Portugal. Math. (N.S.)
Vol. 65, Fasc. 4, 2008, 497–507

Portugaliae Mathematica

A remark on multiple solutions for a nonlinear eigenvalue problem

Gen-Qiang Wang and Sui Sun Cheng

(Communicated by Luís Sanchez)

Abstract. Multiple solutions are shown to exist for the system $Bu = \lambda F(u)$, where $\lambda > 0$, B is a positive definite matrix and F is 'superquadratic' or 'subquadratic'. We make use of Clark's theorem in critical point theory to obtain our results and provide examples to show that the results are sharp.

Mathematics Subject Classification (2000). Primary 47J30; Secondary 39A10.
Keywords. Nonlinear system, steady state, Clark's theorem, critical point.

1. Introduction

To motivate what follows, consider $n+2$ artificial neuron units $U_0, U_1, \dots, U_{n+1}$ placed evenly in a linear fashion. Let $u_i^{(t)}$ denote the state value (or electron content) of the i-th neuron unit during the time period $t \in \mathbb{N} = \{0, 1, 2, \dots\}$. We assume that the neuron units U_0 and U_{n+1} are 'grounded' so that their state values are maintained at the zero level, while the other neuron units is activated by its two neighbors so that the change of state values between two consecutive time periods of the i-th unit is given by

$$u_i^{(t+1)} - u_i^{(t)} = \alpha(u_{i-1}^{(t)} - u_i^{(t)}) + \alpha(u_{i+1}^{(t)} - u_i^{(t)}) + \beta f_i(u_i^{(t)}), \qquad i = 1, 2, \dots, n,$$

where $\alpha, \beta \geq 0$, and each f_i is a real valued function defined on $\mathbb{R}$ and stands for the bias mechanism inherent in the i-th neuron unit. The parameter α stands for the 'diffusion' constant, while β measures the strength of bias mechanism. By letting A be the $(n \times n)$-matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ & \cdots & \cdots & \cdots & \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \tag{1}$$

and $u^{(t)} = (u_1^{(t)}, u_2^{(t)}, \ldots, u_n^{(t)})^{\dagger}$, we may then write

$$u^{(t+1)} - u^{(t)} = -\alpha A u^{(t)} + \beta F(u^{(t)}), \qquad t \in \mathbb{N}, \tag{2}$$

where $F\big((x_1, x_2, \ldots, x_n)^{\dagger}\big) = \big(f_1(x_1), f_2(x_2), \ldots, f_n(x_n)\big)^{\dagger}$. By imposing the initial distribution vector $u^{(0)}$, we may calculate $u^{(1)}, u^{(2)}, \ldots$ in a unique manner. The corresponding sequence $\{u^{(t)}\}_{t\in\mathbb{N}}$ is called a solution of (2). In order to understand the dynamics of the neural network (2), we need only to find all solutions or special solutions that may reveal its important characteristics. One particular class of solutions consist of those that are 'time independent', that is, $u^{(t+1)} = u^{(t)} = u$ for $t \in \mathbb{N}$. They are called the steady state solutions and are useful since they are candidates for storing useful informations in the corresponding digital devices.

Clearly, if $\alpha = 0$, then the steady state solutions $\{u\}_{t\in\mathbb{N}}$ are solutions of

$$\beta F(u) = 0. \tag{3}$$

If $\beta = 0$, then they are solutions of

$$\alpha A u = 0. \tag{4}$$

If $\alpha \neq 0$ and $\beta \neq 0$, then they are solutions of

$$Au = \lambda F(u) \tag{5}$$

where $\lambda = \beta/\alpha > 0$.

There are now numerous studies related to nonlinear systems or linear systems of the form (3) or (4) and hence we will not add any discussions here. There are also studies related to the nonlinear system (5) (see for example [1], [6]–[8]). Our interest here, however, is the existence of multiple nontrivial real solutions when the a priori parameter λ falls within some appropriate range. Such an interest is meaningful since the parameter λ acts as an analog control.

In this paper, we will employ Clark's theorem in the critical point theory to show the existence of $2n$ nontrivial real solutions of

$$Bu = \lambda F(u), \qquad \lambda > 0, \tag{6}$$

where B is a *general symmetric positive definite matrix* which is not necessary of the form (1). The definiteness assumption on B is not vacuous since (1) is such a matrix.

Let us state Clark's theorem [3], [1] as follows.

Lemma 1.1 ([3], Clark's Theorem 9.1). *Let E be a real Banach space and I a functional in $C^1(E, \mathbb{R})$ which is even, bounded from below, and satisfies the Palais–*

Smale condition. Suppose further that $I(0) = 0$*, there is a set* $K \subset E$ *such that* K *is homeomorphic to* $\mathbb{S}^{j-1}$ *by an odd map, and* $\sup_k I < 0$*. Then* I *possesses at least* j *distinct pairs of nontrivial critical points.*

We remark that a continuously differentiable functional $J \in C^1(E, \mathbb{R})$ is said to satisfy the Palais–Smale condition (P–S condition) if any sequence $\{u_n\} \subset E$ for which $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence in E.

2. Main results

Let the eigenvalues of B be $\lambda_1, \dots, \lambda_n$ ordered by $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. The main results are the following theorems.

Theorem 2.1. *Assume that* $f_k \in C(\mathbb{R}, \mathbb{R})$ *for* $k \in \{1, \dots, n\}$ *and satisfies the conditions:*

(H$_1$) *For any* $z \ne 0$ *and* $k \in \{1, \dots, n\}$, $f_k(-z) = -f_k(z) \ne 0$ *and* $f_k(z) = o(z)$ *as* $z \to 0$.

(H$_2$) *There exist positive constants* a_1, a_2 *and* M *such that*

$$\int_0^z f_k(s)\, ds \ge a_1 |z|^2 - a_2 \quad \textit{for } |z| \ge M \textit{ and } k \in \{1, \dots, n\}.$$

Then for each $\lambda > \frac{\lambda_n}{2a_1}$, (6) *possesses at least* $2n$ *nontrivial real solutions.*

Theorem 2.2. *Assume that* $f_k \in C(\mathbb{R}, \mathbb{R})$ *for* $k \in \{1, \dots, n\}$ *and satisfies the conditions:*

(H$_3$) *For any* $z \ne 0$ *and* $k \in \{1, \dots, n\}$, $f_k(-z) = -f_k(z) \ne 0$ *and*

$$\lim_{z \to 0} \frac{\int_0^z f_k(s)\, ds}{z^2} = \infty.$$

(H$_4$) *There exist positive constants* a_1, a_2 *and* $M > 0$ *such that*

$$\int_0^z f_k(s)\, ds \le a_1 |z|^2 + a_2 \textit{ for } |z| \ge M \quad \textit{and} \quad k \in \{1, \dots, n\}.$$

Then for each $\lambda < \frac{1}{2a_1} \lambda_1$, (6) *possesses at least* $2n$ *nontrivial real solutions.*

Before proving the above results, let us first consider several simple examples.

Example 2.3. The equation

$$cx = \lambda f(x), \qquad c > 0,\ \lambda > 0, \tag{7}$$

is of the form (6). The function f defined by $f(x) = x^3$ satisfies $f(-x) = -x^3 = -f(x)$ for $x \neq 0$, $f(x)/x = x^2 \to 0$ as $x \to 0$, and

$$\int_0^t f(x)\,dx = \frac{t^4}{4} \geq at^2 - b, \qquad |t| \geq 2\sqrt{a},$$

for any $a, b > 0$. By Theorem 2.1, we see that (7) has two nontrivial real solutions for any λ with $\lambda > c/2a$. Since a can be taken in an arbitrary manner, we see that (7) has two nontrivial solutions for any $\lambda > 0$. Indeed, the solutions are given precisely by $x = 0, \pm\sqrt{c/\lambda}$. Thus Theorem 2.1 is sharp when $n = 1$.

Example 2.4. The system

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x^3 \\ y^3 \end{pmatrix}, \qquad \lambda > 0, \tag{8}$$

is of the form (6). Furthermore, it is equivalent to a system of two equations of the form (7). By Example 2.3, each equation has two nontrivial solutions. Since 0 is also a solution, we see that (8) has at least 8 nontrivial solutions. Indeed, the solutions of (8) are given by

$$(0,0),\ (0, \pm\sqrt{2/\lambda}),\ (\pm\sqrt{1/\lambda}, 0),\ (\pm\sqrt{1/\lambda}, \pm\sqrt{2/\lambda}),\ (\pm\sqrt{1/\lambda}, \mp\sqrt{2/\lambda}).$$

Example 2.5. The equation

$$cx = \lambda x^{1/3}, \qquad c > 0,\ \lambda > 0, \tag{9}$$

is of the form (6). The function defined by $f(x) = x^{1/3}$ satisfies $f(-x) = (-x)^{1/3} = -x^{1/3}$ for $x \neq 0$,

$$\int_0^t f(x)\,dx = \frac{3}{4}t^{4/3} \leq at^2 + b, \qquad |t| \geq \left(\frac{3}{4a}\right)^{3/2}$$

for any $a, b > 0$, and

$$\lim_{t \to 0} \frac{1}{t^2}\int_0^t f(x)\,dx = +\infty.$$

By Theorem 2.2, equation (9) has at least two nontrivial real solutions for any $\lambda \in (0, c/2a)$ and hence any $\lambda > 0$. On the other hand, the solutions of (9) are given precisely by $x = 0,\ \pm(\lambda/c)^{3/2}$. Thus Theorem 2.2 is also sharp (when $n = 1$).

In order to prove both Theorems, we first reformulate our problem as a critical point problem. Consider the functional $I : \mathbb{R}^n \to \mathbb{R}$ defined by

$$I(u) = -\frac{1}{2}u^{\dagger}Bu + \lambda\sum_{k=1}^{n}\int_0^{u_k} f_k(s)\,ds, \qquad u = (u_1, \dots, u_n)^{\dagger} \in \mathbb{R}^n. \tag{10}$$

Since

$$\frac{\partial I(u)}{\partial u_k} = -(Bu)_k + \lambda f_k(u_k), \qquad k \in \{1, \dots, n\}, \tag{11}$$

we see that a column vector $w = (w_1, w_2, \dots, w_n)^{\dagger}$ is a critical point of the functional I corresponding to λ if and only if w is a solution of (6) corresponding to λ.

Lemma 2.6. *If* (H_2) *holds, then for each* $\lambda > \frac{\lambda_n}{2a_1}$ *the functional* I *defined by* (10) *is bounded from below in* $\mathbb{R}^n$.

Proof. According to (H_2), if we let

$$a_2' = \max_{1 \le k \le n}\left\{\left|\int_0^z f_k(s)\,ds - a_1|z|^2 + a_2\right| : |z| \le M\right\}$$

and $a = a_2 + a_2'$, then for any $z \in \mathbb{R}$ and $k \in \{1, \dots, n\}$ we have

$$\int_0^z f_k(s)\,ds \ge a_1|z|^2 - a. \tag{12}$$

Thus, for any $u = (u_1, \dots, u_n)^{\dagger} \in \mathbb{R}^n$, we have

$$\begin{aligned} I(u) &= -\frac{1}{2}u^T Bu + \lambda\sum_{k=1}^{n}\int_0^{u_k} f_k(s)\,ds \\ &\ge -\frac{1}{2}\lambda_n\|u\|_2^2 + a_1\lambda\sum_{k=1}^{n}|u_k|^2 + n\lambda a \\ &\ge -\frac{1}{2}\lambda_n\|u\|_2^2 + a_1\lambda\|u\|_2^2 + na\lambda \\ &= \left(a_1\lambda - \frac{1}{2}\lambda_n\right)\|u\|_2^2 + n\lambda a. \end{aligned} \tag{13}$$

Since $\lambda > \frac{\lambda_n}{2a_1}$, we see that

$$I(u) \geq n\lambda a \tag{14}$$

for any $u = (u_1, \dots, u_n)^\dagger \in \mathbb{R}^n$. The proof is complete. □

Lemma 2.7. *If* (H_2) *holds, then for* $\lambda > \frac{\lambda_n}{2a_1}$ *the functional* I *defined by* (10) *satisfies the P–S condition.*

Indeed, this follows from the fact that the functional I is in fact coercive, since we are dealing with a finite dimensional space.

We now turn to the proof Theorem 2.1. It suffices to find $2n$ nontrivial critical points of the functional I defined by (10). First of all, by (H_1), we see that $I(0) = 0$. Moreover, $I \in C^1(\mathbb{R}^n, \mathbb{R})$ and $I(-u) = I(u)$ for any $u \in \mathbb{R}^n$, that is, I is even. Furthermore, by Lemma 2.6 and Lemma 2.7, I is bounded from below in $\mathbb{R}^n$ and satisfies the P–S condition. It is easy to see from (H_1) that

$$\lim_{z \to 0} \frac{\int_0^z f_k(s)\,ds}{z^2} = 0, \qquad k \in \{1, \dots, n\}. \tag{15}$$

Thus there exist positive constants δ and $\alpha < \frac{\lambda_1}{2\lambda}$ such that for any $z \in \mathbb{R}$ with $|z| \leq \delta$,

$$\int_0^z f_k(s)\,ds \leq \alpha |z|^2. \tag{16}$$

Let $K = \{u \in \mathbb{R}^n : \|u\|_2 = \delta\} \subset \mathbb{R}^n$. It is easy to see that K is homeomorphic to $\mathbb{S}^{n-1}$ by an odd map, and that for any $u \in K$ satisfying $\|u\|_2 = \delta$,

$$I(u) = -\frac{1}{2} u^\dagger B u + \lambda \sum_{k=1}^n \int_0^{u_k} f_k(s)\,ds \leq -\left(\frac{1}{2}\lambda_1 - \lambda\alpha\right) \|u\|_2^2 = -\sigma < 0,$$

where $\sigma = \left(\frac{1}{2}\lambda_1 - \lambda\alpha\right)\delta^2 > 0$. Thus by Lemma 1.1, we know that I possesses at least two nontrivial critical points. The proof is complete.

Next we turn to the proof of Theorem 2.2. Consider

$$J(u) = -I(u) = \frac{1}{2} u^\dagger B u - \lambda \sum_{k=1}^n \int_0^{u_k} f_k(s)\,ds, \qquad u \in \mathbb{R}^n. \tag{17}$$

Clearly, a column vector $w = (w_1, w_2, \dots, w_n)^\dagger$ is a critical point of the functional J corresponding to λ if and only if w is a solution of (6) corresponding to λ.

Lemma 2.8. *If* (H_4) *holds, then for* $\lambda < \frac{1}{2a_1}\lambda_1$ *the functional* J *defined by* (17) *is bounded from bounded in* $\mathbb{R}^n$.

The proof is similar to the proof of Lemma 2.6 and hence is skipped.

Lemma 2.9. *If* (H_4) *holds, then for* $\lambda < \frac{1}{2a_1}\lambda_1$ *the functional* J *defined by* (17) *satisfies the P–S condition.*

Indeed, this follows from the fact that the functional I is coercive, since we are dealing with finite dimensional space.

To prove Theorem 2.2, it suffices to find $2n$ nontrivial critical points of the functional J defined by (17). First of all, by (H_3), we see that $J(0) = 0$, $J \in C^1(\mathbb{R}^n, \mathbb{R})$ and $I(-u) = I(u)$ for any $u \in \mathbb{R}^n$, that is, J is even. Furthermore, by Lemmas 2.8 and 2.9, J is bounded from below in $\mathbb{R}^n$ and satisfies the P–S condition. It is easy to see from (H_3) that

$$\lim_{z\to 0} \frac{\int_0^z f_k(s)\,ds}{z^2} = \infty, \qquad k \in \{1, \dots, n\}. \tag{18}$$

Therefore there exist positive constants δ and $\alpha > \frac{\lambda_n}{2\lambda}$ such that for any $z \in \mathbb{R}$ and $|z| \le \delta$,

$$\int_0^z f_k(s)\,ds \ge \alpha|z|^2, \qquad k \in \{1, \dots, n\}. \tag{19}$$

Let $K = \{u \in \mathbb{R}^n \mid \|u\|_2 = \delta\} \subset \mathbb{R}^n$. It is easy to see that K is homeomorphic to $\mathbb{S}^{n-1}$ by an odd map, and that for any $u \in K$ satisfying $\|u\|_2 = \delta$,

$$J(u) = \frac{1}{2}u^\dagger Bu - \lambda \sum_{k=1}^n \int_0^{u_k} f_k(s)\,ds \le \left(\frac{1}{2}\lambda_n - \lambda\alpha\right)\|u\|_2^2 = -\sigma < 0,$$

where $\sigma = \left(\lambda\alpha - \frac{1}{2}\lambda_n\right)\delta^2 > 0$. Thus by Lemma 1.1, we know that J possesses at least $2n$ nontrivial critical points. The proof is complete.

3. Remarks and further examples

The assumptions (H_1) and (H_2) are basically the same as those in [7]. However, in [7], the authors assert that there are 2^n solutions with all components in $\mathbb{R}\backslash\{0\}$. Unfortunately, there is an error in the proof as explained by Ackermann in his review; see MR2183553. The same error appears also in [8] and [6]. Therefore the validity of the results in [2]–[4] remains unknown.

Now we give a counterexample which shows that the conclusions of 'Theorems' 2 and 3 in [7] are not true (see also 'Theorem' 4 in [6]).

Example 3.1. Consider the system

$$\left.\begin{aligned} 2x_1 - x_2 &= 2x_1^3, \\ -x_1 + 2x_2 - x_3 &= \frac{1}{2}x_2^3, \\ -x_2 + 2x_3 &= \frac{2}{81}x_3^3, \end{aligned}\right\} \tag{20}$$

which is of the form (6), where

$$B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

and $\lambda = 1$, $f_1(z) = 2z^3$, $f_2(z) = \frac{1}{2}z^3$, $f_3(z) = \frac{2}{81}z^3$. It is easy to see that the conditions of 'Theorem' 2 of [7] are satisfied. If 'Theorem' 2 of [7] holds, then (20) has a solution $(x_1, x_2, x_3)^\dagger$ that satisfies $x_1 < 0$, $x_2 < 0$ and $x_3 > 0$. However, we see from the third equation of (20) that

$$\frac{2}{81}x_3^3 - 2x_3 = -x_2 > 0,$$

which implies that $x_3 > 9$. From the first equation of (20), we have

$$2x_1^3 - 2x_1 = -x_2 > 0,$$

which leads to $-1 < x_1 < 0$. Thus, $x_1 + x_3 > 8$. By the second equality of (20), we see that

$$-\frac{1}{2}x_2^3 = x_1 - 2x_2 + x_3 > x_1 + x_3 > 8.$$

Therefore $x_2 < -(16)^{1/3}$. On the other hand, from the first equality of (20), we get

$$x_2 = 2x_1 - 2x_1^3 > 2x_1 > -2 > -(16)^{1/3}.$$

This is a contradiction. Thus, the conclusion of 'Theorem' 2 of [7] fails.

Note that (20) can be written as

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2x_1^3 \\ \frac{1}{2}x_2^3 \\ \frac{2}{81}x_3^3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{21}$$

Now if we let $x = 2x_1^3$, $y = \frac{1}{2}x_2^3$, $z = \frac{2}{81}x_3^3$, then (21) can be written as

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt[3]{2}}\sqrt[3]{x} \\ \sqrt[3]{2}\sqrt[3]{y} \\ \sqrt[3]{\frac{81}{2}}\sqrt[3]{z} \end{pmatrix}. \tag{22}$$

It is easy to see that the system (22) satisfying all conditions of 'Theorem' 3 of [7]. But it cannot have a solution $(x, y, z)^\dagger$ satisfying $x < 0$, $y < 0$ and $z > 0$. This leads to a counterexample for Theorem 3 in [7] and shows that the existence of one 'positive', one 'negative' and $2^n - 2$ 'sign changing' solutions is not true in general.

In the review by Ackermann, all the solutions of the system

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} cx^3 \\ y^3 \end{pmatrix} \tag{23}$$

are stated (and can be checked easily):

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} +\frac{1}{2}\sqrt{6} \\ -\frac{1}{2}\sqrt{6} \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{2}\sqrt{6} \\ +\frac{1}{2}\sqrt{6} \end{pmatrix}.$$

Since the eigenvalues of the coefficient matrix are 1 and 3, and the functions $f_1(x)$ and $f_2(x)$ given by $f_1(x) = f_2(x) = 2x^3$ satisfy the assumptions of Theorem 2.1, the conclusion of Theorem 2.1 is also sharp when $n = 2$.

There are other aspects of our theorems as far as sharpness is concerned. Let us consider two more nontrivial examples.

Example 3.2. In (7), let $c = 1$ and let f be defined by

$$f(x) = \begin{cases} x^3 & \text{if } |x| < \frac{1}{2}, \\ \frac{1}{4}x & \text{if } |x| \geq \frac{1}{2}. \end{cases} \tag{24}$$

Then $f(-x) = -f(x)$ for $x \neq 0$. Furthermore, when $0 < |x| < \frac{1}{2}$, $f(x)/x = x^2 \to 0$ as $x \to 0$, and

$$\int_0^t f(x)\, dx \geq \frac{1}{8}t^2 - \frac{1}{64}, \qquad |t| \geq \frac{1}{2},$$

by Theorem 2.1, we see that (7) has 2 nontrivial real solutions for any λ that satisfies $\lambda > 4$. Indeed, the solutions are given precisely by $x = 0, \pm\sqrt{1/\lambda}$. Furthermore in case $\lambda \leq 4$, the conditions of Theorem 2.1 cannot be satisfied. But we can easily see that (7) has only the trivial solution. Therefore the condition $\lambda > \lambda_n/2a_1$ is sharp when $n = 1$.

Example 3.3. Consider

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} f(x) \\ f(y) \end{pmatrix}, \tag{25}$$

where

$$f(x) = \begin{cases} x^3 & \text{if } |x| < \frac{1}{2}, \\ \frac{1}{4}x & \text{if } |x| \geq \frac{1}{2}. \end{cases} \tag{26}$$

The conditions of Theorem 2.1 are satisfied (with $\lambda_n = 3$). Thus (25) has four nontrivial real solutions for any λ that satisfies $\lambda > 12$. Indeed, the solutions are given precisely by

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{\lambda}\sqrt{\lambda} \\ \frac{1}{\lambda}\sqrt{\lambda} \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{\lambda}\sqrt{\lambda} \\ -\frac{1}{\lambda}\sqrt{\lambda} \end{pmatrix}, \quad \begin{pmatrix} +\frac{1}{\lambda}\sqrt{3\lambda} \\ -\frac{1}{\lambda}\sqrt{3\lambda} \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{\lambda}\sqrt{3\lambda} \\ +\frac{1}{\lambda}\sqrt{3\lambda} \end{pmatrix}.$$

There are reasons to assume a more general positive definite matrix B in (6). Indeed, if we assume that the neuron units U_0 and U_{n+1} are not 'grounded', but U_0 is kept at the level $U_1/2$ and U_{n+1} at $U_{n+1}/2$, then the corresponding matrix changes from A to the $(n \times n)$-matrix

$$\begin{pmatrix} \frac{3}{2} & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ & \cdots & \cdots & \cdots & \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & \frac{3}{2} \end{pmatrix}.$$

More generally, we may require that $U_0 = \alpha U_1$ and $U_{n+1} = \beta U_n$. Then the corresponding $(n \times n)$-matrix is

$$\begin{pmatrix} 2-\alpha & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ & \cdots & \cdots & \cdots & \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2-\beta \end{pmatrix}.$$

Such a matrix can be regarded as a tridiagonal matrix with perturbations, and their spectral properties can be found in various studies (see e.g. [4], [5]). There are also problems related to the vibration of particles attached to strings and nets which also lead us to positive definite matrices (see [1] for general discussions).

References

[1] S. S. Cheng, *Partial difference equations*. Adv. Discrete Math. Appl. 3, Taylor & Francis, London 2003. Zbl 1016.39001 MR 2193620

[2] D. C. Clark, A variant of the Lusternik-Schnirelman theory. *Indiana Univ. Math. J.* **22** (1972), 65–74. Zbl 0228.58006 MR 0296777

[3] P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations. CBMS Regional Conf. Ser. in Math. 65, Amer. Math. Soc., Providence, RI, 1986. Zbl 0609.58002 MR 0845785

[4] W.-C. Yueh, Eigenvalues of several tridiagonal matrices. *Appl. Math. E-Notes* **5** (2005), 66–74. Zbl 1068.15006

[5] W.-C. Yueh and S. S. Cheng, Synchronization in an artificial neural network. *Chaos Solitons Fractals* **30** (2006), 734–747. Zbl 05315352 MR 2238697

[6] G. Zhang, Existence of non-zero solutions for a nonlinear system with a parameter. *Nonlinear Anal.* **66** (2007), 1410–1416. Zbl 1113.65056 MR 2294448

[7] G. Zhang and S. S. Cheng, Existence of solutions for a nonlinear system with a parameter. *J. Math. Anal. Appl.* **314** (2006), 311–319. Zbl 1087.39021 MR 2183553

[8] G. Zhang and Z. Yang, Existence of 2^n nontrivial solutions for discrete two-point boundary value problems. *Nonlinear Anal.* **59** (2004), 1181–1187. Zbl 1062.39020 MR 2098512

Received June 14, 2007; revised October 17, 2007

G.-Q. Wang, Department of Computer Science, Guangdong Polytechnic Normal University, Guangzhou, Guangdong 510665, P. R. China

E-mail: w7633@hotmail.com

S. S. Cheng, Department of Mathematics, Tsing Hua University, Hsinchu, Taiwan 30043 R. O. China

E-mail: sscheng@math.nthu.edu.tw