Portugal. Math. (N.S.)
Vol. 65, Fasc. 4, 2008, 431–445

Portugaliae Mathematica

(Quasi)periodic solutions in (in)finite dimensional Hamiltonian systems with applications to celestial mechanics and wave equation

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(Communicated by Susanna Terracini)

Abstract. We describe a general method, based on a Lyapunov–Schmidt reduction and perturbative techniques, recently used by the authors to find periodic and quasi-periodic solutions both in finite and in infinite dimensional Hamiltonian systems. We also illustrate some concrete applications to celestial mechanics and nonlinear wave equation.

Mathematics Subject Classification (2000). Primary 34C25, 35L05, 70F10, 34C27; Secondary 37K50, 37J40, 70K43.

Keywords. Nearly-integrable Hamiltonian systems, periodic solutions lower dimensional elliptic tori, N-body problem, wave equation.

Introduction

In this note we deal with four topics:

Spatial planetary three-body problem. We consider one "star" and two "planets", modeled by three massive points, interacting through gravity in a three-dimensional space. Near the limiting solutions given by the two planets revolving around the star on Keplerian ellipses with *small eccentricity* and *small non-zero mutual inclination*, the system is proved to have *two-dimensional, elliptic, quasi-periodic solutions*, provided that the masses of the planets are small enough compared to the mass of the star and the osculating Keplerian major semiaxes belong to a two-dimensional set of density close to one.

Planar planetary many-body problem. As above, but one "star" and N "planets", the interior two ones bigger than the others (as in the exterior solar system). Near the limiting solutions given by the N planets revolving around the star on

*Supported by MIUR *Variational Methods and Nonlinear Differential Equations.*

Keplerian ellipses with *small eccentricity* and *zero mutual inclination*, the system is proved to have *N-dimensional, elliptic, quasi-periodic solutions*.

Periodic orbits approaching lower dimensional elliptic KAM tori. By a general Birkhoff–Lewis–Conley–Zehnder-type result, we prove the existence of *infinitely many periodic solutions with larger and larger minimal period*, accumulating onto elliptic invariant tori of Hamiltonian systems.

As an application, periodic orbits close to the quasi-periodic ones of the above planetary problems are constructed.

Long time periodic orbits for the nonlinear wave equation. A Birkhoff–Lewis result for the nonlinear wave equation is given, which yields the existence of solutions of longer and longer period accumulating to zero, which is an elliptic equilibrium of the associated infinite dimensional Hamiltonian system.

The above results are discussed in full detail in [9], [10], [6], [16], [11], [12] and [13].

1. Quasi-periodic solutions

1.1. Maximal dimensional KAM tori. The main object of KAM Theory is the persistence of maximal invariant tori, i.e., tori of dimension n in nearly integrable Hamiltonian systems with n degrees of freedom of the type

$$H(I,\varphi) = h(I) + \varepsilon f(I,\varphi)$$

with $I \in \mathbb{R}^n$, $\varphi \in \mathbb{T}^n$.

When $\varepsilon = 0$, the system is completely integrable and the motion lies on the invariant tori $\{I_0\} \times \mathbb{T}^n$, traveled with (quasi)periodic motion

$$\varphi(t) = \varphi(0) + \omega_0 t,$$

where $\omega_0 = \nabla h(I_0)$.

When $\varepsilon \neq 0$, most of these tori survive, according to the arithmetic properties of the frequency, and their motions are conjugated to linear quasi-periodic ones.

KAM theory then found rich applications in planetary celestial mechanics, where ε represents the order of the ratio between the mass of the planets and the mass of the star. Namely, maximal KAM tori have been constructed in [1] for the planar three-body problem, in [27] and [32] for the spatial three-body problem and in [22] for the spatial many-body problem.

1.2. Lower dimensional KAM tori. KAM theory is also capable to detect the conservation of lower dimensional tori. In our case we deal with linearly stable (i.e., elliptic) tori, the conservation of which, *under suitable non-degeneracy assumptions*, was stated in [28], [21], [25], [29] and [20].

These lower dimensional elliptic tori may be visualized by looking at the Hamiltonian

$$H(I, \phi, p, q) = \omega \cdot I + \Omega \cdot J$$

with $(I, \phi) \in \mathbb{R}^n \times \mathbb{T}^n$ and $(p, q) \in \mathbb{R}^m \times \mathbb{R}^m$ conjugated variables and

$$J_i = \frac{p_i^2 + q_i^2}{2}.$$

The corresponding motions are

$$\begin{cases} I(t) = I_0, \\ \phi(t) = \phi_0 + \omega t, \\ \big(p(t), q(t)\big) = (p_0, q_0) \cos \Omega t. \end{cases}$$

and so the n-dimensional tori

$$\mathcal{T}(I_0) = \{I = I_0, \phi \in \mathbb{T}^n, p = q = 0\}$$

are invariant and traveled by the linear frequency ω, while the motions around $\mathcal{T}(I_0)$ revolve with the elliptic frequency Ω.

The KAM theory then is concerned with the preservation of such tori when the Hamiltonian presents higher order terms: for this some *rational independence* between ω and Ω is needed to avoid resonances.

In celestial mechanics, this independence becomes a touchy business, since the system is often degenerate and the frequencies are related among each other.

1.3. Elliptic two-dimensional invariant tori for the planetary three-body problem. Delaunay and Poincaré described the spatial three-body problem as a four-dimensional Hamiltonian system. The integrable limit obtained by neglecting the interaction between the small planets gives rise to decoupled Keplerian ellipses—in particular we look at the limiting cases of planets revolving on circles. This integrable limit possess motions lying on $\mathbb{T}^2$ (namely, the topological product of the two Keplerian orbits) and so *the number of linear frequencies is less than the number of degrees of freedom.*

In [24], the question on whether these tori survive is settled. As a matter of fact, the persistence of two-dimensional invariant tori for the planetary three-body problem for *large* mutual inclinations is proved in [24]. These orbits are *unstable* (i.e., partially hyperbolic) and they do not have any physical planetary analogue: as stated in [24], "the solutions found are of the elliptic-hyperbolic type and hence are unstable. It would be desirable to establish similarly the existence of such solutions in the stable case. In this case, however, there is an essential difficulty ... however, there are good reasons to conjecture that in general the stable solutions need not persist ...".

The difficulty Jefferys and Moser refer to is likely to be connected with, at least, the fact that the system is *properly degenerate* since on one hand it has *four* degrees of freedom and on the other hand the integrable limit depends only on *two* action variables. Also, the limiting solutions lie on *two*-dimensional tori, traveled with quasi-periodic frequencies which *depend* (due to Kepler's law) on the major semiaxes of the osculating ellipses, so such dependence may produce resonances.

These problems may be overcome since the major semiaxis are related to the two *non-degenerate* actions, so it is possible to keep track of their influence on the system. Then the question raised in [24] may be answered as follows:

Theorem 1.1 ([9]). *The spatial three-body problem possesses orbits lying on two-dimensional elliptic invariant tori traveled with irrational frequencies corresponding to a small eccentricity-inclination regime, provided that the two Keplerian major semiaxes belong to a two-dimensional set of nearly-full measure.*

The eccentricities are of order ε^c.

The inclinations may be chosen in a range from ε^c *up to order one in* ε, *and during the motions they oscillate of order* ε^c.

The semiaxes may be chosen as a set whose complement has measure of order ε^c *and during the motions they oscillate of order* ε^c.

The constants c above may be explicitly determined. The proof of Theorem 1.1 consists in three steps:

- The system is written in Delaunay-Poincaré variables as

$$H = -\frac{1}{2}\sum_j \frac{k_j}{\Lambda_j^2} + \varepsilon f(\Lambda, \lambda, \eta, \xi)$$

  with $j = 1, 2$.

  From the celestial mechanics viewpoint, k_j is constant, Λ_j^2 has size of the order of the major semiaxis of the j-th planet, (η_j, ξ_j) has size of order $\Lambda_j e_j^2$, where e_j is the eccentricity, and is oriented along the argument of the perihelion of the j-th planet.

- An averaging procedure over the fast angles (conjugated to the two non-degenerate actions) removes the degeneracy up to a high enough order, yielding a Hamiltonian of the form

$$H = h(I) + \varepsilon g(I, p, q) + o(\epsilon^3).$$

  The linear part can be diagonalized and the eigenvalues can be estimated.

- The quantitative version of the nondegeneracy conditions, required for the persistence of elliptic tori, is checked perturbatively.

1.4. N-dimensional elliptic invariant tori for the planar $(N+1)$-body problem. A result analogous to Theorem 1.1 holds for N planets in the planar case. The assumption that the system is planar is needed here to remove the degeneracies caused by further symmetries, since in contrast to the case $N = 2$ there is no reduction of the nodes available.

For concreteness, we focus on the case when the exterior planets are order of δ, smaller than the two interior ones, for a small $\delta \in (0, 1)$. In this case, which mimics the exterior solar system, the eigenvalues of the secular dynamics may be computed asymptotically in ϵ and δ and so a quantitative non-degeneracy condition may be checked due to the different scales involved.

We refer to [10] for further details.

2. Periodic solutions

2.1. Periodic orbits close to elliptic tori and applications to the three-body problem. The importance of periodic solutions in Hamiltonian systems was remarked by Poincaré: "D'ailleurs, ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissons essayer de pénétrer dans une place jusqu'ici réputée inabordable ...".

Poincaré also conjectured that periodic orbits approximate any trajectory: "... voici un fait que je n'ai pu démontrer rigoureusement, mais qui me parait pourtant très vraisemblable. Étant données des équations de la forme définie dans le n. 13[1] et une solution particulière quelconque de ces équations, on peut toujours trouver une solution périodique (dont la période peut, il est vrai, être très longue), telle que la différence entre les deux solutions soit aussi petite qu'on le veut, pendant un temps aussi long qu'on le veut."

A partial answer for such a conjecture was given in [31], by proving periodic orbits to be dense in any regular and compact energy surface, *generically in the C^2 category*.

The conjecture is *open* for given systems, and, in particular, for the three-body problem.

As an intermediate step towards this conjecture, one may try to find periodic orbits approaching invariant manifolds. Periodic orbits accumulating on elliptic periodic points and on elliptic periodic orbits are constructed in [14], while [17] provided periodic orbits accumulating on maximal KAM tori. These results may be rephrased by saying that in a system with n degrees of freedom, under suitable

[1]Formula n. 13 mentioned by Poincaré is the Hamilton equation.

assumptions, there is a plethora of periodic orbits accumulating on elliptic tori of dimension 0, 1 and n.

A natural question is whether it is possible to obtain analogous results for elliptic tori of intermediate dimension k, with $2 \le k \le n-1$. Such question is addressed by the following

Theorem 2.1 ([6]). *Under suitable non-degeneracy and non-resonance assumptions between linear and elliptic frequencies, there are infinitely many periodic orbits, whose minimal period goes to infinity, accumulating on elliptic invariant tori of any dimension.*

Theorem 2.1 finds a natural application to the three-body problem via the elliptic tori constructed in Theorem 1.1:

Theorem 2.2 ([6]). *The spatial planetary three-body problem affords, for ε small enough, infinitely many periodic solutions, with larger and larger minimal period, accumulating onto the elliptic invariant tori of Theorem* 1.1.

Also, near the unperturbed circular periodic motions with minimal period T, there correspond at least two geometrically distinct T-periodic trajectories (*provided that ε and T are subject to some inequalities*).

Similar applications to the planar N-body case are given in [16].

The proof of Theorem 2.1 is based on the following technique. By an *averaging procedure*, close to the torus we want to approach, the system is a small perturbation of the integrable Hamiltonian

$$H_{\rm int} = \omega \cdot I + \frac{\nu}{2} RI \cdot I + \frac{1}{2}\sum_i \Omega_i (p_i^2 + q_i^2) + \frac{\nu}{2}\sum_i (QI)_i (p_i^2 + q_i^2), \tag{1}$$

where R and Q are suitable matrices and $\nu > 0$ is a small parameter measuring the distance from the torus. The solutions of $H_{\rm int}$ are

$$\left.\begin{aligned} I(t) &= I_0, \\ \phi(t) &= \phi_0 + \tilde{\omega} t + \eta^2 Q^t (p_0^2 + q_0^2) t/2, \\ \big(p_i(t), q_i(t)\big) &= (p_{i0}, q_{i0}) \cos \tilde{\Omega}_i t, \end{aligned}\right\} \tag{2}$$

where

$$\tilde{\omega} := \omega + \nu R I_0$$

is the vector of the *shifted linear frequencies* and

$$\tilde{\Omega} := \Omega + \nu Q I_0,$$

is the vector of the *shifted elliptic frequencies*.

The manifold $\{p = q = 0\}$ is invariant for the H_{int}-flow and is filled up by the N-dimensional tori

$$\{I = I_0, \phi \in \mathbb{T}^N, p = q = 0\},$$

on which the flow $t \mapsto (I_0, \phi_0 + \tilde{\omega} t, 0, 0)$ is T-periodic, $T > 0$, if and only if

$$\tilde{\omega} \in \frac{2\pi}{T}\mathbb{Z}^N. \tag{3}$$

To satisfy (3), we choose

$$I_0 = I_0(T) := -R^{-1}\{\omega T/2\pi\}, \qquad v = v(T) := 2\pi/T,$$

where $\{\cdot\}$ denotes the fractional part. The infinite T-periodic orbits of the family

$$\mathscr{F} := \{I(t) = I_0, \phi(t) = \phi_0 + \tilde{\omega} t, p(t) = q(t) = 0\}$$

will not all persist for the flow of the complete Hamiltonian H. But if $\mathscr{F}$ is *isolated* (namely there are no other T-periodic solutions close to it), i.e.,

$$2\pi\ell - \tilde{\Omega}_i(T)T = 2\pi\ell - \Omega_i T + 2\pi(QR^{-1})_i\{\omega T/2\pi\} \neq 0 \qquad \text{for all } \ell, i \tag{4}$$

(recall (2)), we can prove existence of solutions of H bifurcating from $\mathscr{F}$. The "geometric" condition (4) clearly appears as a *non-resonance condition* between the linear frequencies ω and the elliptic ones Ω.

We will search T-periodic solutions of H of the form

$$\big(I(t), \phi(t), p(t), q(t)\big) = \big(I_0(T), \phi_0 + \tilde{\omega}(T)t, 0\big) + \zeta(t),$$

where $\phi_0 \in \mathbb{T}^N$ is a parameter to determinate, and

$$\zeta(t) = \big(J(t), \psi(t), p(t), q(t)\big)$$

is a small analytic and T-periodic curve with ψ having zero average. The Hamilton equations then reduce to the following functional equation for ζ and ϕ_0:

$$L\zeta = N(\zeta; \phi_0) \tag{5}$$

where L is the linear operator

$$L\zeta := \big(\dot{J}, \dot{\psi} - \eta^2 RJ, \dot{q} - \tilde{\Omega}(T)p, \dot{p} + \tilde{\Omega}(T)q\big)$$

and N is a suitable nonlinearity. The kernel $\mathscr{K}$ and the range $\mathscr{R}$ of L are, respectively,

$$\mathscr{K} = \{\psi \equiv \text{const}\}$$

and

$$\mathscr{R} = \left\{ \int_0^T \tilde{\psi} = 0 \right\}.$$

Then we use a Lyapunov–Schmidt reduction: equation $L\zeta = N(\zeta; \phi_0)$ splits into the *kernel equation*

$$0 = \Pi_{\mathscr{K}} N(\zeta; \phi_0) \tag{6}$$

and the *range equation*

$$L\zeta = \Pi_{\mathscr{R}} N(\zeta; \phi_0). \tag{7}$$

The "geometric" condition (4) is equivalent to the "analytic" fact that L is invertible on $\mathscr{R}$. If we choose the one-dimensional parameter T (the period) such that (4) holds, the range equation becomes $\zeta = L^{-1}\Pi_{\mathscr{R}} N(\zeta; \phi_0)$; for any fixed ϕ_0 in the kernel $\mathscr{K} \sim \mathbb{T}^N$, we solve the range equation by contractions.

Then we insert the solutions of the range equation into the kernel equation. It turns out that the resulting equation for ϕ_0 is the Euler–Lagrange equation of the *reduced action functional* (namely the action functional evaluated on the solutions of the range equation). Since ϕ_0 runs on a torus, elementary finite-dimensional analysis produces critical points of the reduced action functional and so the desired periodic orbits.

2.2. The wave equation. The first real breakthrough in the study of periodic solutions of the (one-dimensional) wave equation, the "vibrating string", was due to Rabinowitz at the end of the 1960s. He rephrased the problem as a *variational* one and proved existence under suitable assumption on the nonlinearity (e.g. monotonicity).

Many authors—Brezis, Nirenberg, Coron, Hofer, etc.—have used and developed variational methods to study the problem.

The variational approach has the advantage of being *global* and of imposing only few restrictions on the strength of the nonlinearity. On the other hand it does impose a very strong restriction on the allowed periods: they must be *rational* multiples of the string length.

This fact is due to the appearance of a *small divisors problem*. It should be remarked that in finite dimension small divisors appear only in searching quasi-periodic solutions. To understand why they come out in looking for periodic solutions in infinite dimension, let us consider the eigenvalues of the homogenous problem with Dirichlet boundary condition

$$\begin{cases} \Box u = u_{tt} - u_{xx} = 0, \\ u(t,0) = u(t,\pi) = 0, \\ u(t+T,x) = u(t,x) \end{cases}$$

(string length $= \pi$). The eigenvalues are

$$\lambda_{i\ell} = \omega^2 \ell^2 - i^2, \qquad i \geq 1,\ \ell \in \mathbb{Z}.$$

where $\omega = 2\pi/T$. There are two cases:

(1) $\omega \in \mathbb{Q}$: 0 is an eigenvalue of infinite multiplicity, namely $\Box$ has an infinite dimensional kernel, but on the cokernel $\Box^{-1}$ exists bounded (and compact);

(2) $\omega \in \mathbb{R}\backslash\mathbb{Q}$: 0 is not an eigenvalue, namely $\Box$ has no kernel, but $\lambda_{i\ell}$ accumulate to 0, namely $\Box^{-1}$ is unbounded (*small divisors problem*).

To deal with small divisors, a different approach was developed by Kuksin, Wayne, Craig, Bourgain, Pöschel, and others at the end of the 1980s. They used the fact that the wave equation is an *infinite dimensional Hamiltonian system* and modified the classical KAM ideas to work in this infinite dimensional context. This perturbative approach has the advantage of allowing *irrational* periods, but it is *local* in nature, namely it requires weak nonlinearity (equivalently small amplitude solutions).

The Hamiltonian structure. We look for periodic in time solutions of the one-dimensional nonlinear wave equation with Dirichlet boundary conditions (vibrating string):

$$\left.\begin{aligned} u_{tt} - u_{xx} + \mu u + f(u) = 0, \\ u(t,0) = u(t,\pi) = 0, \end{aligned}\right\} \tag{8}$$

where $\mu > 0$ is the "mass" and f, with $f(0) = f'(0) = 0$, is the nonlinearity.

Introducing $v = u_t$, the Hamiltonian is

$$H(v,u) = \int_0^\pi \left(\frac{v^2}{2} + \frac{u_x^2}{2} + \mu \frac{u^2}{2} + g(u) \right) dx,$$

where $g = \int_0^u f(s)\, ds$. The Hamilton equations are

$$u_t = \frac{\partial H}{\partial v} = v, \qquad v_t = -\frac{\partial H}{\partial u} = u_{xx} - \mu u - f(u).$$

Introducing coordinates

$$\mathfrak{q} = (\mathfrak{q}_1, \mathfrak{q}_2, \ldots) \in \ell^{a,s}, \qquad \mathfrak{p} = (\mathfrak{p}_1, \mathfrak{p}_2, \ldots) \in \ell^{a,s}$$

with

$$\ell^{a,s} := \left\{ \mathfrak{q} \mid \|\mathfrak{q}\|_{a,s}^2 := \sum_i |\mathfrak{q}_i|^2 i^{2s} e^{2ai} < \infty \right\}$$

by means of the relations

$$v(x) = \sum_i \sqrt{\omega_i}\,\mathfrak{p}_i \chi_i(x), \qquad u(x) = \sum_i \frac{\mathfrak{q}_i}{\sqrt{\omega_i}} \chi_i(x),$$

where $\chi_i(x) := \sqrt{2/\pi}\sin ix$ and

$$\omega_i := \sqrt{i^2 + \mu},$$

the Hamiltonian takes the form

$$H = \frac{1}{2}\sum_i \omega_i(\mathfrak{q}_i^2 + \mathfrak{p}_i^2) + P(\mathfrak{q}).$$

The origin is an *elliptic equilibrium*. Note the term $P = O(\mathfrak{q}^4)$ is regularizing, namely it *gains a derivative*

$$\nabla P : \ell^{a,s} \to \ell^{a,s+1}.$$

The orbits of the quadratic Hamiltonian $\sum_{i\geq 1} \omega_i(\mathfrak{q}_i^2 + \mathfrak{p}_i^2)/2$ are superpositions of the harmonic oscillations $\mathfrak{q}_i(t) = A_i \cos(\omega_i t + \varphi_i)$, where $A_i \geq 0$, $\varphi_i \in \mathbb{R}$ and ω_i are the amplitude, the phase and the frequency of the i-th harmonic oscillator, respectively.

Analogously, the solutions of the linear equations $u_{tt} - u_{xx} + \mu u = 0$ are

$$u(t,x) = \sum_{i\geq 1} a_i \cos(\omega_i t + \varphi_i) \sin ix,$$

with $a_i = A_i\sqrt{2/\pi\omega_i} \geq 0$. These solutions are

(1) *periodic* if *one* mode is excited,

(2) *quasi-periodic* if $N \geq 2$ modes are excited,

(3) *almost-periodic* if *infinitely* many modes are excited.

Only the orbit in (1) is periodic, since for all $\mathscr{I} := \{i_1, \ldots, i_N\} \subset \mathbb{N}^+$, $N \geq 2$, the frequency vector $\omega := (\omega_{i_1}, \ldots, \omega_{i_N})$ is *rationally independent*.

To prove nonlinear continuations of the periodic orbits (1) above means to extend the Lyapunov Center Theorem for finite dimensional systems to this infinite dimensional situation. Such results can be found in [33], [26], [19], [15], [2] and [18]. The case *completely resonant* case $\mu = 0$ (namely the extensions of the Weinstein and Moser theorems) can be found in [4], [7], [23], [8], [5].

The Birkhoff–Lewis periodic orbits discussed above for finite dimensional systems are completely different; in particular they have no analogs in the linear case.

In extending the results of [6] to the infinite dimensional case, one faces problems: the extension of the Birkhoff Normal Form and the small divisors.

A first result was obtained in [3] for the beam equation and the NLS equation. We remark that in this case the small divisors are not really small and no KAM analysis is necessary.

Let us consider the nonlinear wave equation in (8). Let us suppose that f is real analytic and odd, namely $f = \sum_{m \geq 3} f_m u^m$ with $f_3 \neq 0$, and let us fix a *finite* subset $\mathscr{I} = \{i_1, \dots, i_N\} \subset \mathbb{N}$, i.e., $\mathscr{I} = \{1, \dots, N\}$ (low modes).

Following [30], we perform a canonical transformation that puts H in partial Birkhoff Normal Form:

$$H = \Lambda + \bar{G} + \hat{G} + K$$

where

- $\Lambda = \sum_{i \geq 1} \omega_i (\mathfrak{p}_i^2 + \mathfrak{q}_i^2)/2$ is the quadratic part,
- $\bar{G} + \hat{G}$ is of order 4 in $\mathfrak{p}$, $\mathfrak{q}$,
- K is of order 6 in $\mathfrak{p}$, $\mathfrak{q}$,
- $\bar{G}$ depends only on the "actions" $(\mathfrak{p}_i^2 + \mathfrak{q}_i^2)/2$,
- $\hat{G}$ depends only on $\mathfrak{p}_i$, $\mathfrak{q}_i$, $i > N$ (high modes).

Since we look for small amplitude solutions, we introduce the perturbative parameter v (measuring the distance from the origin). We put action-angle variables (I, ϕ) on the low modes:

$$(\mathfrak{p}_i, \mathfrak{q}_i) = \sqrt{v}(\sqrt{I_i} \sin \phi_i, \sqrt{I_i} \cos \phi_i), \qquad i \leq N,$$
$$(\mathfrak{p}_i, \mathfrak{q}_i) = \sqrt{v}(p_i, q_i), \qquad i > N.$$

In the new variables (I, ϕ, p, q) the Hamiltonian is a small perturbation of the integrable Hamiltonian H_{int} defined in (1), where the index i runs over all the integer greater than N, $\omega = (\omega_i)_{i \leq N}$, $\Omega = (\omega_i)_{i > N}$, $A \in \mathrm{Mat}(N \times N)$ and $B \in \mathrm{Mat}(\infty \times N)$. Notice that the "twist" condition $\det A \neq 0$ holds (since $f_3 \neq 0$).

The crucial fact here is that, since now $i > N$, the quantities in the "geometric" condition (4) accumulate to zero; these are exactly the small divisors we have to deal with.

We will quantitatively impose that the admissible periods T satisfy

$$|2\pi\ell - \Omega_i T + (R^{-1}Q)_i\{\omega T\}| \geq \frac{\text{const}}{|i|^\tau} \tag{9}$$

for all $\ell \in \mathbb{Z}$, $i > N$, and for a suitable $\tau \geq 1$ ("Melnikov condition").

We have two cases: $\tau = 1$ or $\tau > 1$.

In [11], [12] the *strong* non-resonance condition $\tau = 1$ was imposed. In this case only a zero measure set of periods T satisfies (9). The great advantage is that, since $\tau = 1$, the small divisors cause the loss of only one "derivative", which is compensated by the fact that N "gains one derivative". Therefore the range equation can be solved by the standard Implicit Function Theorem (or Fixed Point Theorem), as in the finite dimensional situation.

The weaker condition $\tau > 1$ imposed in [13] allows a large set of admissible periods T satisfying (9). However the loss of $\tau > 1$ derivatives is no more compensated by the nonlinearity that gains only one derivative, and a Nash–Moser superconvergent procedure is necessary.

As it is well known, the crucial point in the Nash–Moser procedure is *the inversion of the linearized operator in a neighborhood of the origin*. Then one proceeds by the usual Newton superconvergent iterative scheme. However, due to that excision procedure used to control the small divisors at any steps, here one has also to *solve the bifurcation equation on a Cantor set*. This last problem is difficult (see [8] and [5]). To overcome it, invert the above procedure [13]:

(i) first we *solve the bifurcation equation* by suitable *symmetries of the Hamiltonian* (recall that f is odd),

(ii) then we solve the resulting range equation by a Nash–Moser scheme.

It remains to solve the inversion of the linearized operator. The present case presents two further technical difficulties:

- the linearized operator is a first order *non self-adjoint* operator requiring non standard spectral analysis,
- the linearized operator is not a *Toeplitz* operator (namely it does not simply act as a convolution operator).

We conclude recalling the main result of [13]:

Theorem 2.3 ([13]). *Fix $\mu > 0$ and let f be a real analytic, odd function of the form $f(u) = \sum_{m\geq 3} f_m u^m$, $f_3 \neq 0$. Let $N \geq 2$.*

Then there exists a Cantor set $\mathscr{C}$ *of asymptotically full measure*

$$\lim_{v_0 \to 0^+} \frac{\operatorname{meas}(\mathscr{C} \cap (0, v_0])}{v_0} = 1$$

such that for all $v = 2\pi/T \in \mathscr{C}$ *there exists a* T*-periodic analytic solution* $u(t,x)$ *of* (8) *satisfying*

$$u(t,x) = \sqrt{v} \sum_{i \le N} a_i \cos(\tilde{\omega}_i t) \sin ix + O(v), \qquad \tilde{\omega}_i - \omega_i = O(v),$$

for suitable $a_i > 0$, $\tilde{\omega}_i \in \mathbb{R}$.

Moreover, fix $0 < \rho < 1/2$. *Then, except a zero measure set of* μ'*s, the minimal period* $T^{\min}$ *of the* T*-periodic orbit satisfies*

$$T^{\min} \ge \text{const} \cdot T^{\rho}.$$

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Received July 2, 2007; revised July 25, 2007

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