1}.$$

For the last term, observe that $\gamma\Phi_{-\lambda+2} \in L^1(\mathbb{R})$, and so this term is bounded by

$$|1-\lambda|(2-\lambda) \| \psi'' \|_{L^\infty} \|(\gamma\Phi_{-\lambda+2})\|_{L^1(\mathbb{R})} \|T\|_{L^1(0,2\pi)} \lesssim \|T\|_{\lambda,1}.$$

This shows one inclusion.

Next, we must show that $(\Phi_{-\lambda}\gamma) * T$ is in $L^1(0, 2\pi)$ if for all $\psi \in \mathcal{D}(\mathbb{R})$, we have $\Phi_{-\lambda} * (T\psi) \in L^1(0, 2\pi)$. We choose $\psi = 1$ on $(-2\pi, 2\pi)$ and observe that in the interval $(0, 2\pi)$ one has

$$(\Phi_{-\lambda}\gamma) * T = (\Phi_{-\lambda}\gamma) * (T\psi) = \Phi_{-\lambda} * (T\psi) + (\Phi_{-\lambda}(\gamma - 1)) * (T\psi),$$

which is a sum of integrable functions, since, as above, the second term is locally integrable. This concludes the proof of Lemma 3.2. \square

3.2. Key L^1 estimates on the distributions Ψ_λ . The basic idea for the following results is that the singularities of the formal expressions defining the distributions Φ_λ and Ψ_λ coincide. We now obtain a continuous embedding of the space $W_{\text{per}}^{-\lambda, 1}(0, 2\pi)$ into the space of periodic distributions T for which $\Psi_\lambda * T \in L^1(0, 2\pi)$. Thus, we seek for estimates of the form

$$\|\Psi_\lambda * T\|_{L^1(0, 2\pi)} \lesssim \|T\|_{-\lambda, 1}.$$

Proposition 3.3. *Suppose that $\lambda \geq 0$ and $\lambda \notin -\mathcal{E}$.*

(1) *If $T \in W_{\text{per}}^{\lambda, 1}(0, 2\pi)$, then*

$$\|\Psi_{-\lambda} * T\|_{L^1(0, 2\pi)} \lesssim \|T\|_{\lambda, 1} := \|(\gamma\Phi_{-\lambda}) * T\|_{L^1(0, 2\pi)} + \|T\|_{L^1(0, 2\pi)}. \tag{34}$$

(2) *If $T \in W_{\text{per}}^{-\lambda, 1}(0, 2\pi)$, then*

$$\|\Psi_\lambda * T\|_{L^1(0, 2\pi)} \lesssim \|T\|_{-\lambda, 1} := \|(\gamma\Phi_\lambda) * T\|_{L^1(0, 2\pi)} + \|\gamma_\lambda * T\|_{L^1(0, 2\pi)}. \tag{35}$$

Proof. Consider first the estimate (34). From Proposition 2.4, we find

$$\Psi_{-\lambda} = \frac{\Gamma(1/2 - \lambda)}{\sqrt{\pi}} (1 + |\xi|)^{-\lambda-1} (\tau_{1,1}(\gamma\Phi_{-\lambda}) + \tau_{-1,1}(\gamma\Phi_{-\lambda})),$$

which gives

$$\begin{aligned} \|\Psi_{-\lambda} * T\|_{L^1} &\lesssim \|((1 + |\xi|)^{-\lambda-1} \tau_{1,1}(\gamma\Phi_{-\lambda})) * T\|_{L^1} \\ &\quad + \|((1 + |\xi|)^{-\lambda-1} \tau_{-1,1}(\gamma\Phi_{-\lambda})) * T\|_{L^1}. \end{aligned}$$

Now, since T is periodic, the first term (for instance; the other term is treated similarly) equals

$$\|\tau_{1,1}((1 + |\xi - 1|)^{-\lambda-1}(\gamma\Phi_{-\lambda}) * T)\|_{L^1} = \|(\alpha_{-\lambda}\gamma\Phi_{-\lambda}) * T\|_{L^1},$$

where $\alpha_{-\lambda}(\xi) := (2 - \xi)^{-\lambda-1}$. Next, from Lemma 3.1, one has $(\gamma\Phi_{-\lambda+\mu}) * T \in L^1(0, 2\pi)$ for any $\mu > 0$, since $T \in W_{\text{per}}^{\lambda, 1}(0, 2\pi)$. Therefore we find, Taylor developing $\alpha_{-\lambda}$ around $\xi = 0$, and proceeding as in the proof of Lemma 3.2,

$$\|(\gamma\Phi_{-\lambda}\alpha_{-\lambda}) * T\|_{L^1} \leq \|\gamma\Phi_{-\lambda}(\alpha_{-\lambda}(0) + \zeta\alpha'_{-\lambda}(0) + \dots + \mathcal{O}(\zeta^{k+1})) * T\|_{L^1}$$

and thus

$$\begin{aligned} \|(\gamma\Phi_{-\lambda}\alpha_{-\lambda}) * T\|_{L^1} &\leq |\alpha_{-\lambda}(0)| \|\gamma\Phi_{-\lambda} * T\|_{L^1} + |\alpha'_{-\lambda}(0)| \|\gamma\Phi_{-\lambda+1} * T\|_{L^1} + \dots \\ &\quad + |\alpha_{-\lambda}^{(k)}(0)| \|\gamma\Phi_{-\lambda+k} * T\|_{L^1} + C\|\gamma\Phi_{-\lambda+k+1}\|_{L^1} \|T\|_{L^1}, \end{aligned}$$

with k large enough so that $-\lambda + k + 1 > 0$. This completes the proof of (34).

Finally, splitting the distribution Ψ_λ , we see that, to show (35), it is enough to estimate $\|(\Phi_\lambda\alpha_\lambda\gamma) * T\|_{L^1(0,2\pi)}$. Using Lemma 2.2 and Remark 2.3 we get

$$\|(\Phi_\lambda\alpha_\lambda\gamma) * T\|_{L^1(0,2\pi)} \leq \|\Phi_{-\lambda}\gamma * \Phi_\lambda\alpha\gamma * \Phi_\lambda\gamma * T\|_{L^1(0,2\pi)} + \|\gamma_\lambda * \Phi_\lambda\alpha\gamma * T\|_{L^1(0,2\pi)}$$

thus

$$\begin{aligned} &\|(\Phi_\lambda\alpha_\lambda\gamma) * T\|_{L^1(0,2\pi)} \\ &\leq (\alpha_\lambda(0) + \|\Phi_1 A_\lambda\|_{L^1(\mathbb{R})}) \|(\Phi_\lambda\gamma) * T\|_{L^1(0,2\pi)} + \|\Phi_\lambda\alpha\gamma\|_{L^1(0,2\pi)} \|\gamma_\lambda * T\|_{L^1(0,2\pi)} \\ &\lesssim \|T\|_{-\lambda,1}. \end{aligned}$$

This completes the proof of Proposition 3.3. □

3.3. Key L^2 estimates on the distributions Ψ_λ . The additional structure provided by the Fourier transform in L^2 allows us to apply a completely different method to derive estimates on the distributions Ψ_λ . In fact, we show that the space of periodic distributions which are in $L^2(0, 2\pi)$ after convolution with the distributions $\Psi_{-\lambda}$ is precisely the classical Sobolev space $H_{\text{per}}^\lambda(0, 2\pi)$.

We begin by recalling the definition of the Sobolev spaces $H_{\text{per}}^\lambda(0, 2\pi)$. First, note that if T is a periodic distribution, then we may define its Fourier coefficients $c_n, n \in \mathbb{Z}$ and its continuous Fourier transform is given by

$$\hat{T} = \sum_{n \in \mathbb{Z}} \delta_{\xi=2\pi n} c_n.$$

The rate of decay of $|c_n|$ is a measure of the regularity and integrability properties of T . For instance, $T \in L^2(0, 2\pi)$ iff $\sum_{n \in \mathbb{Z}} |c_n|^2 < \infty$, and moreover one has $\sum_{n \in \mathbb{Z}} |c_n|^2 = \|T\|_{L^2(0,2\pi)}^2$. The Sobolev spaces $H_{\text{per}}^\lambda(0, 2\pi)$ are defined for $\lambda \in \mathbb{R}$ as

$$H_{\text{per}}^\lambda(0, 2\pi) := \left\{ T \in \mathcal{D}'_{\text{per}}(0, 2\pi) : \sum_{n \in \mathbb{Z}} (1 + |n|^2)^\lambda |c_n|^2 < \infty \right\},$$

with norm given by

$$\|T\|_{H^\lambda} = \sum_{n \in \mathbb{Z}} (1 + |n|^2)^\lambda |c_n|^2.$$

Note that this definition makes sense for all $\lambda \in \mathbb{R}$, giving a single, coherent definition of a scale of Sobolev spaces.

Proposition 3.4. *For every $\lambda \in \mathbb{R}$, $\lambda \notin -\mathcal{E}$, one has*

$$\{T \in \mathcal{D}'_{\text{per}}(0, 2\pi) : \Psi_{-\lambda} * T \in L^2(0, 2\pi)\} = H^\lambda_{\text{per}}(0, 2\pi),$$

and

$$\|\Psi_{-\lambda} * T\|_{L^2(0, 2\pi)} \lesssim \|T\|_{H^\lambda}.$$

Proof. Suppose that $\Psi_{-\lambda} * T \in L^2(0, 2\pi)$. Since it is periodic, we must have $F(\Psi_{-\lambda} * T) = \sum_{n \in \mathbb{Z}} \delta_{2\pi n} d_n$, for some $d_n \in \mathbb{C}$ with $\sum_{n \in \mathbb{Z}} |d_n|^2 < \infty$. On the other hand, since T is periodic with Fourier coefficients c_n ,

$$F(\Psi_{-\lambda} * T) = F(\Psi_{-\lambda})F(T) = \hat{\Psi}_{-\lambda} \sum_{n \in \mathbb{Z}} \delta_{2\pi n} c_n = \sum_{n \in \mathbb{Z}} \delta_{2\pi n} a_n c_n,$$

where $a_n := \hat{\Psi}_{-\lambda}(2\pi n)$. Now the key point is that the explicit formula of $\hat{\Psi}_{-\lambda}(s)$ is known [6]:

$$\hat{\Psi}_{-\lambda}(s) = \Gamma(1/2 - \lambda)(s/2)^{\lambda+1/2} J_{-\lambda-1/2}(s),$$

where J_α denotes the Bessel functions. For our purposes, it is enough to know that J_α satisfies

$$s^{-\alpha} J_\alpha(s) \in C^\infty(\mathbb{R}), \quad \text{and} \quad |J_\alpha(s)| \sim 1/\sqrt{s}, \quad s \rightarrow \infty$$

for all α . From this it is easily deduced that

$$|a_n| \lesssim (1 + |n|)^\lambda.$$

Now observe that there exist constants $A, B > 0$ such that

$$A \sum_{n \in \mathbb{Z}} (1 + |n|)^{2\lambda} |c_n|^2 \leq \sum_{n \in \mathbb{Z}} (1 + |n|^2)^\lambda |c_n|^2 \leq B \sum_{n \in \mathbb{Z}} (1 + |n|)^{2\lambda} |c_n|^2.$$

The first statement of the proposition follows immediately, since it is equivalent to saying

$$\sum_{n \in \mathbb{Z}} \delta_{2\pi n} |d_n|^2 < \infty \iff \sum_{n \in \mathbb{Z}} (1 + |n|^2)^\lambda |c_n|^2 < \infty,$$

while the desired estimate follows from Parseval’s identity. This completes the proof of Proposition 3.4. \square

4. Euler–Poisson–Darboux equation (generic case)

4.1. General formula. We are in a position to now validate a formal expansion near the coordinate singularity. The solutions are suitable translates and rescalings of the distributions Ψ_ω investigated in the previous sections. Here, we provide new regularity estimates for the solutions of the EPD equation involving the scale of Sobolev-type spaces presented in the previous section.

Consider the Euler-Poisson-Darboux equation

$$Q_{tt} + \frac{2\omega}{t} Q_t - Q_{\theta\theta} = 0, \tag{36}$$

in which ω is a constant. If Q satisfies (36), we write $\mathcal{P}^\omega(Q) = 0$. Let $\omega \in \mathbb{R}$. Then, it is easy to check that the two functions

$$(t^2 - \theta^2)_+^{-\omega}, \quad t^{1-2\omega}(t^2 - \theta^2)_+^{\omega-1}$$

are solutions of the equation (36). In consequence, at least *formally*, the general solutions are given by the convolution

$$\begin{aligned} Q(t, \theta) = & C \int_{\mathbb{R}} u_1(\theta') (t^2 - (\theta - \theta')^2)_+^{-\omega} d\theta' \\ & + C' \int_{\mathbb{R}} u_2(\theta') t^{1-2\omega} (t^2 - (\theta - \theta')^2)_+^{\omega-1} d\theta' \end{aligned} \tag{37}$$

in which the constants C, C' are arbitrary. The change of variables $x = (\theta' - \theta)/t$ yields

$$\begin{aligned} Q(t, \theta) = & C \int_{-1}^1 u_1(\theta + tx) t^{1-2\omega} (1 - x^2)^{-\omega} dx \\ & + C' \int_{-1}^1 u_2(\theta + tx) (1 - x^2)^{\omega-1} dx. \end{aligned} \tag{38}$$

When $\omega \in (0, 1)$ (so that both integrals exist), this can be written as

$$Q^\omega(t, \theta) = t^{1-2\omega} \langle \Psi_{1-\omega}, u_1(\theta + t \cdot) \rangle + \langle \Psi_\omega, u_2(\theta + t \cdot) \rangle, \tag{39}$$

by choosing $C = C_{-\omega}$ and $C' = C_{\omega-1}$ (cf. (26)). This may be written as a convolution in two equivalent ways, as follows (recall the scaling operator $\sigma_a \phi(x) = \phi(ax)$).

$$\begin{aligned} Q^\omega(t) &= t^{1-2\omega}\sigma_{1/t}(\Psi_{1-\omega} * \sigma_t u_1) + \sigma_{1/t}(\Psi_\omega * \sigma_t u_2) \\ &= t^{-2\omega}(\sigma_{1/t}\Psi_{1-\omega}) * u_1 + t^{-1}(\sigma_{1/t}\Psi_\omega) * u_2. \end{aligned} \tag{40}$$

Any of these two expressions is well defined for any value of ω except $\omega \in \mathcal{E}$. Moreover, we may take $u_{1,2}$ to be any distribution, and the convolution still makes sense. When $\omega \in (0, 1)$ and $u_{1,2}$ are bounded functions, these expressions reduce to the explicit solutions presented above.

We must still check that these convolutions are indeed weak solutions (in the sense of distributions) to the equation (36).

Proposition 4.1. *Given $\omega \in \mathbb{R} \setminus \mathcal{E}$ and any two distributions $u_1, u_2 \in \mathcal{D}'_{\text{per}}(0, 2\pi)$, the formula (39) defines a distributional solution $Q \in \mathcal{D}'_{\text{per}}(\mathbb{R}_+ \times (0, 2\pi))$ of (36), that is,*

$$\mathcal{P}^\omega(Q) = 0$$

in the sense of distributions.

Proof. From (40), we see that Q is a solution of the equation (36) provided

$$\mathcal{P}^\omega(t^{-2\omega}\sigma_{1/t}\Psi_{1-\omega}) + \mathcal{P}^\omega(t^{-1}\sigma_{1/t}\Psi_\omega) = 0,$$

because since \mathcal{P}^ω is a linear operator, we have $\mathcal{P}^\omega(f * g) = (\mathcal{P}^\omega f) * g$.

Consider for instance the second term, and set $\xi = x/t$. Then, we have

$$\xi_t = -\frac{1}{t}\xi, \quad \xi_x = \frac{1}{t}, \quad \Psi_\omega(\xi)_x = -\frac{1}{t}\xi\Psi'_\omega(\xi),$$

where the notation $\Psi_\omega(\xi)$ is just shorthand for $\sigma_{1/t}\Psi_\omega$. Using these properties, we easily compute

$$\mathcal{P}^\omega\left(\frac{1}{t}\Psi_\omega(\xi)\right) = \frac{1}{t^3}((2 - 2\omega)\Psi_\omega(\xi) + (4 - 2\omega)\xi\Psi'_\omega(\xi) - (1 - \xi^2)\Psi''_\omega(\xi)).$$

Next, we use the relations valid in the sense of distributions,

$$\begin{aligned} \Psi'_\omega(\xi) &= (1 - 2\omega)\xi\Psi_{\omega-1}(\xi), \\ \Psi''_\omega(\xi) &= (1 - 2\omega)\Psi_{\omega-1}(\xi) + (1 - 2\omega)(3 - 2\omega)\xi^2\Psi_{\omega-2}(\xi), \\ \Psi_\omega(\xi) &= \frac{1 - 2\omega}{2 - 2\omega}(1 - \xi^2)\Psi_{\omega-1}(\xi) \end{aligned}$$

(cf. Proposition 2.4 and (27)) to write the expression above in terms of, say, $\Psi_{\omega-2}$ only. The result is readily seen to be zero. □

4.2. Key estimates. Now that we know that $Q^\omega(t, \theta)$ is a solution (in the sense of distributions) to the EPD equation (36), it is natural to ask in what sense are the initial data approached. Equivalently, one seeks to validate certain asymptotic expansions. In our case, if the functions u_1, u_2 are smooth and $\omega \in (0, 1)$, one can check by Taylor expansion that the solution Q^ω given by (40) satisfies the asymptotic expansion

$$Q^\omega(t, \theta) - t^{1-2\omega}u_1(\theta) - u_2(\theta) = o(1), \quad t \rightarrow 0. \tag{41}$$

Our main objective in this section is twofold: on the one hand, we determine what is the minimal regularity one must suppose on the data u_1, u_2 so that the solution Q^ω is an integrable function. On the other hand, we generalize the asymptotic expansion above to non-smooth data u_1, u_2 and all values of the parameter ω (except $\omega \in \mathcal{E}$ — see the next section). Both these questions are answered using the Sobolev-type spaces presented in the previous section.

It is convenient to consider the following decomposition of Q^ω ,

$$\begin{aligned} Q^\omega &= t^{1-2\omega}\sigma_{1/t}(\Psi_{1-\omega} * \sigma_t u_1) + \sigma_{1/t}(\Psi_\omega * \sigma_t u_2) \\ &= Q_1^\omega(t; u_1) + Q_2^\omega(t; u_2). \end{aligned} \tag{42}$$

Also, denote $\Psi_\omega^t := (1/t)\sigma_{1/t}\Psi_\omega$, and observe that

$$\sigma_{1/t}(\Psi_\omega * \sigma_t u) = \Psi_\omega^t * u.$$

We begin by ensuring that the solution Q^ω is integrable, by choosing the appropriate spaces for the data u_1, u_2 .

Lemma 4.2. *For all $\omega \in \mathbb{R} \setminus \mathcal{E}$ one has the following estimates: If $\omega \geq 0$ and $u \in W_{\text{per}}^{-\omega, 1}(0, 2\pi)$, then*

$$\begin{aligned} \|\sigma_{1/t}(\Psi_\omega * \sigma_t u)\|_{L^1(0, 2\pi)} &\lesssim t^{-\omega} \|u\|_{-\omega, 1}, \\ \|\sigma_{1/t}(\Psi_\omega * \sigma_t u)\|_{-\omega, 1} &\lesssim \|u\|_{-\omega, 1}. \end{aligned}$$

If $\omega < 0$ and $u \in W_{\text{per}}^{-\omega, 1}(0, 2\pi)$, then

$$\begin{aligned} \|\sigma_{1/t}(\Psi_\omega * \sigma_t u)\|_{L^1(0, 2\pi)} &\lesssim t^{-\omega} \|u\|_{-\omega, 1} + \|u\|_{L^1(0, 2\pi)} \\ &= \mathcal{O}(1) \|u\|_{-\omega, 1}. \end{aligned}$$

Proof. Throughout the proof, we suppose that ω is not a negative integer or zero, since in that case the proof is easier.

Consider the case $\omega > 0$ and fix $u \in W_{\text{per}}^{-\omega, 1}(0, 2\pi)$. First, note that

$$\|\sigma_{1/t}(\Psi_\omega * \sigma_t u)\|_{L^1(0, 2\pi)} = t \|\Psi_\omega * \sigma_t u\|_{L^1(0, 2\pi/t)}.$$

Next, remark that the distribution $\sigma_t u$ is $2\pi/t$ -periodic, so we may apply the estimate (34) in Proposition 3.3 to obtain

$$\begin{aligned} t\|\Psi_\omega * \sigma_t u\|_{L^1(0, 2\pi/t)} &\lesssim t\|\sigma_t u\|_{W_{\text{per}}^{-\omega, 1}(0, 2\pi/t)} \\ &= t\|\Phi_\omega \gamma * \sigma_t u\|_{L^1(0, 2\pi/t)} + t\|\gamma_\omega * \sigma_t u\|_{L^1(0, 2\pi/t)} \\ &= t^{-1}\|\sigma_{1/t}(\Phi_\omega \gamma) * u\|_{L^1(0, 2\pi)} + t^{-1}\|\sigma_{1/t} \gamma_\omega * u\|_{L^1(0, 2\pi)} \\ &= t^{-\omega}\|(\Phi_\omega \sigma_{1/t} \gamma) * u\|_{L^1(0, 2\pi)} + t^{-1}\|\sigma_{1/t} \gamma_\omega * u\|_{L^1(0, 2\pi)}. \end{aligned}$$

Now using (18) in Lemma 2.2 we get

$$\begin{aligned} t\|\Psi_\omega * \sigma_t u\|_{L^1(0, 2\pi/t)} &\lesssim t^{-\omega}\|\Phi_{-\omega} \gamma * (\Phi_\omega \sigma_{1/t} \gamma) * \Phi_\omega \gamma * u\|_{L^1(0, 2\pi)} \\ &\quad + t^{-\omega}\|(\Phi_\omega \sigma_{1/t} \gamma)\|_{L^1(\mathbb{R})}\|\gamma_\omega * u\|_{L^1(0, 2\pi)} \\ &\quad + t^{-1}\|\Phi_{-\omega} \gamma * \sigma_{1/t} \gamma_\omega\|_{L^1(\mathbb{R})}\|\Phi_\omega \gamma * u\|_{L^1(0, 2\pi)} \\ &\quad + t^{-1}\|\sigma_{1/t} \gamma_\omega\|_{L^1(\mathbb{R})}\|\gamma_\omega * u\|_{L^1(0, 2\pi)}. \end{aligned} \tag{43}$$

To treat the first term, we note that according to Lemma 2.2,

$$\begin{aligned} \Phi_{-\omega} \gamma * (\Phi_\omega \sigma_{1/t} \gamma) &= \Phi_{-\omega} \gamma * \Phi_\omega \gamma + \Phi_{-\omega} \gamma * [\Phi_\omega(\sigma_{1/t} \gamma - \gamma)] \\ &= \delta + \gamma_\omega + \Phi_{-\omega} \gamma * \Phi_\omega(\sigma_{1/t} \gamma - \gamma). \end{aligned}$$

This means that

$$\begin{aligned} t^{-\omega}\|\Phi_{-\omega} \gamma * (\Phi_\omega \sigma_{1/t} \gamma) * \Phi_\omega \gamma * u\|_{L^1(0, 2\pi)} \\ \leq t^{-\omega}\|u\|_{-\omega, 1} + t^{-\omega}\|\Phi_{-\omega} \gamma * \Phi_\omega(\sigma_{1/t} \gamma - \gamma)\|_{L^1(\mathbb{R})}\|\Phi_\omega \gamma * u\|_{L^1(0, 2\pi)}, \end{aligned}$$

so that we need to bound the term $\|\Phi_{-\omega} \gamma * \Phi_\omega(\sigma_{1/t} \gamma - \gamma)\|_{L^1(\mathbb{R})}$ uniformly with respect to t . Note that this is, for each t , in $\mathcal{D}(\mathbb{R})$. To achieve this, reproduce the proof of Lemma 2.2 replacing $\Phi_\omega(\sigma_{1/t} \gamma - \gamma)$ for $\Phi_\omega \gamma$. We present the case $\omega \in (1, 2)$, which employs all the arguments necessary to treat the general case. We find, then,

$$\Phi_{-\omega} \gamma * \Phi_\omega(\sigma_{1/t} \gamma - \gamma)(s) = \tilde{\gamma}_\omega(s, t),$$

with $\tilde{\gamma}_\omega$ given by (21) with $(\sigma_{1/t} \gamma - \gamma)(sr)$ and its derivatives replacing $\gamma(sr)$ and its derivatives at every occurrence (but not $\gamma(s(1-r))$). To bound the term under analysis, consider for instance the term

$$\int_0^\infty s \left| \int_0^1 \frac{(1-r)^{1-\lambda} r^{\lambda+1}}{\Gamma(2-\lambda)\Gamma(\lambda)} \gamma(s(1-r))(\sigma_{1/t} \gamma - \gamma)''(sr) dr \right| ds,$$

which is easily seen to be the most singular. Now, $\gamma(s(1-r)) \leq 1$, and the function of r inside the integral has a finite integral, from the properties of the beta function. Therefore, it is enough to bound the term

$$\int_0^\infty s |(\sigma_{1/t}\gamma)''(s)| ds.$$

Now note that the integrand is actually supported in $(t(1-a), t(1+a))$, where a is the fixed constant given by (16), and that

$$\max_s |(\sigma_{1/t}\gamma)''(s)| = \max_s |t^{-2}\gamma''(s)| = (at)^{-2}.$$

These facts give

$$\int_0^\infty s |(\sigma_{1/t}\gamma)''(s)| ds \lesssim \int_{t(1-a)}^{t(1+a)} (at)^{-2} s ds = \mathcal{O}(1).$$

This takes care of the first term in (43). Next, observe that the facts

$$\frac{1}{t} \|\sigma_{1/t}f\|_{L^1(\mathbb{R})} = \|f\|_{L^1(\mathbb{R})}, \quad t^{-\omega}\Phi_\omega\sigma_{1/t}\gamma = \frac{1}{t}\sigma_{1/t}(\Phi_\omega\gamma)$$

allow us to bound the second and fourth terms of (43) by $Ct^{-\omega}\|u\|_{-\omega,1}$. Finally, for the third term we find

$$\begin{aligned} t^{-1}\|\Phi_{-\omega}\gamma * \sigma_{1/t}\gamma_\omega\|_{L^1(\mathbb{R})} &= t^{-\omega}\|\Phi_{-\omega}\sigma_t\gamma * \gamma_\omega\|_{L^1(\mathbb{R})} \\ &\leq t^{-\omega}\|\Phi_{-\omega}\gamma * \gamma_\omega\|_{L^1(\mathbb{R})} + t^{-\omega}\|\Phi_{-\omega}(\gamma - \sigma_t\gamma) * \gamma_\omega\|_{L^1(\mathbb{R})}. \end{aligned}$$

Now, since the function γ_ω has compact support (see Lemma 2.2), the first term is actually in $\mathcal{D}(\mathbb{R})$. Thus,

$$\begin{aligned} t^{-\omega}\|\Phi_{-\omega}(\gamma - \sigma_t\gamma) * \gamma_\omega\|_{L^1(\mathbb{R})} &\leq t^{-\omega}\|\Phi_{-\omega}(\gamma - \sigma_t\gamma)\|_{L^1(\mathbb{R})}\|\gamma_\omega\|_{L^1(\mathbb{R})} \\ &\leq t^{-\omega} \int_{1-a}^{(1+a)/t} \Phi_{-\omega}\|\gamma_\omega\|_{L^1(\mathbb{R})} \\ &\lesssim (1 - t^{-\omega})\|\gamma_\omega\|_{L^1(\mathbb{R})}. \end{aligned}$$

This allows us to bound the third term in (43) by $t^{-\omega}\|u\|_{-\omega,1}$. This completes the proof of the first inequality of the lemma. For the second inequality, observe that

$$\begin{aligned} \|\sigma_{1/t}(\Psi_\omega * \sigma_t u)\|_{-\omega,1} &= \|\Psi_\omega^t * \Phi_\omega\gamma * u\|_{L^1(0,2\pi)} + \|\gamma_\omega * \Psi_\omega^t * u\|_{L^1(0,2\pi)} \\ &\lesssim \|u\|_{-\omega,1}, \end{aligned}$$

Since Ψ_ω^t is integrable for $\omega > 0$. This completes the derivation of the second inequality of the lemma.

Suppose, then, that $\omega < 0$. Again by Proposition 3.3, and methods similar to the foregoing proof, we find

$$\begin{aligned} \|\sigma_{1/t}(\Psi_\omega * \sigma_t u)\|_{L^1(0,2\pi)} &\leq t\|\sigma_t u\|_{W_{\text{per}}^{-\omega,1}(0,2\pi/t)} \\ &= t\|(\gamma\Phi_\omega) * \sigma_t u\|_{L^1(0,2\pi/t)} + \|u\|_{L^1(0,2\pi)} \\ &\leq t^{-\omega}\|(\gamma\Phi_\omega) * u\|_{L^1(0,2\pi)} \\ &\quad + t^{-\omega}\|\Phi_\omega(\gamma - \sigma_{1/t}\gamma)\|_{L^1(\mathbb{R})}\|u\|_{L^1(0,2\pi)} + \|u\|_{L^1(0,2\pi)}. \end{aligned}$$

The third estimate of the lemma now follows from

$$t^{-\omega}\|\Phi_\omega(\gamma - \sigma_{1/t}\gamma)\|_{L^1(\mathbb{R})} \leq t^{-\omega} \int_{t(1-a)}^{1+a} \frac{x^{\omega-1}}{\Gamma(\omega)} = \frac{\mathcal{O}(1)}{\Gamma(\omega+1)}(t^{-\omega} - 1)$$

and from $\|(\gamma\Phi_\omega) * u\|_{L^1(0,2\pi)} = |u|_{-\omega,1}$. This completes the proof of Lemma 4.2. □

The following lemma will be the basis of our proof of the asymptotic expansions in Theorem 4.4.

Lemma 4.3. *Let $\omega \in \mathbb{R} \setminus \mathcal{E}$. If $\omega \geq 0$ and $u \in W_{\text{per}}^{-\omega+1,1}(0,2\pi)$, then*

$$\|\sigma_{1/t}(\Psi_\omega * \sigma_t u) - u\|_{-\omega,1} \lesssim t\|u\|_{-\omega+1,1}. \tag{44}$$

If $\omega \leq 0$ and $u \in W_{\text{per}}^{k,1}(0,2\pi)$, with $k \in \mathbb{N}_0$ and $\omega + k \in (0,1]$, then

$$\|\sigma_{1/t}(\Psi_\omega * \sigma_t u) - u\|_{L^1(0,2\pi)} \lesssim t\|u'\|_{k-1,1}. \tag{45}$$

Proof. Once again, if ω is a non-positive integer, the proof is simpler and we omit it. To obtain (44), we set $\Psi_\omega^t := (1/t)\sigma_{1/t}\Psi_\omega$, and recall the definition of the negative Sobolev spaces and norms (32). We find

$$\begin{aligned} &\|\Psi_\omega^t * u - u\|_{-\omega,1} \\ &\leq \|\Psi_\omega^t * (\gamma\Phi_\omega * u) - \gamma\Phi_\omega * u\|_{L^1(0,2\pi)} + \|\Psi_\omega^t * (\gamma_\omega * u) - \gamma_\omega * u\|_{L^1}. \end{aligned}$$

Now observe that, for $f \in W_{\text{per}}^{1,1}(0,2\pi)$ and $\omega \geq 0$,

$$\begin{aligned} \|\Psi_\omega^t * f - f\|_{L^1(0,2\pi)} &= \int_0^{2\pi} \left| \int_{-1}^1 \Psi_\omega(x)(f(\theta + tx) - f(\theta)) dx \right| d\theta \\ &\leq \int_{-1}^1 t|x|\Psi_\omega(x) \int_0^{2\pi} \left| \frac{f(\theta + tx) - f(\theta)}{tx} \right| dx d\theta. \end{aligned}$$

Thus

$$\|\Psi_\omega^t * f - f\|_{L^1(0,2\pi)} \leq t \|x\Psi_\omega\|_{L^1(\mathbb{R})} \|f'\|_{L^1(0,2\pi)}. \tag{46}$$

The function $\gamma\Phi_\omega * u$ is in $W_{\text{per}}^{1,1}(0, 2\pi)$, by Lemma 3.1, and $\gamma_\omega * u$ is smooth. Therefore, we may apply (46) and find

$$\begin{aligned} & \|\Psi_\omega^t * u - u\|_{-\omega,1} \\ & \leq t \|x\Psi_\omega\|_{L^1(\mathbb{R})} (\|(\gamma\Phi_\omega * u)'\|_{L^1(0,2\pi)} + \|(\gamma_\omega * u)'\|_{L^1(0,2\pi)}) \\ & \leq t \|x\Psi_\omega\|_{L^1(\mathbb{R})} (\|\gamma\Phi_{\omega-1} * u\|_{L^1(0,2\pi)} + \|\gamma'\Phi_\omega * u\|_{L^1(0,2\pi)} + \|\gamma'_\omega * u\|_{L^1(0,2\pi)}). \end{aligned}$$

To complete the proof of (44), we only need to bound the last two terms by $\|u\|_{-\omega+1,1}$. For this, note that γ'_ω and $\gamma'\Phi_\omega$ are in $\mathcal{D}(\mathbb{R})$. Let $\rho \in \mathcal{D}(\mathbb{R})$. We have from Lemma 2.2,

$$\begin{aligned} \|\rho * u\|_{L^1(0,2\pi)} & \leq \|\gamma\Phi_{\omega-1} * \gamma\Phi_{1-\omega} * \rho * u\|_{L^1(0,2\pi)} + \|\gamma_{\omega-1} * \rho * u\|_{L^1(0,2\pi)} \\ & \leq \|\rho * \gamma\Phi_{1-\omega}\|_{L^1(\mathbb{R})} \|\gamma\Phi_{\omega-1} * u\|_{L^1(0,2\pi)} + \|\rho\|_{L^1(\mathbb{R})} \|\gamma_{\omega-1} * u\|_{L^1(0,2\pi)} \\ & = C(\rho, \omega) \|u\|_{-\omega+1,1}. \end{aligned}$$

This completes the proof of the estimate (44).

We now turn to the proof of (45). Let $\omega < 0$ and $k \in \mathbb{N}_0$ such that $\omega + k \in (0, 1]$. First, observe that

$$t^{-1} \sigma_{1/t} \Psi_\omega^{(j)} = t^{j-1} (\sigma_{1/t} \Psi_\omega)^{(j)}.$$

Using this fact and the relations (28) we find

$$\begin{aligned} \Psi_\omega^t * u & = t^{-1} \sigma_{1/t} \left(\sum_{j=0}^k \beta_j \Psi_{\omega+k}^{(j)} \right) * u + t^{-1} \sigma_{1/t} \left(\sum_{j=2}^{k+1} \eta_j \Psi_{\omega+k+1}^{(j)} \right) * u \\ & = t^{-1} (\sigma_{1/t} \Psi_{\omega+k}) * u + \sum_{j=1}^k \beta_j t^{j-1} (\sigma_{1/t} \Psi_{\omega+k})^{(j)} * u \\ & \quad + \sum_{j=1}^k \eta_{j+1} t^{j-1} (\sigma_{1/t} \Psi'_{\omega+k+1})^{(j)} * u. \end{aligned}$$

Thus

$$\begin{aligned} \Psi_\omega^t * u & = t^{-1} (\sigma_{1/t} \Psi_{\omega+k}) * u + \sum_{j=1}^k \beta_j t^{j-1} (\sigma_{1/t} \Psi_{\omega+k}) * u^{(j)} \\ & \quad + \sum_{j=1}^k \eta_{j+1} t^{j-1} (\sigma_{1/t} \Psi'_{\omega+k+1}) * u^{(j)}. \end{aligned}$$

Finally, this gives

$$\begin{aligned} \|\Psi_\omega^t * u - u\|_{L^1} &\leq \|\Psi_{\omega+k}^t * u - u\|_{L^1} + \sum_{j=1}^k \beta_j t^{j-1} \|\sigma_{1/t} \Psi_{\omega+k} * u^{(j)}\|_{L^1} \\ &\quad + \sum_{j=1}^k \eta_{j+1} t^{j-1} \|\sigma_{1/t} \Psi'_{\omega+k+1} * u^{(j)}\|_{L^1} \end{aligned}$$

and so

$$\begin{aligned} \|\Psi_\omega^t * u - u\|_{L^1} &\leq t \|x \Psi_{\omega+k}\|_{L^1(\mathbb{R})} \|u'\|_{L^1(0, 2\pi)} \\ &\quad + \sum_{j=1}^k (\beta_j \|\Psi_{\omega+k}\|_{L^1(\mathbb{R})} + \eta_{j+1} \|\Psi'_{\omega+k+1}\|_{L^1(\mathbb{R})}) t^j \|u^{(j)}\|_{L^1(0, 2\pi)} \\ &\lesssim t \|u'\|_{k-1, 1}, \end{aligned}$$

where we have also used (46). This completes the proof of Lemma 4.3. □

We now validate the asymptotic expansion (41) for non-smooth data, in the appropriate fractional Sobolev spaces. We see that in order to ensure their validity, we must take the data $u_{1,2}$ in a space which is more regular, as is natural.

Theorem 4.4. *Let $\omega \in \mathbb{R} \setminus \mathcal{E}$ and $\bar{\omega} = \min(\omega - 1, -\omega)$. Let Q^ω be the solution of the EPD equation given by (42).*

1. *If $(u_1, u_2) \in W_{\text{per}}^{\omega-1, 1} \times W_{\text{per}}^{-\omega, 1}$, then for every $t > 0$, the operator*

$$(u_1, u_2) \mapsto (Q_1^\omega(t; u_1), Q_2^\omega(t; u_2))$$

maps $W_{\text{per}}^{\omega-1, 1} \times W_{\text{per}}^{-\omega, 1}$ into $(L^1(0, 2\pi))^2$, and the following estimates hold:

$$\|Q^\omega(t)\|_{L^1(0, 2\pi)} \lesssim \begin{cases} t^{-\omega} \|u_1\|_{\omega-1, 1} + t^{-\omega} \|u_2\|_{-\omega, 1} + \|u_2\|_{L^1}, & \omega \leq 0 \\ t^{-\omega} (\|u_1\|_{\omega-1, 1} + \|u_2\|_{-\omega, 1}), & \omega \in (0, 1) \\ t^{1-2\omega} \|u_1\|_{\omega-1, 1} + t^{-\omega} \|u_2\|_{-\omega, 1}, & \omega \geq 1, \end{cases} \quad (47)$$

$$\|Q^\omega(t)\|_{\bar{\omega}, 1} \lesssim t^{1-2\omega} \|u_1\|_{\omega-1, 1} + \|u_2\|_{-\omega, 1}. \quad (48)$$

2. *Suppose, in addition, that $(u_1, u_2) \in W_{\text{per}}^{\omega, 1} \times W_{\text{per}}^{1-\omega, 1}$, and let $k \in \mathbb{Z}$ be such that $\omega + k \in (0, 1]$. Then for every $t > 0$, we have the asymptotic expansions*

$$\|Q^\omega(t) - t^{1-2\omega} u_1(t) - u_2\|_{\bar{\omega}, 1} \lesssim \begin{cases} t^{2-2\omega} \|u_1\|_{\omega, 1} + t \|u_2\|_{k, 1}, & \omega \leq 0 \\ t^{2-2\omega} \|u_1\|_{\omega, 1} + t \|u_2\|_{1-\omega, 1}, & \omega \in (0, 1) \\ t^{2-2\omega} \|u_1\|_{-k, 1} + t \|u_2\|_{1-\omega, 1}, & \omega \geq 1. \end{cases} \quad (49)$$

We now estimate the spatial derivatives of the solutions.

Theorem 4.5. *Let $\omega \in \mathbb{R} \setminus \mathcal{E}$ and $\bar{\omega} = \min(\omega - 1, -\omega)$. Let Q^ω be the solution of the EPD equation given by (42).*

1. *If $(u_1, u_2) \in W_{\text{per}}^{\omega-1,1} \times W_{\text{per}}^{-\omega,1}$, then for every $t > 0$, the operator*

$$(u_1, u_2) \mapsto ((Q_1^\omega)_\theta(t; u_1), (Q_2^\omega)_\theta(t; u_2))$$

maps $W_{\text{per}}^{\omega-1,1} \times W_{\text{per}}^{-\omega,1}$ into $(W_{\text{per}}^{-1,1}(0, 2\pi))^2$, and the estimates (47) in Theorem 4.4 hold with $\|Q^\omega(t)\|_{L^1(0,2\pi)}$ replaced by $\|Q_\theta^\omega(t)\|_{-1,1}$.

2. *Suppose, in addition, that $(u_1, u_2) \in W_{\text{per}}^{\omega,1} \times W_{\text{per}}^{1-\omega,1}$, and let $k \in \mathbb{Z}$ be such that $\omega + k \in (0, 1]$. Then, for every $t > 0$, $Q^\omega(t) \in L^1(0, 2\pi)$ and*

$$\|Q_\theta^\omega(t)\|_{L^1(0,2\pi)} \lesssim \begin{cases} t^{-\omega}\|u_1\|_{\omega,1} + t^{-\omega}\|u_2\|_{1-\omega,1} + \|u_2\|_{L^1}, & \omega \leq 0 \\ t^{-\omega}(\|u_1\|_{\omega,1} + \|u_2\|_{1-\omega,1}), & \omega \in (0, 1) \\ t^{1-2\omega}\|u_1\|_{\omega,1} + t^{-\omega}\|u_2\|_{1-\omega,1}, & \omega \geq 1. \end{cases}$$

Moreover, the following asymptotic expansion holds

$$\|Q_\theta^\omega(t) - t^{1-2\omega}u_1'(t) - u_2'\|_{\bar{\omega}-1,1} \lesssim \begin{cases} t^{2-2\omega}\|u_1\|_{\omega,1} + t\|u_2\|_{k,1}, & \omega \leq 0 \\ t^{2-2\omega}\|u_1\|_{\omega,1} + t\|u_2\|_{1-\omega,1}, & \omega \in (0, 1) \\ t^{2-2\omega}\|u_1\|_{-k,1} + t\|u_2\|_{1-\omega,1}, & \omega \geq 1. \end{cases}$$

For the time derivatives, we have the following result.

Theorem 4.6. *Let $\omega \in \mathbb{R} \setminus \mathcal{E}$, $k \in \mathbb{Z}$ such that*

$$\omega + k \in (0, 1), \quad \bar{\omega} = \min(\omega - 1, -\omega).$$

Let Q^ω be the solution of the EPD equation given by (42). Then there is a locally bounded function $\eta(\omega)$ with $\eta(-n) = 0$ for all $n \in \mathbb{N}_0$ such that

1. *If $(u_1, u_2) \in W_{\text{per}}^{\omega,1} \times W_{\text{per}}^{1-\omega,1}$, one has*

$$\left\| \frac{t}{1-2\omega} \partial_t Q^\omega \right\|_{L^1(0,2\pi)} \lesssim \begin{cases} t^{1-\omega}\|u_1\|_{\omega,1} + t^{-\omega}\|u_2\|_{1-\omega,1} + \eta(\omega)\|u_2\|_{L^1}, & \omega \leq 0 \\ t^{1-2\omega}\|u_1\|_{\omega,1} + t^{-\omega}\|u_2\|_{1-\omega,1}, & \omega > 0. \end{cases}$$

2. *Suppose that $(u_1, u_2) \in W_{\text{per}}^{\omega+1,1} \times W_{\text{per}}^{1-\omega,1}(0, 2\pi)$. Then, one has the asymptotic expansions*

$$\begin{aligned} & \left\| \frac{t}{1-2\omega} \partial_t Q^\omega - t^{1-2\omega}u_1 \right\|_{\min(\omega,0)} \\ & \lesssim \begin{cases} t^{2-2\omega}\|u_1\|_{\omega+1,1} + t^{-\omega}\|u_2\|_{1-\omega,1} + \eta(\omega)\|u_2\|_{L^1}, & \omega \leq 0 \\ t^{2-2\omega}\|u_1\|_{1-k,1} + \|u_2\|_{1-\omega,1}, & \omega \in (0, 1) \\ t^{2-2\omega}\|u_1\|_{1-k,1} + t^{1-\omega}\|u_2\|_{1-\omega,1}, & \omega \geq 1. \end{cases} \end{aligned}$$

Proof of Theorem 4.4. The asymptotic expansions (49) are a consequence of the following estimates, which in turn are obtained immediately from lemmas 4.2 and 4.3: If $\omega \leq 0$, then

$$\begin{aligned} \|Q_1^\omega - t^{1-2\omega}u_1\|_{\omega-1,1} &\lesssim t^{2-2\omega}\|u_1\|_{\omega,1}, \\ \|Q_2^\omega - u_2\|_{L^1(0,2\pi)} &\lesssim t\|u_2\|_{k,1}. \end{aligned}$$

If $\omega \in (0, 1)$, then

$$\begin{aligned} \|Q_1^\omega - t^{1-2\omega}u_1\|_{\omega-1,1} &\lesssim t^{2-2\omega}\|u_1\|_{\omega,1}, \\ \|Q_2^\omega - u_2\|_{-\omega,1} &\lesssim t\|u_2\|_{1-\omega,1}. \end{aligned}$$

If $\omega \geq 1$, then

$$\begin{aligned} \|Q_1^\omega - t^{1-2\omega}u_1\|_{L^1(0,2\pi)} &\lesssim t^{2-2\omega}\|u_1\|_{-k,1}, \\ \|Q_2^\omega - u_2\|_{-\omega,1} &\lesssim t\|u_2\|_{1-\omega,1}. \end{aligned}$$

Also, the estimates (47), (48) are a consequence of Lemma 4.2. □

Theorem 4.5 is checked similarly. For the proof of Theorem 4.6, note the following relation

$$\partial_t \Psi_\omega^t = \frac{2\omega - 1}{t} (\Psi_{\omega-1}^t - \Psi_\omega^t),$$

which holds in the sense of distributions and is used to compute the time derivatives of the solution. The estimates in Theorem 4.6 are then deduced from lemmas 4.2 and 4.3. The only thing to note is the function $\eta(\omega)$. It comes from performing the estimates in Theorem 4.6 in the particular cases $\omega = -n$. In these cases, cancellation of the terms of first order $\|u_2\|_{L^1}$ occurs, which accounts for the function η . See below for some explicit calculations.

Finally, similar results may be obtained for data in the spaces $H_{\text{per}}^\lambda(0, 2\pi)$ studied in Section 3.3. We omit these results for the sake of brevity.

4.3. Some particular values of ω . The results of the preceding theorems are clarified by considering some particular values of the parameter ω , where explicit calculations may be done. Foremost is the case $\omega = 0$, which corresponds to the wave equation. In that case, the explicit solution is well known and is given by

$$Q^0 = \frac{t}{2} \int_{-1}^1 u_1(\theta + tx) dx + \frac{1}{2} (u_2(\theta + t) + u_2(\theta - t)). \tag{50}$$

Alternatively, one may use the formula for Ψ_0 given by (22) and the definition of Q^ω in (42) to arrive at (50), when u_1 is integrable. However, the formulas (22), (42) allow for distributions u_1 with less regularity, namely in the space $W_{\text{per}}^{-1,1}(0, 2\pi)$. We must now write

$$t\Psi_1^t * u_1(\theta)$$

instead of the integral in (50) (recall that $\Psi_\omega^t = (1/t)\sigma_{1/t}\Psi_\omega$). The kernel Ψ_1 is simply $\chi_{(-1,1)}/2$.

In the case $\omega = 0$, the second claim of Theorem 4.4 states that if $(u_1, u_2) \in L^1(0, 2\pi) \times W_{\text{per}}^{1,1}(0, 2\pi)$, then

$$\|Q^0 - tu_1 - u_2\|_{-1,1} \lesssim t^2\|u\|_{L^1} + t\|u_2\|_{1,1}.$$

This should be compared with the fact that using elementary methods one may take (more regular) data $(u_1, u_2) \in W_{\text{per}}^{1,1}(0, 2\pi) \times W_{\text{per}}^{2,1}(0, 2\pi)$, for which (50) makes sense, and obtain the estimate

$$\|Q^0 - tu_1 - u_2\|_{L^1} \lesssim t^2\|u\|_{1,1} + t\|u_2\|_{2,1}.$$

Here, we see that in order to be able to consider data with less regularity, the asymptotic development takes place in a larger space. The interest in this case lies in the fact that the estimate for u_1 is not easily obtainable by elementary techniques. Indeed, since u_1 is just a distribution, one cannot work directly with integral expressions such as (50). Our analysis is therefore necessary to obtain regularity estimates.

We now examine the case $\omega = -1$, where similar observations apply. According to the formulas (22), (42), the solution of the EPD equation (42) is now given by

$$Q^{-1} = t^3\Psi_2^t * u_1 + \frac{1}{2}(u_2(\theta + t) + u_2(\theta - t)) + \frac{t}{2}(u_2'(\theta + t) - u_2'(\theta - t)),$$

with $\Psi_2(x) = (1 - x^2)_+/2$ and $(u_1, u_2) \in W_{\text{per}}^{-2,1}(0, 2\pi) \times W_{\text{per}}^{1,1}(0, 2\pi)$. By Theorem 4.4, Q^{-1} is in $L^1(0, 2\pi)$. Other explicit solutions may be computed similarly.

5. Euler–Poisson–Darboux equation (exceptional exponents)

The expression (39) handles separately the two fundamental kernels, but does not show clearly how the solution depends on the parameter ω . In fact, we now rewrite (39) in a different form allowing us to pass to the limit when $\omega \rightarrow 1/2$, using the continuity results in Lemma 2.4.

To achieve this, observe that according to Lemma 2.4,

$$\frac{\Psi_\omega - \Psi_{1-\omega}}{1 - 2\omega} \rightarrow \Psi_{1/2} \ln(K_{1/2}(1 - x^2))$$

in the sense of distributions when $\omega \rightarrow 1/2$. In addition, as we will see in Section 6, this distribution is a solution of the EPD equation with $\omega = 1/2$. Thus, given initial data ψ, q , we need only choose u_1, u_2 in (39) appropriately so that the above quotient will arise. This leads us to the choice

$$u_1(x) = \frac{\psi(x)}{1 - 2\omega}, \quad u_2(x) = q(x) - \frac{\psi(x)}{1 - 2\omega}.$$

Thus, given two functions q, ψ , the solution of (36) for $\omega \in (0, 1) \setminus \{1/2\}$ is given by

$$\begin{aligned} Q^\omega(t, \theta) &:= \frac{t^{1-2\omega}}{1 - 2\omega} \langle \Psi_{1-\omega}, \psi \rangle + \left\langle \Psi_\omega, q - \frac{\psi}{1 - 2\omega} \right\rangle \\ &= \langle \Psi_\omega, q \rangle + \frac{t^{1-2\omega} - 1}{1 - 2\omega} \langle \Psi_{1-\omega}, \psi \rangle + \left\langle \frac{\Psi_{1-\omega} - \Psi_\omega}{1 - 2\omega}, \psi \right\rangle. \end{aligned} \tag{51}$$

Taking the limit $\omega \rightarrow 1/2$ in (51) we obtain

$$Q^{1/2}(t, \theta) = \langle \Psi_{1/2}, q(\theta + t \cdot) \rangle + \langle \Psi_{1/2} \ln(tK_{1/2}(1 - x^2)), \psi(\theta + t \cdot) \rangle, \tag{52}$$

which is the solution for $\omega = 1/2$.

Considering the other exceptional values $\omega = -1/2, -3/2, \dots$, we can generalize (51) as follows:

Proposition 5.1. *Let $u(t, \theta)$ be a solution of the EPD equation (36), that is, $\mathcal{P}^\omega(u) = 0$. Then the function*

$$v(t, \theta) = \mathcal{G}^\omega(u(t, \theta)) := (2\omega - 1)u(t, \theta) + t \frac{\partial}{\partial t} u(t, \theta) \tag{53}$$

satisfies $\mathcal{P}^{\omega-1}(v) = 0$.

This result can be checked by direct substitution of v into the equation (36) and is valid for all $\omega \in \mathbb{R}$. Therefore, it may be used to define solutions of $\mathcal{P}^{\omega-1}$ from solutions of \mathcal{P}^ω . For instance, take $\omega - 1 = -1/2$. Recall that $\Psi_{-1/2}$ is not

defined, so it is not a solution of $\mathcal{P}^{-1/2}$. Let us apply Lemma 5.1 with u given by (52). For the first term we find

$$\begin{aligned} \mathcal{G}^{1/2}(\langle \Psi_{1/2}, q(\theta + t \cdot) \rangle) &= t \frac{\partial}{\partial t} \langle \Psi_{1/2}, q(\theta + t \cdot) \rangle \\ &= t \langle x \Psi_{1/2}, q'(\theta + t \cdot) \rangle = \frac{t^2}{2} \langle \Psi_{3/2}, q''(\theta + t \cdot) \rangle \end{aligned}$$

and for the second,

$$\begin{aligned} \mathcal{G}^{1/2}(\langle \Psi_{1/2} \ln(tK_{1/2}(1 - x^2)), \psi(\theta + t \cdot) \rangle) &= t \frac{\partial}{\partial t} \langle \Psi_{1/2} \ln(tK_{1/2}(1 - x^2)), \psi(\theta + t \cdot) \rangle \\ &= \langle \Psi_{1/2}, \psi(\theta + t \cdot) \rangle + t \langle x \Psi_{1/2} \ln(tK_{1/2}(1 - x^2)), \psi'(\theta + t \cdot) \rangle. \end{aligned}$$

So that a solution of $\mathcal{P}^{-1/2}(u) = 0$ is given by

$$\begin{aligned} Q^{-1/2}(t, \theta) &= \frac{t^2}{2} \langle \Psi_{3/2}, q''(\theta + t \cdot) \rangle \\ &\quad + \langle \Psi_{1/2}, \psi(\theta + t \cdot) \rangle + t \langle x \Psi_{1/2} \ln(tK_{1/2}(1 - x^2)), \psi'(\theta + t \cdot) \rangle. \end{aligned}$$

Clearly, this procedure may be iterated to find solutions of $\mathcal{P}^\omega = 0$ for $\omega = -3/2, -5/2, \dots$. Such solutions consist of a term $Ct^{1-2\omega} \langle \Psi_{1-\omega}, q^{(2k)} \rangle$, with $1 - \omega - k \in (0, 1)$, which vanishes when $t \rightarrow 0$ since $1 - 2\omega > 0$, a term $C \langle \Psi_{1/2}, \psi \rangle$, and terms of the form $Ct^a \langle x^b \Psi_{1/2} \ln(tK_{1/2}(1 - x^2)), \psi^{(k)} \rangle$, with $t^a \rightarrow 0$. Therefore, for $\omega = -3/2, -5/2, \dots$, we have $Q^\omega(t, \theta) \rightarrow C\psi(\theta)$. An explicit formula for this constant can be found in [2]. Note that only when $\omega = 1/2$ does the solutions blows-up as $\ln t$ when $t \rightarrow 0$.

6. A special case of interest

Here, we search for functions $P : \mathbb{R}_+ \times [0, 2\pi]$ that are periodic in space and satisfy the equation

$$P_{tt} + \frac{1}{t} P_t - P_{\theta\theta} = 0. \tag{54}$$

We begin the discussion by constructing solutions with bounded variation when the data have bounded variation (Theorem 6.1), and next we determine the optimal regularity assumption on the data ensuring that solutions have bounded variation.

It is straightforward to check that, given arbitrary smooth and periodic functions $v, \varphi : [0, 2\pi] \rightarrow \mathbb{R}$ representing the singular behavior of the solution on the line $t = 0$, the explicit formula

$$\begin{aligned}
 P(t, \theta) &= \frac{1}{\pi} \int_{\theta-t}^{\theta+t} v(\theta') \ln \left(\frac{K_{1/2}}{t} (t^2 - (\theta - \theta')^2) \right) (t^2 - (\theta - \theta')^2)^{-1/2} d\theta' \\
 &\quad + \frac{1}{\pi} \int_{\theta-t}^{\theta+t} \varphi(\theta') (t^2 - (\theta - \theta')^2)^{-1/2} d\theta'
 \end{aligned} \tag{55}$$

makes sense and yields a 2π -periodic solution of the equation (54). Here,

$$\begin{aligned}
 \ln K_{1/2} &:= -(1/\pi) \int_{\theta-t}^{\theta+t} \ln \left(1 - \frac{(\theta - \theta')^2}{t^2} \right) (t^2 - (\theta - \theta')^2)^{-1/2} d\theta' \\
 &= -(1/\pi) \int_{-1}^1 (1 - x^2)^{-1/2} \ln(1 - x^2) dx
 \end{aligned}$$

is a normalization constant. By formally expanding $P(t, \theta)$ when $t \rightarrow 0$ and assuming that the data v and φ are smooth, we can check that the following expansion holds in the pointwise sense,

$$\lim_{t \rightarrow 0} (P(t, \theta) - v(\theta) \ln t - \varphi(\theta)) = 0, \quad \theta \in [0, 2\pi].$$

We are interested in extending this result to data and solutions with weaker regularity, especially solutions that are solely in the space $BV_{\text{per}}(0, 2\pi)$ of 2π -periodic functions with bounded total variation. We will see that the condition at $t = 0$ must be relaxed and holds only in the L^1 sense.

To recast (55) in the distributional framework developed in Section 2, we perform the change of variable $x = (\theta - \theta')/t$ and rewrite the formula (55) in the form

$$P(t, \theta) = \frac{1}{\pi} \int_{-1}^1 \frac{v(\theta + tx)}{\sqrt{1 - x^2}} \ln(tK_{1/2}(1 - x^2)) dx + \frac{1}{\pi} \int_{-1}^1 \frac{\varphi(\theta + tx)}{\sqrt{1 - x^2}} dx. \tag{56}$$

Recalling the definition of the distributions Ψ_ω , (22), we note that the distribution $\Psi_{1/2}$ is a regular function and we may write

$$\begin{aligned}
 P(t, \theta) &= \langle \Psi_{1/2} \ln(tK_{1/2}(1 - x^2)), (\tau_{t,\theta})v \rangle + \langle \Psi_{1/2}, (\tau_{t,\theta})\varphi \rangle, \\
 \ln K_{1/2} &:= -\langle \Psi_{1/2} \ln(1 - x^2), 1 \rangle,
 \end{aligned} \tag{57}$$

where we recall that

$$(\tau_{t,\theta})v(x) = v(\theta + tx), \quad x \in [0, 2\pi].$$

We investigate the regularity of P in terms of the regularity of the data v, φ and validate the above expansion of P near the line $t = 0$. In turn, this result may be used to find regularity estimates for the exceptional solutions of (36) constructed in Proposition 5.1.

Theorem 6.1. *Given any data $v, \varphi \in BV_{\text{per}}(0, 2\pi)$, the explicit formula (55) defines a solution $P \in L^\infty(\varepsilon, \infty, BV_{\text{per}}(0, 2\pi))$ (for every $\varepsilon > 0$) of the Euler-Poisson-Darboux equation (55) which satisfies the given data at $t = 0$ in the L^1 sense, as follows: define \tilde{P} by*

$$\tilde{P}(t, \theta) := P(t, \theta) - v(\theta) \ln t - \varphi(\theta).$$

Then, \tilde{P} satisfies the time and space estimates ($0 < t < t'$)

$$\text{TV}(\tilde{P}(t)) \leq 2(|\ln t| + \ln K_0) \text{TV}(v) + 2 \text{TV}(\varphi) \tag{58}$$

$$\left\| \frac{\tilde{P}(t) - \tilde{P}(t')}{t - t'} \right\|_{L^1(0, 2\pi)} \leq \frac{2}{\pi} \left(|\ln t| + \frac{t'}{t} + \ln K_0 + 2 \right) \text{TV}(v) + \frac{2}{\pi} \text{TV}(\varphi). \tag{59}$$

Moreover, P_t exists in a classical sense for almost all (t, θ) , and

$$\|tP_t(t) - v\|_{L^1(0, 2\pi)} \lesssim t|\ln t| \text{TV}(v) + t \text{TV}(\varphi). \tag{60}$$

Also, one has

$$\|\tilde{P}(t)\|_{L^1(0, 2\pi)} \leq \frac{2t}{\pi} (|\ln t| + \ln K_0 + 2) \text{TV}(v) + \frac{2t}{\pi} \text{TV}(\varphi) \tag{61}$$

and

$$\frac{\tilde{P}_\theta}{|\ln t|} \xrightarrow{*} 0 \text{ in the weak-star sense of measures as } t \rightarrow 0. \tag{62}$$

Since (54) is a linear equation we immediately deduce from (61) the following continuous dependence property

$$\begin{aligned} \|P(t) - P'(t)\|_{L^1(0, 2\pi)} &\leq (\ln t) \|v - v'\|_{L^1(0, 2\pi)} + \|\varphi - \varphi'\|_{L^1(0, 2\pi)} \\ &\quad + \frac{2t}{\pi} (|\ln t| + |\ln K_0| + 2) \text{TV}(v - v') \\ &\quad + \frac{2t}{\pi} \text{TV}(\varphi - \varphi'), \end{aligned} \tag{63}$$

valid for any two solutions P, P' associated with data v, φ and v', φ' , respectively.

Proof. The identities

$$\begin{aligned} \frac{1}{\pi} \int_{-1}^1 |x|(1-x^2)^{-1/2} dx &= \frac{2}{\pi}, \\ \frac{1}{\pi} \int_{-1}^1 |x \ln(1-x^2)|(1-x^2)^{-1/2} dx &= \frac{4}{\pi}, \quad \int_{-1}^1 (1-x^2)^{-1/2} dx = \pi \end{aligned}$$

will be used throughout this proof. Consider a continuous test function ψ on S^1 . Let $\tau_a f$ denote the function $\theta \mapsto f(\theta + a)$. Then, it is easy to see that if $f \in BV(S^1)$, we have

$$\langle \partial_\theta(\tau_a f), \psi \rangle \langle f_\theta, \tau_{-a} \psi \rangle.$$

Therefore, we find

$$\begin{aligned} \langle \tilde{P}_\theta, \psi \rangle &= \langle P_\theta - v_\theta \ln t - \varphi_\theta, \psi \rangle \\ &= \frac{1}{\pi} \int_{-1}^1 \langle \tau_{xt}(v_\theta) - v_\theta, \psi \rangle \ln t (1-x^2)^{-1/2} dx \\ &\quad + \frac{1}{\pi} \int_{-1}^1 \langle \tau_{xt}(v_\theta), \psi \rangle \ln(K_0(1-x^2)) (1-x^2)^{-1/2} dx \\ &\quad + \frac{1}{\pi} \int_{-1}^1 \langle \tau_{xt}(\varphi_\theta) - \varphi_\theta, \psi \rangle (1-x^2)^{-1/2} dx \end{aligned}$$

thus

$$\begin{aligned} \langle \tilde{P}_\theta, \psi \rangle &= \langle P_\theta - v_\theta \ln t - \varphi_\theta, \psi \rangle \\ &= \frac{1}{\pi} \int_{-1}^1 \langle v_\theta, \tau_{-xt} \psi - \psi \rangle \ln t (1-x^2)^{-1/2} dx \\ &\quad + \frac{1}{\pi} \int_{-1}^1 \langle v_\theta, \tau_{-xt} \psi \rangle \ln((1-x^2)K_0) (1-x^2)^{-1/2} dx \\ &\quad + \frac{1}{\pi} \int_{-1}^1 \langle \varphi_\theta, \tau_{-xt} \psi - \psi \rangle (1-x^2)^{-1/2} dx. \end{aligned}$$

To derive the estimate (58) we observe that

$$\begin{aligned} \text{TV}(\tilde{P}(t)) &= \sup_{\substack{\psi \in C(S^1) \\ \|\psi\|_\infty=1}} |\langle \tilde{P}_\theta(t), \psi \rangle| \\ &\leq \frac{1}{\pi} \sup_{\|\psi\|_\infty=1} \int_{-1}^1 |\langle v_\theta, \tau_{-tx} \psi - \psi \rangle \ln t| (1-x^2)^{-1/2} dx \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\pi} \sup_{\|\psi\|_\infty=1} \int_{-1}^1 |\langle v_\theta, \tau_{-tx}\psi \rangle \ln(K_0(1-x^2))| (1-x^2)^{-1/2} dx \\
 & + \frac{1}{\pi} \sup_{\|\psi\|_\infty=1} \int_{-1}^1 |\langle \varphi_\theta, \tau_{-tx}\psi - \psi \rangle| (1-x^2)^{-1/2} dx
 \end{aligned}$$

thus

$$\text{TV}(\tilde{P}(t)) \leq 2 \text{TV}(v) + 2 \text{TV}(v) \ln K_0 + 2 \text{TV}(\varphi).$$

Consider now the estimate (62). Since

$$\frac{1}{\pi} \int_{-1}^1 \ln(K_0(1-x^2))(1-x^2)^{-1/2} dx = 0, \tag{64}$$

we may write

$$\begin{aligned}
 |\langle \tilde{P}_\theta, \varphi \rangle| & \leq \frac{1}{\pi} \int_{-1}^1 |\langle v_\theta, \tau_{-xt}\psi - \psi \rangle \ln(tK_0(1-x^2))| (1-x^2)^{-1/2} dx \\
 & + \frac{1}{\pi} \int_{-1}^1 |\langle \varphi_\theta, \tau_{-xt}\psi - \psi \rangle| (1-x^2)^{-1/2} dx
 \end{aligned}$$

thus

$$\begin{aligned}
 |\langle \tilde{P}_\theta, \varphi \rangle| & \leq \text{TV}(v) \frac{1}{\pi} \int_{-1}^1 \sup_{\theta \in S^1} |\psi(\theta - xt) - \psi(\theta)| |\ln(t(1-x^2)K_0)| (1-x^2)^{-1/2} dx \\
 & + \text{TV}(\varphi) \frac{1}{\pi} \int_{-1}^1 \sup_{\theta \in S^1} |\psi(\theta - xt) - \psi(\theta)| (1-x^2)^{-1/2} dx.
 \end{aligned}$$

Now, given any $\varepsilon > 0$, we may find δ small enough so that $|\ln t|^{-1} \leq 1$ if $t < \delta$, and (since ψ is uniformly continuous), $\sup_{\theta \in S^1} |\psi(\theta - \theta') - \psi(\theta)| < \varepsilon$ for all $|\theta'| < \delta$. Take $t < \delta$, so that $|tx| \leq t < \delta$. We obtain

$$|\langle \tilde{P}_\theta, \psi \rangle| / |\ln t| \leq \varepsilon \text{TV}(\varphi) + \varepsilon \text{TV}(v)(1 + 2|\ln K_0|) \lesssim \varepsilon,$$

which gives (62).

To derive (59), we observe that

$$\begin{aligned}
 \frac{P(t, \theta) - P(t', \theta)}{t - t'} & = \frac{1}{\pi} \int_{-1}^1 \frac{\varphi(\theta + tx) - \varphi(\theta + t'x)}{t - t'} (1-x^2)^{-1/2} dx \\
 & + \frac{1}{\pi} \int_{-1}^1 \frac{v(\theta + tx) \ln t - v(\theta + t'x) \ln t'}{t - t'} (1-x^2)^{-1/2} dx
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\pi} \int_{-1}^1 \frac{v(\theta + tx) - v(\theta + t'x)}{t - t'} \ln(K_0(1 - x^2))(1 - x^2)^{-1/2} dx \\
 & =: A + B + C.
 \end{aligned}$$

For the first term on the right-hand side above we can write

$$\begin{aligned}
 \|A\|_{L^1(0,2\pi)} & \leq \frac{1}{\pi} \int_{-1}^1 \int_0^{2\pi} \left| \frac{\varphi(\theta + tx) - \varphi(\theta + t'x)}{xt - xt'} \right| |x|(1 - x^2)^{-1/2} dx \\
 & \leq \text{TV}(\varphi) \frac{1}{\pi} \int_{-1}^1 |x|(1 - x^2)^{-1/2} dx = \frac{2}{\pi} \text{TV}(\varphi).
 \end{aligned}$$

On the other hand, a straightforward computation yields

$$\begin{aligned}
 B - \frac{\ln t - \ln t'}{t - t'} v(\theta) \frac{1}{\pi} \int_{-1}^1 \frac{v(\theta + tx) - v(\theta + t'x)}{t - t'} \ln t(1 - x^2)^{-1/2} dx \\
 + \frac{1}{\pi} \int_{-1}^1 \frac{\ln t - \ln t'}{t - t'} (v(\theta + t'x) - v(\theta))(1 - x^2)^{-1/2} dx
 \end{aligned}$$

and thus

$$\begin{aligned}
 \left\| B - \frac{\ln t - \ln t'}{t - t'} v(\cdot) \right\|_{L^1(0,2\pi)} & \leq |\ln t| \text{TV}(v) \frac{1}{\pi} \int_{-1}^1 |x|(1 - x^2)^{-1/2} dx \\
 & \quad + \frac{1}{\pi} \int_{-1}^1 \frac{1}{t} \int_0^{2\pi} \left| \frac{v(\theta + t'x) - v(\theta)}{t'x} \right| t'x |x|(1 - x^2)^{-1/2} dx \\
 & = \frac{2}{\pi} \left(|\ln t| + \frac{t'}{t} \right) \text{TV}(v).
 \end{aligned}$$

Finally, for third term we have

$$\|C\|_{L^1(0,2\pi)} \leq \frac{4}{\pi} \text{TV}(v) + \frac{2}{\pi} |\ln K_0| \text{TV}(v).$$

Combining the above estimates gives (59).

To show (61), we note that

$$\begin{aligned}
 \tilde{P}(t, \theta) & = \frac{1}{\pi} \int_{-1}^1 (\varphi(\theta + tx) - \varphi(\theta))(1 - x^2)^{-1/2} dx \\
 & \quad + \frac{1}{\pi} \int_{-1}^1 (v(\theta + tx) - v(\theta)) \ln t(1 - x^2)^{-1/2} dx \\
 & \quad + \frac{1}{\pi} \int_{-1}^1 (v(\theta + tx) - v(\theta)) \ln(K_0(1 - x^2))(1 - x^2)^{-1/2} dx,
 \end{aligned}$$

where we have used (64). Integrating over the interval $[0, 2\pi]$ we obtain

$$\begin{aligned} \|\tilde{\mathcal{P}}(t)\|_{L^1(0, 2\pi)} &\leq \frac{1}{\pi} \int_{-1}^1 \int_0^{2\pi} \left| \frac{\varphi(\theta + tx) - \varphi(\theta)}{tx} \right| t|x|(1-x^2)^{-1/2} d\theta dx \\ &\quad + \frac{1}{\pi} \int_{-1}^1 \int_0^{2\pi} \left| \frac{v(\theta + tx) - v(\theta)}{tx} \right| t|x \ln t|(1-x^2)^{-1/2} d\theta dx \\ &\quad + \frac{1}{\pi} \int_{-1}^1 \int_0^{2\pi} \left| \frac{v(\theta + tx) - v(\theta)}{tx} \right| t|x \ln((1-x^2)K_0)|(1-x^2)^{-1/2} d\theta dx. \end{aligned}$$

Hence

$$\|\tilde{\mathcal{P}}(t)\|_{L^1(0, 2\pi)} \leq \frac{2t}{\pi} \text{TV}(\varphi) + \frac{2t}{\pi} |\ln t| \text{TV}(v) + \frac{4t}{\pi} \text{TV}(v) + \frac{2t}{\pi} |\ln K_0| \text{TV}(v),$$

which is (61). This completes the proof of Theorem 6.1. \square

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References

- [1] M. L. Bernardi, Second order abstract differential equations with singular coefficients. *Ann. Mat. Pura Appl.* (4) **130** (1982), 257–286. [Zbl 0486.35066](#) [MR 663974](#)
- [2] E. K. Blum, The solutions of the Euler-Poisson-Darboux equation for negative values of the parameter. *Duke Math. J.* **21** (1954), 257–269. [Zbl 0057.08001](#) [MR 0063542](#)
- [3] F. J. Bureau, Divergent integrals and partial differential equations. *Comm. Pure Appl. Math.* **8** (1955), 143–202. [Zbl 0064.09204](#) [MR 0068085](#)
- [4] R. W. Carroll and R. E. Showalter, *Singular and degenerate Cauchy problems*. Math. Sc. Eng. 127, Academic Press, New York 1976. [MR 0460842](#)
- [5] J.-P. Dias and M. Povoas, A variant of the Euler-Poisson-Darboux equation arising in quantum gravity. *C. R. Acad. Sci. Paris Sér. I Math.* **315** (1992), 169–173. [Zbl 0769.35047](#) [MR 1197231](#)
- [6] I. M. Gelfand and G. E. Shilov, *Generalized functions*. Vol. 1, Academic Press, New York 1964. [Zbl 0115.33101](#) [MR 0435831](#)
- [7] L. Hörmander, *The analysis of linear partial differential operators I*. Grundlehren Math. Wiss. 256, Springer-Verlag, Berlin 1983. [Zbl 0521.35001](#) [MR 0717035](#)

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