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On the solution set of some classes of nonconvex nonclosed differential inclusions

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Abstract. We consider a nonconvex and nonclosed semilinear differential inclusion and prove the arcwise connectedness of the set of its mild solutions. A similar result is provided for a class of nonconvex, nonclosed second order differential inclusions.

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1. Introduction

This paper is concerned with the following semilinear differential inclusion

$$x' \in Ax + F\big(t, x, H(t, x)\big), \qquad x(0) = x_0, \tag{1.1}$$

where X is a real separable Banach space, $\mathscr{P}(X)$ is the family of all subsets of X, $I = [0, T]$, $F(\cdot, \cdot, \cdot) : I \times X^2 \to \mathscr{P}(X)$, $H(\cdot, \cdot) : I \times X \to \mathscr{P}(X)$ and A is the infinitesimal generator of a strongly continuous semigroup $\{G(t); t \geq 0\}$ on X.

When F does not depend on the last variable, (1.1) reduces to

$$x' \in Ax + F(t, x), \qquad x(0) = x_0. \tag{1.2}$$

Existence results and qualitative properties of the mild solutions of problem (1.2) may be found in [3], [6], [7], [9], [13] etc. In all these papers the set-valued map F is assumed to be at least closed-valued. Such an assumption is quite natural in order to obtain good properties of the solution set, but it is interesting to investigate the problem when the right-hand side of the multivalued equation may have nonclosed values.

Following the approach in [12] we consider problem (1.1), where F and H are closed-valued multifunctions Lipschitzian with respect to the second variable and F is contractive in the third variable. Obviously, the right-hand side of the differential inclusion in (1.1) is in general neither convex nor closed. We prove the arcwise connectedness of the solution set of (1.1). The main tool is a result ([11], [12]) concerning the arcwise connectedness of the fixed point set of a class of nonconvex nonclosed set-valued contractions.

Afterwards this result is extended to second-order differential inclusions of the form

$$x'' \in Ax + F(t, x, H(t, x)), \quad x(0) = x_0, \quad x'(0) = y_0,$$

where F and H are as above and A is the infinitesimal generator of a strongly continuous cosine family of operators $\{C(t); t \geq 0\}$ on X. We note that several existence results concerning mild solutions for the Cauchy problem

$$x'' \in Ax + F(t, x), \quad x(0) = x_0, \quad x'(0) = y_0,$$

can be found in [1], [2], [4], [5], etc.

The paper is organized as follows: in Section 2 we recall some preliminary results that we use in the sequel and in Section 3 we prove our main result.

2. Preliminaries

Let Z be a metric space with the distance d_Z and let 2^Z be the family of all nonempty closed subsets of Z. For $a \in Z$ and $A, B \in 2^Z$ we set $d_Z(a, B) = \inf_{b \in B} d_Z(a, b)$ and $d_Z^*(A, B) = \sup_{a \in A} d_Z(a, B)$. Denote by D_Z the Pompeiu–Hausdorff generalized metric on 2^Z defined by

$$D_Z(A, B) = \max\{d_Z^*(A, B), d_Z^*(B, A)\}, \quad A, B \in 2^Z.$$

In what follows, when the product $Z = Z_1 \times Z_2$ of metric spaces Z_i, $i = 1, 2$, is considered, it is assumed that Z is equipped with the distance $d_Z((z_1, z_2), (z_1', z_2')) = \sum_{i=1}^{2} d_{Z_i}(z_i, z_i')$.

Let X be a nonempty set and let $F : X \to 2^Z$ be a set-valued map from X to Z. The range of F is the set $F(X) = \bigcup_{x \in X} F(x)$. Let $(X, \mathscr{F})$ be a measurable space. The multifunction $F : X \to 2^Z$ is called measurable if $F^{-1}(\Omega) \in \mathscr{F}$ for any open set $\Omega \subset Z$, where $F^{-1}(\Omega) = \{x \in X; F(x) \cap \Omega \neq \emptyset\}$. Let (X, d_X) be a metric space. The multifunction F is called Hausdorff continuous if for any $x_0 \in X$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $x \in X$, $d_X(x, x_0) < \delta$ implies that $D_Z(F(x), F(x_0)) < \epsilon$.

Let $(T, \mathscr{F}, \mu)$ be a finite, positive, nonatomic measure space and let $(X, |\cdot|_X)$ be a Banach space. We denote by $L^1(T, X)$ the Banach space of all (equivalence classes of) Bochner integrable functions $u : T \to X$ endowed with the norm

$$|u|_{L^1(T,X)} = \int_T |u(t)|_X \, d\mu$$

A nonempty set $K \subset L^1(T,X)$ is called decomposable if, for every $u, v \in K$ and every $A \in \mathscr{F}$, one has

$$\chi_A \cdot u + \chi_{T \setminus A} \cdot v \in K,$$

where χ_B, $B \in \mathscr{F}$ indicates the characteristic function of B.

A metric space Z is called an absolute retract if, for any metric space X and any nonempty closed set $X_0 \subset X$, every continuous function $g : X_0 \to Z$ has a continuous extension $g : X \to Z$ over X. It is obvious that every continuous image of an absolute retract is an arcwise connected space.

In what follows we recall some preliminary results that are the main tools in the proof of our result.

Let $(T, \mathscr{F}, \mu)$ be a finite, positive, nonatomic measure space, S a separable Banach space and let $(X, |\cdot|_X)$ be a real Banach space. To simplify the notation we write E in place of $L^1(T,X)$.

Lemma 2.1 ([12]). *Assume that $\phi : S \times E \to 2^E$ and $\psi : S \times E \times E \to 2^E$ are Hausdorff continuous multifunctions with nonempty, closed, decomposable values satisfying the following conditions:*

a) *There exists $L \in [0,1)$ such that, for every $s \in S$ and every $u, u' \in E$,*

$$D_E\big(\phi(s,u), \phi(s,u')\big) \le L|u - u'|_E.$$

b) *There exists $M \in [0,1)$ such that $L + M < 1$ and for every $s \in S$ and every $(u,v), (u',v') \in E \times E$,*

$$D_E\big(\psi(s,u,v), \psi(s,u',v')\big) \le M(|u - u'|_E + |v - v'|_E).$$

Set $\mathrm{Fix}\big(\Gamma(s,\cdot)\big) = \{u \in E; u \in \Gamma(s,u)\}$, *where* $\Gamma(s,u) = \psi\big(s,u,\phi(s,u)\big)$, $(s,u) \in S \times E$. *Then the following holds:*

1) *For every $s \in S$ the set* $\mathrm{Fix}\big(\Gamma(s,\cdot)\big)$ *is nonempty and arcwise connected.*

2) *For any $s_i \in S$ and any $u_i \in \mathrm{Fix}\big(\Gamma(s,\cdot)\big)$, $i = 1, \dots, p$ there exists a continuous function $\gamma : S \to E$ such that $\gamma(s) \in \mathrm{Fix}\big(\Gamma(s,\cdot)\big)$ for all $s \in S$ and $\gamma(s_i) = u_i$, $i = 1, \dots, p$.*

Lemma 2.2 ([12]). *Let $U : T \to 2^X$ and $V : T \times X \to 2^X$ be two nonempty closed-valued multifunctions satisfying the following conditions:*

a) *U is measurable and there exists $r \in L^1(T)$ such that $D_X\big(U(t), \{0\}\big) \le r(t)$ for almost all $t \in T$.*

b) *The multifunction* $t \to V(t,x)$ *is measurable for every* $x \in X$.

c) *The multifunction* $x \to V(t,x)$ *is Hausdorff continuous for all* $t \in T$.

Then for any $v : T \to X$*, a measurable selection from* $t \to V(t, U(t))$*, there exists a selection* $u \in L^1(T,X)$ *such that* $v(t) \in V(t, u(t))$, $t \in T$.

In what follows $I = [0, T]$, X is a real separable Banach space with norm $|\cdot|$ and with the corresponding metric $d(\cdot,\cdot)$. We consider $\{G(t)\}_{t\geq 0} \subset L(X,X)$ a strongly continuous semigroup of bounded linear operators from X to X having the infinitesimal generator A and a set valued map $F(\cdot,\cdot)$ defined on $I \times X$ with nonempty closed subsets of X, which define the following differential inclusion

$$x' \in Ax + F(t,x), \qquad x(0) = x_0. \tag{2.1}$$

It is well known that, in general, the Cauchy problem

$$x' = Ax + f(t,x), \qquad x(0) = x_0$$

may not have a classical solution and that a way to overcome this difficulty is to look for continuous solutions of the integral equation

$$x(t) = G(t)x_0 + \int_0^t G(t-u)f(u, x(u))\,du.$$

This is why the concept of mild solution is convenient for solving (2.1).

A continuous mapping $x(\cdot) \in C(I,X)$ is called a *mild trajectory* of (2.1) if there exists a (Bochner) integrable function $f(\cdot) \in L^1(I,X)$ such that

$$f(t) \in F(t, x(t)) \ a.e. \ (I),$$

$$x(t) = G(t)x_0 + \int_0^t G(t-u)f(u)\,du \qquad \text{for all } t \in I,$$

i.e., $f(\cdot)$ is a (Bochner) integrable selection of the set-valued map $F(\cdot, x(\cdot))$ and $x(\cdot)$ is the mild solution of the initial value problem

$$x' = Ax + f(t), \qquad x(0) = x_0.$$

We shall use the following notation for the solution set of (2.1):

$$\mathcal{S}_1(x_0) = \{x(\cdot);\ x(\cdot) \text{ is a mild solution of } (2.1)\}. \tag{2.2}$$

Denote by $B(X)$ the Banach space of bounded linear operators from X into X. We recall that a family $\{C(t); t \in R\}$ of operators in $B(X)$ is a strongly continuous cosine family if the following conditions are satisfied:

(i) $C(0) = I$, where I is the identity operator in X,

(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t, s \in R$,

(iii) the map $t \to C(t)x$ is strongly continuous for all $x \in X$.

The strongly continuous sine family $\{S(t); t \in R\}$ associated to a strongly continuous cosine family $\{C(t); t \in R\}$ is defined by

$$S(t)x := \int_0^t C(s)x\,ds, \qquad x \in X,\ t \in R.$$

The infinitesimal generator $A : X \to X$ of a cosine family $\{C(t); t \in R\}$ is defined by

$$Ax = \left(\frac{d^2}{dt^2}\right) C(t)x|_{t=0}.$$

Fore more details on strongly continuous cosine and sine family of operators we refer to [8], [10], [14].

In what follows, A is infinitesimal generator of a cosine family $\{C(t); t \in R\}$ and $F(\cdot,\cdot) : I \times X \to \mathscr{P}(X)$ is a set-valued map with nonempty closed values, which define the following Cauchy problem associated to a second-order differential inclusion:

$$x'' \in Ax + F(t,x), \qquad x(0) = x_0, \qquad x'(0) = y_0. \tag{2.3}$$

A continuous mapping $x(\cdot) \in C(I,X)$ is called a *mild solution* of problem (2.3) if there exists a (Bochner) integrable function $f(\cdot) \in L^1(I,X)$ such that

$$f(t) \in F\big(t, x(t)\big)\ a.e.\ (I),$$

$$x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-u)f(u)\,du \qquad \text{for all } t \in I,$$

i.e., $f(\cdot)$ is a (Bochner) integrable selection of the set-valued map $F\big(\cdot, x(\cdot)\big)$ and $x(\cdot)$ is the mild solution of the Cauchy problem

$$x'' = Ax + f(t), \qquad x(0) = x_0, \qquad x'(0) = y_0.$$

We make the following notation:

$$\mathscr{S}_2(x_0, y_0) = \{x(\cdot);\ x(\cdot) \text{ is a mild solution of } (2.3)\}. \tag{2.4}$$

In order to study problems (2.1) and (2.3) we introduce the following hypothesis.

Hypothesis 2.3. *$F : I \times X \times X \to \mathscr{P}(X)$ and $H : I \times X \to \mathscr{P}(X)$ are two set-valued maps with nonempty closed values, satisfying*

i) *The set-valued maps $t \to F(t,u,v)$ and $t \to H(t,u)$ are measurable for all $u, v \in X$.*

ii) *There exist $l(\cdot) \in L^1(I,R)$ such that, for every $u, u' \in X$,*

$$D\big(H(t,u), H(t,u')\big) \le l(t)|u - u'| \text{ a.e. } (I).$$

iii) *There exist $m(\cdot) \in L^1(I,R)$ and $\theta \in [0,1)$ such that, for every $u, v, u', v' \in X$,*

$$D\big(F(t,u,v), F(t,u',v')\big) \le m(t)|u - u'| + \theta|v - v'| \text{ a.e. } (I).$$

iv) *There exist $f, g \in L^1(I,R)$ such that*

$$d\big(0, F(t,0,0)\big) \le f(t), \quad d\big(0, H(t,0)\big) \le g(t) \text{ a.e. } (I).$$

In what follows, $N(t) = \max\{l(t), m(t)\}$, $t \in I$ and $N^*(t) = \int_0^t N(s)\,ds$.

Given $\alpha \in R$, we denote by L^1 the Banach space of all (equivalence classes of) Lebesgue measurable functions $\sigma : I \to X$ endowed with the norm

$$|\sigma|_1 = \int_0^T e^{-\alpha N^*(t)} |\sigma(t)|\,dt.$$

3. Main results

Even if the multifunction from the right-hand side of (1.1) has, in general, non-closed, nonconvex values, the solution sets $\mathscr{S}_1(x_0)$, $\mathscr{S}_2(x_0, y_0)$ defined in (2.2) and in (2.4), respectively, have some meaningful properties stated in the theorems below.

We consider first the semilinear differential inclusion

$$x' \in Ax + F\big(t, x, H(t,x)\big), \quad x(0) = x_0, \tag{3.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup $\{G(t); t \in I\}$ on X. Let $M_1 \ge 1$ be such that $|G(t)| \le M_1$ for all $t \in I$.

Theorem 3.1. *Consider A the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{G(t)\}_{t \ge 0}$ on the real separable Banach space X, assume that F and H satisfy Hypothesis 2.3 and let $\alpha > \frac{2M_1}{1-\theta}$. Then*

1) *For every $x_0 \in X$, the solution set $\mathscr{S}_1(x_0)$ of (3.1) is nonempty and arcwise connected in the space $C(I,X)$.*

2) *For any* $\xi_i \in X$ *and any* $x_i \in \mathcal{S}_1(\xi_i)$, $i = 1, \dots, p$, *there exists a continuous function* $s : X \to C(I, X)$ *such that* $s(\xi) \in \mathcal{S}_1(\xi)$ *for any* $\xi \in X$ *and* $s(\xi_i) = x_i$, $i = 1, \dots, p$.

3) *The set* $\mathcal{S}_1 = \bigcup_{\xi \in X} \mathcal{S}_1(\xi)$ *is arcwise connected in* $C(I, X)$.

Proof. 1) For $\xi \in X$ and $u \in L^1$, set

$$x_\xi(t) = G(t)\xi + \int_0^t G(t-s)u(s)\,ds, \qquad t \in I,$$

and consider $P : X \to C(I, X)$ defined by $P(\xi)(t) = G(t)\xi$.

We prove that the multifunctions $\phi : X \times L^1 \to 2^{L^1}$ and $\psi : X \times L^1 \times L^1 \to 2^{L^1}$ given by

$$\phi(\xi, u) = \bigl\{v \in L^1;\ v(t) \in H\bigl(t, x_\xi(t)\bigr) \text{ a.e. } (I)\bigr\},$$
$$\psi(\xi, u, v) = \bigl\{w \in L^1;\ w(t) \in F\bigl(t, x_\xi(t), v(t)\bigr) \text{ a.e. } (I)\bigr\},$$

with $\xi \in X$, $u, v \in L^1$ satisfy the hypotheses of Lemma 2.1.

Since $x_\xi(\cdot)$ is measurable and H satisfies Hypotheses 2.3 i) and ii), the multifunction $t \to H\bigl(t, x_\xi(t)\bigr)$ is measurable and nonempty closed-valued, it has a measurable selection. Therefore due to Hypothesis 2.3 iv), the set $\phi(\xi, u)$ is nonempty. The fact that the set $\phi(\xi, u)$ is closed and decomposable follows by a simple computation. In the same way we obtain that $\psi(\xi, u, v)$ is a nonempty closed decomposable set.

Pick $(\xi, u), (\xi_1, u_1) \in X \times L^1$ and choose $v \in \phi(\xi, u)$. For each $\varepsilon > 0$ there exists $v_1 \in \phi(\xi_1, u_1)$ such that, for every $t \in I$, one has

$$\begin{aligned} |v(t) - v_1(t)| &\le D\bigl(H\bigl(t, x_\xi(t)\bigr), H\bigl(t, x_{\xi_1}(t)\bigr)\bigr) + \varepsilon \\ &\le l(t)\Bigl[M_1|\xi - \xi_1| + M_1 \int_0^t |u(s) - u_1(s)|\,ds\Bigr] + \varepsilon. \end{aligned}$$

Hence

$$\begin{aligned} |v - v_1|_1 \le{}& M_1|\xi - \xi_1| \int_0^T e^{-\alpha N^*(t)} l(t)\,dt \\ &+ M_1 \int_0^T e^{-\alpha N^*(t)} l(t) \Bigl(\int_0^t |u(s) - u_1(s)|\,ds\Bigr)\,dt + \varepsilon T \\ \le{}& \frac{M_1}{\alpha}|\xi - \xi_1| + \frac{M_1}{\alpha}|u - u_1|_1 + \varepsilon T \end{aligned}$$

for any $\varepsilon > 0$.

This implies that

$$d_{L^1}\big(v, \phi(\xi_1, u_1)\big) \le \frac{M_1}{\alpha}|\xi - \xi_1| + \frac{M_1}{\alpha}|u - u_1|_1$$

for all $v \in \phi(\xi, u)$. Therefore,

$$d^*_{L^1}\big(\phi(\xi, u), \phi(\xi_1, u_1)\big) \le \frac{M_1}{\alpha}|\xi - \xi_1| + \frac{M_1}{\alpha}|u - u_1|_1.$$

Consequently,

$$D_{L^1}\big(\phi(\xi, u), \phi(\xi_1, u_1)\big) \le \frac{M_1}{\alpha}|\xi - \xi_1| + \frac{M_1}{\alpha}|u - u_1|_1,$$

which shows that ϕ is Hausdorff continuous and satisfies the assumptions of Lemma 2.1.

Pick $(\xi, u, v), (\xi_1, u_1, v_1) \in X \times L^1 \times L^1$ and choose $w \in \psi(\xi, u, v)$. Then, as before, for each $\varepsilon > 0$ there exists $w_1 \in \psi(\xi_1, u_1, v_1)$ such that for every $t \in I$

$$\begin{aligned} |w(t) - w_1(t)| &\le D\big(F(t, x_\xi(t), v(t)), F\big(t, x_{\xi_1}(t), v_1(t)\big)\big) + \varepsilon \\ &\le m(t)\Big[M_1|\xi - \xi_1| + M_1\int_0^t |u(s) - u_1(s)|\, ds\Big] + \theta|v(t) - v_1(t)| + \varepsilon. \end{aligned}$$

Hence

$$\begin{aligned} |w - w_1|_1 &\le \frac{M_1}{\alpha}|\xi - \xi_1| + \frac{M_1}{\alpha}|u - u_1|_1 + \theta|v - v_1|_1 + \varepsilon T \\ &\le \frac{M_1}{\alpha}|\xi - \xi_1| + \left(\frac{M_1}{\alpha} + \theta\right)(|u - u_1|_1 + |v - v_1|_1) + \varepsilon T \\ &\le \frac{M_1}{\alpha}|\xi - \xi_1| + \left(\frac{M_1}{\alpha} + \theta\right)d_{L^1 \times L^1}\big((u, v), (u_1, v_1)\big) + \varepsilon T. \end{aligned}$$

As above, we deduce that

$$D_{L^1}\big(\psi(\xi, u, v), \psi(\xi_1, u_1, v_1)\big) \le \frac{M_1}{\alpha}|\xi - \xi_1| + \left(\frac{M_1}{\alpha} + \theta\right)d_{L^1 \times L^1}\big((u, v), (u_1, v_1)\big);$$

for the multifunction ψ is Hausdorff continuous and satisfies the hypothesis of Lemma 2.1.

Define $\Gamma(\xi, u) = \psi\big(\xi, u, \phi(\xi, u)\big)$, $(\xi, u) \in X \times L^1$. According to Lemma 2.1, the set $\mathrm{Fix}\big(\Gamma(\xi, \cdot)\big) = \{u \in L^1; u \in \Gamma(\xi, u)\}$ is nonempty and arcwise connected in

$L^1(I,X)$. Moreover, for fixed $\xi_i \in X$ and $u_i \in \mathrm{Fix}(\Gamma(\xi_i,\cdot))$, $i = 1,\dots,p$, there exists a continuous function $\gamma : X \to L^1$ such that

$$\gamma(\xi) \in \mathrm{Fix}(\Gamma(\xi,\cdot)) \qquad \text{for all } \xi \in X, \tag{3.2}$$

$$\gamma(\xi_i) = u_i, \qquad i = 1,\dots,p. \tag{3.3}$$

We shall prove that

$$\mathrm{Fix}(\Gamma(\xi,\cdot)) = \{u \in L^1;\ u(t) \in F(t, x_\xi(t), H(t, x_\xi(t)))\ a.e.\ (I)\}. \tag{3.4}$$

Denote by $A(\xi)$ the right-hand side of (3.4). If $u \in \mathrm{Fix}(\Gamma(\xi,\cdot))$ then there is $v \in \phi(\xi, v)$ such that $u \in \psi(\xi, u, v)$. Therefore, $v(t) \in H(t, x_\xi(t))$ and

$$u(t) \in F(t, x_\xi(t), v(t)) \subset F(t, x_\xi(t), H(t, x_\xi(t)))\ a.e.\ (I),$$

so that $\mathrm{Fix}(\Gamma(\xi,\cdot)) \subset A(\xi)$.

Let now $u \in A(\xi)$. By Lemma 2.2, there exists a selection $v \in L^1$ of the multi-function $t \to H(t, x_\xi(t))$ satisfying

$$u(t) \in F(t, x_\xi(t), v(t))\ a.e.\ (I).$$

Hence, $v \in \phi(\xi, v)$, $u \in \psi(\xi, u, v)$ and thus $u \in \Gamma(\xi, u)$, which completes the proof of (3.4).

We next note that the function $T : L^1 \to C(I,X)$,

$$T(u)(t) := \int_0^t G(t-s)u(s)\,ds,$$

is continuous and one has

$$\mathcal{S}_1(\xi) = P(\xi) + T(\mathrm{Fix}(\Gamma(\xi,\cdot))), \qquad \xi \in X. \tag{3.5}$$

Since $\mathrm{Fix}(\Gamma(\xi,\cdot))$ is nonempty and arcwise connected in $L^1(I,X)$, the set $\mathcal{S}_1(\xi)$ has the same properties in $C(I,X)$.

2) Let $\xi_i \in X$ and let $x_i \in \mathcal{S}_1(\xi_i)$, $i = 1,\dots,p$, be fixed. By (3.5) there exists $v_i \in \mathrm{Fix}(\Gamma(\xi_i,\cdot))$ such that

$$x_i = P(\xi_i) + T(v_i), \qquad i = 1,\dots,p.$$

If $\gamma : X \to L^1$ is a continuous function satisfying (3.2) and (3.3), we define, for every $\xi \in X$,

$$s(\xi) = P(\xi) + T(\gamma(\xi)).$$

Obviously, the function $s : X \to C(I, X)$ is continuous, $s(\xi) \in \mathcal{S}_1(\xi)$ for all $\xi \in X$ and

$$s(\xi_i) = P(\xi_i) + T\big(\gamma(\xi_i)\big) = P(\xi_i) + T(v_i) = x_i, \qquad i = 1, \dots, p.$$

3) Let $x_1, x_2 \in \mathcal{S}_1 = \bigcup_{\xi \in X} \mathcal{S}_1(\xi)$ and choose $\xi_i \in X$, $i = 1, 2$, such that $x_i \in S(\xi_i)$, $i = 1, 2$. From the conclusion of 2) we deduce the existence of a continuous function $s : X \to C(I, X)$ satisfying $s(\xi_i) = x_i$, $i = 1, 2$, and $s(\xi) \in \mathcal{S}_1(\xi)$, $\xi \in X$. Let $h : [0, 1] \to X$ be a continuous mapping such that $h(0) = \xi_1$ and $h(1) = \xi_2$. Then the function $s \circ h : [0, 1] \to C(I, X)$ is continuous and satisfies

$$s \circ h(0) = x_1, \qquad s \circ h(1) = x_2, \qquad s \circ h(\tau) \in \mathcal{S}_1\big(h(\tau)\big) \subset \mathcal{S}_1, \qquad \tau \in [0, 1]. \qquad \square$$

Remark 3.2. Theorem 3.1 may be considered an extension to the more general problem (3.1) of the result in [12], namely Theorem 3.1. More exactly, in the particular case when $A \equiv 0$, i.e., when (3.1) reduces to the classical differential inclusions of the form

$$x' \in F(t, x), \qquad x(0) = x_0,$$

Theorem 3.1 yields the statement of Theorem 3.1 in [12].

Next we consider the following second-order differential inclusion

$$x'' \in Ax + F\big(t, x, H(t, x)\big), \qquad x(0) = x_0, \qquad x'(0) = y_0, \tag{3.6}$$

where A is the infinitesimal generator of a strongly continuous cosine family of operators $\{C(t); t \in R\}$ on X.

Let $M_2 \geq 0$ be such that $|C(t)| \leq M_2$ for all $t \in I$. Note that $|S(t)| \leq M_2 t$ for all $t \in I$.

Theorem 3.3. *Consider A the infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t \in R}$ on the real separable Banach space X, assume that F and H satisfy Hypothesis* 2.3, *and let $\alpha > \frac{2M_2 T}{1-\theta}$. Then the following holds:*

1) *For every $(x_0, y_0) \in X \times X$, the solution set $\mathcal{S}_2(x_0, y_0)$ of* (3.6) *is nonempty and arcwise connected in the space $C(I, X)$.*

2) *For any $(\xi_i, \mu_i) \in X \times X$ and any $x_i \in \mathcal{S}_2(\xi_i, \mu_i)$, $i = 1, \dots, p$, there exists a continuous function $s : X \times X \to C(I, X)$ such that $s(\xi, \mu) \in \mathcal{S}_2(\xi, \mu)$ for any $(\xi, \mu) \in X \times X$ and $s(\xi_i, \mu_i) = x_i$, $i = 1, \dots, p$.*

3) *The set $\mathcal{S}_2 = \bigcup_{(\xi, \mu) \in X \times X} \mathcal{S}_2(\xi, \mu)$ is arcwise connected in $C(I, X)$.*

Proof. The proof is similar to the one of Theorem 3.1. We point out only the differences.

For $(\xi,\mu) \in X \times X$ and $u \in L^1$, set

$$x_{\xi,\mu}(t) = C(t)\xi + S(t)\mu + \int_0^t S(t-s)u(s)\,ds, \qquad t \in I,$$

and consider $P : X \times X \to C(I,X)$ defined by $P(\xi,\mu)(t) = C(t)\xi + S(t)\mu$.

Consider also $\phi : X^2 \times L^1 \to 2^{L^1}$ and $\psi : X^2 \times L^1 \times L^1 \to 2^{L^1}$ given by

$$\phi\big((\xi,\mu),u\big) = \big\{v \in L^1; v(t) \in H\big(t, x_{\xi,\mu}(t)\big) \ a.e.\ (I)\big\},$$
$$\psi\big((\xi,\mu),u,v\big) = \big\{w \in L^1; w(t) \in F\big(t, x_{\xi,\mu}(t), v(t)\big) \ a.e.\ (I)\big\}.$$

Similar to the proof of Theorem 3.1, we have that ϕ and ψ satisfy the hypotheses of Lemma 2.1 with

$$D_{L^1}\big(\phi\big((\xi,\mu),u\big), \phi\big((\xi_1,\mu_1),u_1\big)\big) \leq \frac{M_2}{\alpha}|\xi-\xi_1| + \frac{M_2T}{\alpha}|\mu-\mu_1| + \frac{M_2T}{\alpha}|u-u_1|_1,$$

$$D_{L^1}\big(\psi\big((\xi,\mu),u,v\big), \psi\big((\xi_1,\mu_1),u_1,v_1\big)\big) \leq \frac{M_2}{\alpha}|\xi-\xi_1| + \frac{M_2T}{\alpha}|\mu-\mu_1| + \left(\frac{M_2T}{\alpha}+\theta\right) d_{L^1\times L^1}\big((u,v),(u_1,v_1)\big).$$

Define $\Gamma\big((\xi,\mu),u\big) = \psi\big((\xi,\mu),u,\phi\big((\xi,\mu),u\big)\big)$, $\big((\xi,\mu),u\big) \in X^2 \times L^1$, and one has

$$\mathrm{Fix}\big(\Gamma\big((\xi,\mu),\cdot\big)\big) = \big\{u \in L^1; u(t) \in F\big(t, x_{\xi,\mu}(t), H\big(t, x_{\xi,\mu}(t)\big)\big) \ a.e.\ (I)\big\}.$$

We introduce the continuous mapping $T : L^1 \to C(I,X)$,

$$T(u)(t) := \int_0^t S(t-s)u(s)\,ds,$$

and one has

$$\mathcal{S}_2(x_0,y_0) = P(x_0,y_0) + T\big(\mathrm{Fix}\big(\Gamma\big((x_0,y_0),\cdot\big)\big)\big).$$

The proof of statements 1), 2) and 3) follows now as in the proof of Theorem 3.1. □

References

[1] M. Benchohra, On second order differential inclusions in Banach spaces. *New Zealand J. Math.* **30** (2001), 5–13. Zbl 1011.34052 MR 1839518

[2] M. Benchohra and S. K. Ntouyas, Existence of mild solutions of second-order initial value problems for differential inclusions with nonlocal conditions. *Atti. Sem. Mat. Fis. Univ. Modena* **49** (2001), 351–361. Zbl 1019.34063 MR 1877925

[3] A. Cernea, Some qualitative properties of the solution set of infinite horizon operational differential inclusions, *Revue Roumaine Math. Pures Appl.* **43** (1998), 317–328. Zbl 1005.34052 MR 1845531

[4] A. Cernea, On a nonconvex second-order differential inclusion. *Atti. Sem. Mat. Fis. Univ. Modena*, to appear.

[5] A. Cernea, Lipschitz-continuity of the solution map of some nonconvex second-order differential inclusions. *Fasc. Mathematici*, to appear.

[6] F. S. de Blasi and G. Pianigiani, Evolution inclusions in non-separable Banach spaces. *Comment. Math. Univ. Carolin.* **40** (1999), 227–250. Zbl 0987.34063 MR 1732644

[7] F. S. De Blasi, G. Pianigiani and V. Staicu, Topological properties of nonconvex differential inclusions of evolution type. *Nonlinear Anal.* **24** (1995), 711–720. Zbl 0828.34010 MR 1319080

[8] H. O. Fattorini, *Second order linear differential equations in Banach spaces*. North-Holland Math. Stud. 108, North-Holland Publishing Co., Amsterdam 1985. Zbl 0564.34063 MR 0797071

[9] H. Frankowska, A priori estimates for operational differential inclusions. *J. Differential Equations* **84** (1990), 100–128. Zbl 0705.34016 MR 1042661

[10] S. Heikkilä and V. Lakshmikantham, *Monotone iterative techniques for discontinuous nonlinear differential equations*. Monographs Textbooks Pure Appl. Math. 181, Marcel Dekker, New York 1994. Zbl 0804.34001 MR 1280028

[11] S. A. Marano, Fixed points of multivalued contractions with nonclosed, nonconvex values. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei* (9) *Mat. Appl.* **5** (1994), 203–212. Zbl 0862.54040 MR 1298263

[12] S. A. Marano and V. Staicu, On the set of solutions to a class of nonconvex nonclosed differential inclusions. *Acta Math. Hungar.* **76** (1997), 287–301. Zbl 0907.34010 MR 1459237

[13] V. Staicu, Continuous selections of solution sets to evolution equations. *Proc. Amer. Math. Soc.* **113** (1991), 403–413. Zbl 0737.34011 MR 1076580

[14] C. C. Travis and G. F. Webb, Cosine families and abstract nonlinear second order differential equations. *Acta Math. Hungarica* **32** (1978), 75–96. Zbl 0388.34039 MR 0499581

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