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Uniqueness at infinity in time for the Maxwell-Schrödinger system with arbitrarily large asymptotic data

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Abstract. We prove the uniqueness of solutions of the Maxwell-Schrödinger system with given asymptotic behaviour at infinity in time. The assumptions include suitable restrictions on the growth of solutions for large time and on the accuracy of their asymptotics, but no restriction on their size. The result applies to the solutions with prescribed asymptotics constructed in a previous paper.

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1. Introduction

This paper is a sequel to a previous paper [6], hereafter referred to as II, where we studied the theory of scattering for the Maxwell-Schrödinger system (MS) in $3+1$ dimensional space time. That system describes the evolution of a charged non-relativistic quantum mechanical particle interacting with the (classical) electromagnetic field it generates. It can be written as follows:

$$\begin{cases} i\partial_t u = -(1/2)\Delta_A u + A_e u \\ \Box A_e - \partial_t(\partial_t A_e + \nabla\cdot A) = |u|^2 \\ \Box A + \nabla(\partial_t A_e + \nabla\cdot A) = \mathrm{Im}\,\bar{u}\nabla_A u \end{cases} \tag{1.1}$$

where u and (A, A_e) are respectively a complex valued function and an $\mathbb{R}^{3+1}$ valued function defined in space time $\mathbb{R}^{3+1}$, $\nabla_A = \nabla - iA$ and $\Delta_A = \nabla_A^2$ are the covariant gradient and covariant Laplacian respectively, and $\Box = \partial_t^2 - \Delta$ is the d'Alembertian. An important property of that system is its gauge invariance, namely the invariance under the transformation

$$(u, A, A_e) \to \big(u\exp(-i\theta), A - \nabla\theta, A_e + \partial_t\theta\big),$$

where θ is an arbitrary real function defined in $\mathbb{R}^{3+1}$. As a consequence of that invariance, the system (1.1) is underdetermined as an evolution system and has to be supplemented by an additional equation, called a gauge condition. In this paper, we shall use exclusively the Coulomb gauge condition, namely $\nabla \cdot A = 0$. Under that condition, the equation for A_e can be solved by

$$A_e = -\Delta^{-1}|u|^2 = (4\pi|x|)^{-1} \star |u|^2 \equiv g(u) \tag{1.2}$$

where $\star$ denotes the convolution in $\mathbb{R}^3$. Substituting (1.2) and the gauge condition into (1.1) yields the formally equivalent system

$$i\partial_t u = -(1/2)\Delta_A u + g(u)u \tag{1.3}$$

$$\square A = P \operatorname{Im} \bar{u} \nabla_A u \tag{1.4}$$

where $P = \mathbb{1} - \nabla\Delta^{-1}\nabla$ is the projector on divergence free vector fields.

The MS system is known to be locally well posed both in the Coulomb gauge and in the Lorentz gauge $\partial_t A_e + \nabla \cdot A = 0$ in sufficiently regular spaces [8] [9], to have weak global solutions in the energy space [7] and to be globally well posed in a space smaller than the energy space [10].

A large amount of work has been devoted to the theory of scattering and more precisely to the existence of wave operators for nonlinear equations and systems centering on the Schrödinger equation and in particular for the Maxwell-Schrödinger system [2] [4] [6] [12] [14]. As in the case of the linear Schrödinger equation, one must distinguish the short range case from the long range case. In the former case, ordinary wave operators are expected and in a number of cases proved to exist, describing solutions where the Schrödinger function behaves asymptotically like a solution of the free Schrödinger equation. In the latter case, ordinary wave operators do not exist and have to be replaced by modified wave operators including an additional phase in the asymptotic behaviour of the Schrödinger function. In that respect, the MS system in $\mathbb{R}^{3+1}$ belongs to the borderline (Coulomb) long range case. We refer to II and [5] for general background and additional references on that matter.

The main step in the construction of the (modified) wave operators consists in solving the local Cauchy problem with infinite initial time. In the long range case where that problem is singular, that step amounts to construct solutions with prescribed (singular) asymptotic behaviour in time. For the MS system in the Coulomb gauge (1.3) (1.4), that step was performed in II by replacing the original system by an auxiliary system, solving the corresponding problem for that system and then returning to the original one. In particular we derived an existence and uniqueness result for solutions of the auxiliary system with prescribed time asymptotics, from which an existence result for solutions of the original system with prescribed time asymptotics follows. However uniqueness was proved only for the auxiliary system, thereby leaving uniqueness for the original one open. The purpose of the present paper is to supplement the previous results with a direct uniqueness result for the original system, expressed in terms of the original functions (u, A).

In order to state that result we first replace the equation (1.4) for A by the associated integral equation with prescribed asymptotic data $(A_+, \dot{A}_+)$, namely

$$A = A_0 - \int_t^\infty dt' \omega^{-1} \sin\big(\omega(t-t')\big) P \operatorname{Im}(\bar{u}\nabla_A u)(t') \tag{1.5}$$

where $\omega = (-\Delta)^{1/2}$ and A_0 is the solution of the free wave equation $\square A_0 = 0$ given by

$$A_0 = (\cos \omega t) A_+ + \omega^{-1} (\sin \omega t) \dot{A}_+. \tag{1.6}$$

In order to ensure the gauge condition $\nabla \cdot A = 0$, we assume that $\nabla \cdot A_+ = \nabla \cdot \dot{A}_+ = 0$. As a consequence $x \cdot A_0$ is also a solution of the free wave equation. The uniqueness result will be stated for the MS system in the form (1.3) (1.5). Since the Cauchy problem for that system is singular at $t = \infty$, especially as regards the function u, the uniqueness result for that system takes a slightly unusual form. Roughly speaking it states that two solutions (u_i, A_i), $i = 1, 2$, coincide provided u_i and $A_i - A_0$ do not blow up too fast and provided $u_1 - u_2$ tends to zero in a suitable sense as $t \to \infty$. In particular that result does not make any reference to the asymptotic data for u, which should characterize its behaviour at infinity.

In order to state the result we need some notation. We denote by $\|\cdot\|_r$ the norm in $L^r \equiv L^r(\mathbb{R}^3)$, $1 \le r \le \infty$ and by $\dot{H}^1 = \dot{H}^1(\mathbb{R}^3)$ the homogeneous Sobolev space

$$\dot{H}^1 = \{v : \nabla v \in L^2 \text{ and } v \in L^6\}.$$

We shall need the space

$$V_\star = \{v : \langle x \rangle^3 v \in L^2, \langle x \rangle^2 \nabla v \in L^2\}, \tag{1.7}$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$, and the dilation operator

$$S = t\partial_t + x \cdot \nabla + \mathbb{1}. \tag{1.8}$$

It follows from the commutation relation $\square S = (S+2)\square$ that SA_0 satisfies the free wave equation if A_0 does. We shall use the notation

$$\tilde{u}(t) = U(-t)u(t) \tag{1.9}$$

where $U(t) = \exp\big(i(t/2)\Delta\big)$ is the unitary group which solves the free Schrödinger equation. We denote non-negative integers by j, k, ℓ.

The main result can be stated as follows.

Proposition 1.1. *Let $1 \le T < \infty$, $I = [T, \infty)$ and $\alpha \ge 0$. Let A_0 be a divergence free solution of the free wave equation satisfying*

$$\|\nabla^k S^j A_0(t)\|_\infty + \|\nabla^k x \cdot A_0(t)\|_\infty \le Ct^{-1} \quad \text{for } 0 \le j + k \le 1 \tag{1.10}$$

for all $t \in I$. Let (u_i, A_i), $i = 1, 2$, be two solutions of the system (1.3) (1.5) *such that $\tilde{u}_i \in L^\infty_{\text{loc}}(I, V_\star)$, $A_i - A_0 \in L^\infty_{\text{loc}}(I, \dot{H}^1)$ and such that*

$$\|x^k \nabla^\ell \tilde{u}_i(t)\|_2 \le C(1 + \ell nt)^\alpha \quad \text{for } 0 \le \ell \le 1,\ 0 \le k + \ell \le 3, \tag{1.11}$$

$$\|\nabla(A_i - A_0)(t)\|_2 \le Ct^{-1/2}(1 + \ell nt)^{2\alpha}, \tag{1.12}$$

$$\|\langle x/t\rangle(u_1 - u_2)(t)\|_2 \le Ch_\star(t) \tag{1.13}$$

for all $t \in I$, where $h_\star \in \mathscr{C}(I, \mathbb{R}^+)$ is such that the function

$$\bar{h}_\star(t) = t(1 + \ell nt)^{3+9\alpha} h_\star(t) \tag{1.14}$$

be non increasing for t sufficiently large and satisfy

$$\int_t^\infty dt' t'^{-1} \bar{h}_\star(t') \le c\bar{h}_\star(t) \tag{1.15}$$

for all $t \in I$.

Then $(u_1, A_1) = (u_2, A_2)$.

Remark 1.1. As mentioned previously, SA_0 and $x \cdot A_0$ are solutions of the free wave equation. The time decay in (1.10) is the optimal decay that can be obtained for solutions of that equation. Sufficient conditions on A_+, $\dot{A}_+$ ensuring that decay are well known (see for instance [13]).

Remark 1.2. Typical functions $h_\star$ satisfying the assumptions of Proposition 1.1 are

$$h_\star(t) = t^{-\lambda}(1 + \ell nt)^\mu,$$

with $\lambda > 1$ and μ real.

Remark 1.3. It will be shown below that the solutions of the system (1.3) (1.5) obtained in II (see especially Proposition 7.2 in II) satisfy the assumptions of

Proposition 1.1 with $\alpha = 3$ and $h_\star(t) = t^{-2}(1 + \ell n t)^4$ so that Proposition 1.1 applies to those solutions.

Proposition 1.1 will be proved by going to the above mentioned auxiliary system and generalizing the uniqueness proof for that system obtained in II (see Proposition 4.2 of II).

This paper is organized as follows. In Section 2, we derive the auxiliary system which will replace the original system (1.3) (1.5). In Section 3, we collect some notation and preliminary estimates. In Section 4, we derive the uniqueness result, first for the auxiliary system and then for the original one.

2. The auxiliary system

In this section we perform a change of unknown functions which is well adapted to the study of the system (1.3) (1.5) for large time and we derive the auxiliary system satisfied by the new functions. The unitary group $U(t)$ which solves the free Schrödinger equation can be written as

$$U(t) = \exp\big(i(t/2)\Delta\big) = M(t)D(t)FM(t) \tag{2.1}$$

where $M(t)$ is the operator of multiplication by the function

$$M(t) = \exp(ix^2/2t), \tag{2.2}$$

F is the Fourier transform and $D(t)$ is the dilation operator defined by

$$D(t) = (it)^{-3/2} D_0(t), \qquad \big(D_0(t)f\big)(x) = f(x/t). \tag{2.3}$$

We first change u to its pseudo-conformal inverse u_c defined by

$$u(t) = M(t)D(t)\overline{u_c(1/t)}, \tag{2.4}$$

or equivalently,

$$\tilde{u}(t) = \overline{F\tilde{u}_c(1/t)}, \tag{2.5}$$

where for any function f of space time

$$\tilde{f}(t,\cdot) = U(-t)f(t,\cdot).$$

Correspondingly we change A to B defined by

$$A(t) = -t^{-1} D_0(t)B(1/t). \tag{2.6}$$

The transformation $(u, A) \to (u_c, B)$ is involutive. Furthermore it replaces the study of (u, A) in a neighborhood of infinity in time by the study of (u_c, B) in a neighborhood of $t = 0$.

Substituting (2.4), (2.6) into (1.3) and commuting the Schrödinger operator with MD, we obtain that

$$\begin{aligned}&\left\{\left(i\partial_t + (1/2)\Delta_A - g(u)\right)u\right\}(t)\\&\quad = t^{-2}M(t)D(t)\left\{\overline{\left(i\partial_t + (1/2)\Delta_B - \check{B} - t^{-1}g(u_c)\right)u_c}\right\}(1/t)\end{aligned}$$

where for any $\mathbb{R}^3$ vector valued function f of space time

$$\check{f}(t, x) = t^{-1}x \cdot f(t, x). \tag{2.7}$$

Furthermore

$$\operatorname{Im}(\bar{u}\nabla_A u)(t) = t^{-3}D_0(t)\{x|u_c|^2 - t \operatorname{Im} \bar{u}_c\nabla_B u_c\}(1/t)$$

by a direct computation, so that the system (1.3) (1.5) becomes

$$\begin{cases} i\partial_t u_c = -(1/2)\Delta_B u_c + \check{B}u_c + t^{-1}g(u_c)u_c & (2.8)\\ B_2 = \mathscr{B}_2(u_c, B) & (2.9)\end{cases}$$

where B_0 is defined by $(2.6)_0$ and

$$B_2 = B - B_0 - B_1, \tag{2.10}$$

$$B_1 = B_1(u_c) \equiv -F_1(Px|u_c|^2), \tag{2.11}$$

$$\mathscr{B}_2(u_c, B) \equiv tF_2(P \operatorname{Im} \bar{u}_c\nabla_B u_c), \tag{2.12}$$

$$F_j(M) \equiv \int_1^\infty dv v^{-2-j}\omega^{-1}\sin\left(\omega(v-1)\right)D_0(v)M(t/v). \tag{2.13}$$

Here we take the point of view that B_1 is an explicit function of u_c defined by (2.11) and that (2.10) is a change of dynamical variable from B to B_2. The equation (2.9) then replaces (1.5).

In order to take into account the long range character of the MS system, we parametrize u_c in terms of a complex amplitude v and a real phase φ by

$$u_c = v\exp(-i\varphi). \tag{2.14}$$

The role of the phase is to cancel the long range terms in (2.8), namely the contribution of B_1 to $\check{B}$ and the term $t^{-1}g(u_c)$. Because of the limited regularity of B_1, it is convenient to split B_1 and B into a short range and a long range part. Let $\chi \in \mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R})$, $0 \le \chi \le 1$, $\chi(\xi) = 1$ for $|\xi| \le 1$, $\chi(\xi) = 0$ for $|\xi| \ge 2$. We define

$$\begin{cases} \check{B}_L = \check{B}_{1L} = F^\star \chi(\cdot t^{1/2}) F \check{B}_1 \\ \check{B}_S = \check{B}_0 + \check{B}_{1S} + \check{B}_2, \quad \check{B}_{1S} = \check{B}_1 - \check{B}_{1L} \end{cases} \tag{2.15}$$

We then obtain the following system for (v, φ, B_2)

$$\begin{cases} i\partial_t v = Hv & (2.16) \\ \partial_t \varphi = t^{-1} g(v) + \check{B}_{1L}(v) & (2.17) \\ B_2 = \mathscr{B}_2(v, K) & (2.18) \end{cases}$$

where

$$H \equiv -(1/2)\Delta_K + \check{B}_S, \tag{2.19}$$

$$K \equiv B + \nabla\varphi \equiv B + s, \tag{2.20}$$

by imposing (2.17) as the equation for φ. Under (2.17), the equation (2.8) becomes (2.16). The system (2.16)–(2.18) is the auxiliary system which replaces the original system (1.3) (1.5).

3. Notation and preliminary estimates

In this section we introduce some notation and collect a number of estimates which will be used throughout this paper. We denote by $\|\cdot\|_r$ the norm in $L^r = L^r(\mathbb{R}^3)$. For any non negative k we denote by $H^k = H^k(\mathbb{R}^3)$ the standard Sobolev spaces

$$H^k = \{u \in \mathscr{S}'(\mathbb{R}^3) : \|u; H^k\| = \|\langle\omega\rangle^k u\|_2 < \infty\},$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$ and $\omega = (-\Delta)^{1/2}$. In addition we will use the associated homogeneous spaces $\dot{H}^k$ with norm $\|u; \dot{H}^k\| = \|\omega^k u\|_2$. It will be understood that $\dot{H}^1 \subset L^6$. For any $k \ge 2$ we shall use the notation

$$\ddot{H}^k = \dot{H}^1 \cap \dot{H}^k.$$

For any Banach space $X \subset \mathscr{S}'(\mathbb{R}^3)$ we use the notation

$$FX = \{u \in \mathscr{S}'(\mathbb{R}^3) : F^{-1}u \in X\}.$$

For any interval I and any Banach space X we denote by $\mathscr{C}(I, X)$ the space of strongly continuous functions from I to X and by $L^\infty(I, X)$ (resp. $L^\infty_{\text{loc}}(I, X)$) the space of measurable essentially bounded (resp. locally essentially bounded) functions from I to X. For any real numbers a and b we use the notation $a \vee b = \text{Max}(a, b)$ and $a \wedge b = \text{Min}(a, b)$.

We next give estimates of the short and long range parts of B_1 defined by (2.15), namely

$$\|\omega^m \check{B}_{1S}\|_2 \le t^{(p-m)/2} \|\omega^p \check{B}_{1S}\|_2 \le t^{(p-m)/2} \|\omega^p \check{B}_1\|_2 \tag{3.1}$$

for $m \le p$ and similarly

$$\|\omega^m \check{B}_{1L}\|_2 \le (2t^{-1/2})^{p-m} \|\omega^p \check{B}_{1L}\|_2 \le (2t^{-1/2})^{p-m} \|\omega^p \check{B}_1\|_2 \tag{3.2}$$

for $m \ge p$.

We now estimate $F_j(M)$ defined by (2.13), (2.3) and $G_j(M)$ defined similarly by:

$$G_j(M) = \int_1^\infty dv v^{-1-j} \cos\big(\omega(v-1)\big) D_0(v) M(t/v). \tag{3.3}$$

From (2.13) it follows that

$$\omega F_j(M) = F_{j+1}(\omega M), \tag{3.4}$$

$$\partial_t F_j(M) = F_{j+1}(\partial_t M), \tag{3.5}$$

$$x \cdot F_j(PM) = F_{j-1}(x \cdot PM). \tag{3.6}$$

The first two identities are obvious, while in (3.6) we have used the identity

$$[x, f(\omega)] \cdot P = 0$$

which holds for any regular function f. In addition a direct computation yields

$$x \cdot PM = P(x \otimes M) - 2\omega^{-2} \nabla \cdot M$$

from which (3.5) can be continued to

$$x \cdot F_j(PM) = F_{j-1}\big(P(x \otimes M) - 2\omega^{-2} \nabla \cdot M\big). \tag{3.7}$$

Clearly the identities (3.4) (3.5) (3.6) (3.7) hold with F_j replaced by G_j. The following lemma provides an expression for the time derivative of $F_j(M)$ which does not contain the time derivative of M.

Lemma 3.1. *Let* $F_j(M)$ *and* $G_j(M)$ *be defined by* (2.13) *and* (3.3) *respectively. Then*

$$t\partial_t F_j(M) = -F_j\big((x\cdot\nabla + j + 1)M\big) + G_j(M). \tag{3.8}$$

Proof. From (3.5) we can write

$$t\partial_t F_j(M) = -\int_1^\infty dv v^{-2-j}\omega^{-1}\sin\big(\omega(v-1)\big)D_0(v)v\partial_v M(t/v). \tag{3.9}$$

Using the commutator identity

$$(v\partial_v + x\cdot\nabla)D_0(v) = D_0(v)v\partial_v$$

we obtain

$$t\partial_t F_j(M) = -\int_1^\infty dv v^{-1-j}\omega^{-1}\sin\big(\omega(v-1)\big)\partial_v\big(D_0(v)M(t/v)\big) - F_j(x\cdot\nabla M)$$

from which (3.8) follows by integration by parts over the v variable. □

In order to estimate F_j and G_j we define

$$I_j(f)(t) = \int_1^\infty dv v^{-j-3/2} f(t/v) \tag{3.10}$$

for any $j\in\mathbb{R}$ and for any non negative function f defined in $\mathbb{R}^+$. The estimates on F_j and G_j are summarized in the following lemma.

Lemma 3.2. *For any* $m, j\in\mathbb{R}$ *the following estimates hold:*

(1)

$$\|\omega^m F_j(M)\|_2 \le cI_{j+m-2}(\|\omega^{m-1}M\|_2 \wedge \|\omega^m M\|_2), \tag{3.11}$$

$$\|\omega^m G_j(M)\|_2 \le cI_{j+m-2}(\|\omega^m M\|_2). \tag{3.12}$$

(2)

$$\|\omega^m x\cdot F_j(PM)\|_2 \le cI_{j+m-3}(\|\langle x\rangle\omega^{m-1}M\|_2), \tag{3.13}$$

$$\|\omega^m x\cdot G_j(PM)\|_2 \le cI_{j+m-3}(\|x\omega^m M\|_2 + \|\omega^{m-1}M\|_2). \tag{3.14}$$

(3) *For any* r, $2\le r\le 4$,

$$\|F_j(M)\|_r \le c\int_1^\infty dv v^{-1+2/r} v^{-j+1/r}\|M(t/v)\|_{r_1} \tag{3.15}$$

and

$$\|G_j(M)\|_r \leq c \int_1^\infty d\nu \nu^{-1+2/r} \nu^{-j+1/r} \|\omega M(t/\nu)\|_{r_1} \tag{3.16}$$

with $3/r_1 = 2 + 1/r$.

Proof. Part (1). From the definition of F_j and G_j, from (3.4) and the analogue for G_j, from the identity

$$\|\omega^m D_0(\nu) v\|_2 = \nu^{-m+3/2} \|\omega^m v\|_2,$$

and from the estimates

$$\left|\sin\big(\omega(\nu-1)\big)\right| \leq 1 \wedge \omega\nu, \qquad \left|\cos\big(\omega(\nu-1)\big)\right| \leq 1$$

we obtain easily (3.11) and (3.12).

Part (2) is an immediate consequence of (3.7), of the analogue for G_j, and of Part (1).

Part (3). From the pointwise estimate [1] [11]

$$\left\|\sin\big(\omega(\nu-1)\big)v\right\|_r \vee \left\|\cos\big(\omega(\nu-1)\big)v\right\|_r \leq c(\nu-1)^{-1+2/r} \|\omega^{2-4/r} v\|_{\bar{r}}$$

with $2 \leq r < \infty$ and $1/r + 1/\bar{r} = 1$, it follows that

$$\|F_j(M)\|_r \leq c \int_1^\infty d\nu (\nu-1)^{-1+2/r} \nu^{-j+1/r} \|\omega^{1-4/r} M(t/\nu)\|_{\bar{r}}$$

and

$$\|G_j(M)\|_r \leq c \int_1^\infty d\nu (\nu-1)^{-1+2/r} \nu^{-j+1/r} \|\omega^{2-4/r} M(t/\nu)\|_{\bar{r}}$$

which imply (3.15) and (3.16) by Sobolev inequalities. □

In order to take into account the time decay of norms of some variables as t tends to zero, we shall introduce a function $h \in \mathscr{C}(I, \mathbb{R}^+)$, where $I = (0, \tau]$ for $0 < \tau \leq 1$, such that the function $\bar{h}(t) \equiv t^{-1}(1 - \ell n t)^\gamma h(t)$ with $\gamma \geq 0$ be non decreasing in I and satisfy

$$\int_0^t dt' t'^{-1} \bar{h}(t') \leq c\bar{h}(t)$$

for some $c > 0$ and for all $t \in I$. By an elementary computation we then obtain

$$I_j\big(t^{-\lambda}(1 - \ell nt)^{\mu} h\big)(t) = t^{-1/2-j} \int_0^t dt' t'^{j-1/2-\lambda}(1 - \ell nt')^{\mu} h(t')$$
$$\leq ct^{-\lambda}(1 - \ell nt)^{\mu} h(t) \tag{3.17}$$

for any real μ, provided that $j + 3/2 > \lambda$.

In all the estimates in this paper we denote by C a constant depending on the unknown functions through the available norms. Absolute constants, denoted by c in this section, will in general henceforth be omitted. The letters j, k, ℓ will always denote non negative integers.

4. Uniqueness

In this section we prove Proposition 1.1. This will be done by replacing the original system (1.3) (1.5) by the auxiliary system (2.16)–(2.18) and deriving first a uniqueness result for the latter. We recall that the functions B_1 and $\mathscr{B}_2$ are defined (cf. (2.11) (2.12)) by

$$B_1(v) \equiv -F_1(Px|v|^2), \tag{4.1}$$
$$\mathscr{B}_2(v, K) = tF_2(P \operatorname{Im} \bar{v}\nabla_K v). \tag{4.2}$$

The latter will be used in general with

$$K = B + s = B_0 + B_1(v) + B_2 + \nabla\varphi. \tag{4.3}$$

We shall need the space

$$V = \{v : v \in H^3 \text{ and } xv \in H^2\} \tag{4.4}$$

with the natural norm, and for $0 < \tau \leq 1$, $I = (0, \tau]$ and $\alpha \geq 0$, we shall make use of the assumption

$(A_+\alpha)$ $\quad v \in L^{\infty}_{\text{loc}}(I, V)$ and

$$\|v(t); V\| \leq CL^{\alpha} \tag{4.5}$$

for all $t \in I$, where $L = 1 - \ell nt$.

We first prepare the uniqueness result for the system (2.16)–(2.18) with two lemmas.

Lemma 4.1. *Let* $0 < \tau \le 1$, $I = (0, \tau]$, $\alpha \ge 0$ *and let* v *satisfy* $(A_+\alpha)$. *Then*

(1) $B_1(v) \in L^\infty_{\mathrm{loc}}(I, \ddot{H}^4)$, $\nabla \check{B}_1(v) \in L^\infty_{\mathrm{loc}}(I, \ddot{H}^2)$, $g \in L^\infty_{\mathrm{loc}}(I, \ddot{H}^5)$, $\partial_t B_1(v) \in L^\infty_{\mathrm{loc}}(I, \ddot{H}^2)$ *and the following estimates hold for all* $t \in I$:

$$\|\nabla^k B_1(v)\|_2 \le C L^{2\alpha} \qquad \text{for } 1 \le k \le 4, \tag{4.6}$$

$$\|\nabla^k \check{B}_1(v)\|_2 \le C t^{-1} L^{2\alpha} \qquad \text{for } 2 \le k \le 3, \tag{4.7}$$

$$\|\nabla^k g(v)\|_2 \le C L^{2\alpha} \qquad \text{for } 1 \le k \le 5, \tag{4.8}$$

$$\|\nabla^k \partial_t B_1(v)\|_2 \le C t^{-1} L^{2\alpha} \qquad \text{for } 1 \le k \le 2. \tag{4.9}$$

Let in addition φ *satisfy* (2.17). *Then* $\nabla \partial_t \varphi \in L^\infty_{\mathrm{loc}}(I, \ddot{H}^2)$ *and*

$$\|\nabla^{k+1} \partial_t \varphi\|_2 \equiv \|\nabla^k \partial_t s\|_2 \le C t^{-1} L^{2\alpha} \qquad \text{for } 1 \le k \le 2. \tag{4.10}$$

Let in addition $\nabla \varphi(t_0) \in \ddot{H}^2$ *for some* $t_0 \in I$. *Then* $\nabla \varphi \in \mathscr{C}(I, \ddot{H}^2)$ *and*

$$\|\nabla^{k+1} \varphi\|_2 \equiv \|\nabla^k s\|_2 \le C L^{1+2\alpha} \qquad \text{for } 1 \le k \le 2. \tag{4.11}$$

(2) *Let in addition* $u = v \exp(-i\varphi)$. *Then* u *satisfies* $(A_+\alpha_1)$ *with* $\alpha_1 = 3 + 7\alpha$.

(3) *Let in addition* B_0 *satisfy*

$$\|\nabla^k B_0(t)\|_\infty \le C t^{-k} \qquad \text{for } 0 \le k \le 1 \tag{4.12}$$

and let $B_2 \in L^\infty(I, \dot{H}^1)$ *satisfy*

$$\|\nabla B_2(t)\|_2 \le C L^{2\alpha} \tag{4.13}$$

for all $t \in I$. *Then* $\mathscr{B}_2(v, K) \in L^\infty_{\mathrm{loc}}(I, H^2)$, $\check{\mathscr{B}}_2(v, K) \in L^\infty_{\mathrm{loc}}(I, \ddot{H}^2)$, $\partial_t \mathscr{B}_2(v, K) \in L^\infty_{\mathrm{loc}}(I, H^1)$ *and the following estimates hold for all* $t \in I$:

$$\|\nabla^k \mathscr{B}_2(v, K)\|_2 \le C L^{2\alpha} \qquad \text{for } 0 \le k \le 2, \tag{4.14}$$

$$\|\nabla^k \check{\mathscr{B}}_2(v, K)\|_2 \le C t^{-1} L^{2\alpha} \qquad \text{for } 1 \le k \le 2, \tag{4.15}$$

$$\|\nabla^k \partial_t \mathscr{B}_2(v, K)\|_2 \le C t^{-1} L^{2\alpha} \qquad \text{for } 0 \le k \le 1 \tag{4.16}$$

where K *is given by* (4.3).

Remark 4.1. The condition $\nabla f \in \ddot{H}^2$ seems to leave some ambiguity on the nature of f. However it implies that $\nabla f \in L^\infty$ by Sobolev inequalities and therefore that $\langle x \rangle^{-1} f \in L^\infty$. This occurs in particular in Part (1) for $\check{B}_1(v)$, $\partial_t \varphi$ and φ for fixed time.

Proof. Part (1). We first derive the estimates (4.6)–(4.11).

It follows from (4.1) and (3.11) that

$$\|\nabla^k B_1(v)\|_2 \le I_{k-1}(\|\nabla^{k-1} x|v|^2\|_2) \le CL^{2\alpha} \qquad \text{for } 1 \le k \le 4$$

by $(A_+\alpha)$ and Hölder and Sobolev inequalities. Similarly from (3.13)

$$\|\nabla^k \check{B}_1(v)\|_2 \le I_{k-2}(\|\langle x\rangle \nabla^{k-1} x|v|^2\|_2) \le CL^{2\alpha} \qquad \text{for } 2 \le k \le 3.$$

(4.8) is obvious. It follows from (3.8) (3.11) (3.12) that

$$\begin{aligned}\|\nabla^k t\partial_t B_1(v)\|_2 &\le I_{k-1}(\|\nabla^{k-1}(x\cdot\nabla + 2)x|v|^2\|_2 + \|\nabla^k x|v|^2\|_2)\\ &\le CL^{2\alpha} \quad \text{for } 1 \le k \le 2.\end{aligned}$$

(4.10) follows from (2.17) (4.7) (4.8) while (4.11) follows from (4.10) by integration over time.

In order to complete the proof, we need to estimate a lower norm of B_1, $\nabla\check{B}_1$, g, $\partial_t B_1$ and $\nabla\partial_t\varphi$ in order to show that those quantities belong to $\dot{H}^1$. We estimate them in L^4 norm by using the special case $r = 4$ of (3.15), (3.16), namely

$$\|F_j(M)\|_4 \le \int_1^\infty d\nu(\nu-1)^{-1/2}\nu^{-j+1/4}\|M(t/\nu)\|_{4/3} \tag{4.17}$$

and similarly for (3.16), and by using the Hardy-Littlewood-Sobolev (HLS) inequality for g. The right hand side of (4.17) and of the other estimates with the appropriate M is then estimated by the use of $(A_+\alpha)$.

Part (2) follows from $(A_+\alpha)$ and (4.11). The required estimates use only the norm of $\nabla\varphi$ in $\ddot{H}^2$ and the worst contribution comes from

$$\|v|\nabla\varphi|^3\|_2 \le \|v\|_\infty \|\nabla\varphi\|_6^3 \le CL^{\alpha+3(1+2\alpha)}$$

in the estimate of $\|\nabla^3 u\|_2$.

Part (3). We first derive the estimates (4.14)–(4.16). We rewrite (4.2) as

$$\mathscr{B}_2(v,K) = tF_2\big(P\big(\operatorname{Im}\bar{v}\nabla v - \big(B_0 + B_1(v) + B_2 + s\big)|v|^2\big)\big). \tag{4.18}$$

From (3.11), we estimate

$$\begin{aligned}\|\nabla^k \mathscr{B}_2(v,K)\|_2 &\le tI_k(\|\bar{v}\nabla v\|_2 + \|B_0 + B_1(v) + s\|_\infty \|v\|_4^2 + \|B_2\|_6\|v\|_6^2)\\ &\le CtL^{1+4\alpha} \qquad \text{for } 0 \le k \le 1,\end{aligned} \tag{4.19}$$

$$\begin{aligned}
&\|\nabla^2 \mathscr{B}_2(v,K)\|_2 \\
&\quad \leq tI_2\big(\|\nabla(\bar{v}\nabla v)\|_2 + 2\|B_0 + B_1(v) + s\|_\infty \|\bar{v}\nabla v\|_2 \\
&\qquad + \big\|\nabla\big(B_0 + B_1(v)\big)\big\|_\infty \|v\|_4^2 \\
&\qquad + \|\nabla(B_2 + s)\|_2(\|v\|_\infty^2 + 2\|\bar{v}\nabla v\|_3)\big) \leq CL^{2\alpha} \qquad (4.20)
\end{aligned}$$

by $(A_+\alpha)$ (4.6) (4.11) (4.12) (4.13). This proves (4.14). Note that in (4.20) the dominant contribution comes from the term with ∇B_0. All the other terms contribute at most $CtL^{1+4\alpha}$ as in (4.19). The proof of (4.15) is similar, with the factor t omitted, with I_k replaced by I_{k-1} and the factor x absorbed by v.

(4.16) follows from (4.2) (3.8) (3.11) (3.12). We obtain

$$\begin{aligned}
\|\nabla^k \partial_t \mathscr{B}_2(v,K)\|_2 &\leq tI_k(\|(x\cdot\nabla + 2)\,\mathrm{Im}\,\bar{v}\nabla_K v\|_2 + \|\nabla\,\mathrm{Im}\,\bar{v}\nabla_K v\|_2) \\
&\leq CL^{2\alpha} \quad \text{for } 0 \leq k \leq 1,
\end{aligned} \qquad (4.21)$$

by $(A_+\alpha)$ (4.6) (4.11) (4.12) (4.13). The dominant contribution comes from

$$\|x\cdot(\nabla B_0)|v|^2\|_2 \leq \|\nabla B_0\|_\infty \|x|v|^2\|_2 \leq Ct^{-1}L^{2\alpha}.$$

In order to complete the proof, in the same way as in Part (1), we estimate the L^4 norm of $\check{B}_2(v,K)$ by using (4.17) with the appropriate M and estimating the right hand side thereof through $(A_+\alpha)$ (4.6) (4.11) (4.12) (4.13). □

Remark 4.2. For $k = 1$, we have in fact obtained the better estimate

$$\|\nabla\mathscr{B}_2(v,K)\|_2 \vee t\|\nabla\check{\mathscr{B}}_2(v,K)\|_2 \leq CtL^{1+4\alpha} \qquad (4.22)$$

in (4.14), (4.15). For $k = 2$, we could also have obtained better estimates by replacing the assumption (4.12) with

$$\|\nabla B_0\|_2 \leq Ct^{-1/2}$$

which is also satisfied if A_0 is a sufficiently regular solution of the free wave equation. However the estimates (4.14) (4.15) are sufficient for later purposes.

We next estimate the difference of two solutions of the auxiliary system (2.16)–(2.18). For two functions or operators of the same nature f_i, $i = 1, 2$, we shall use the notation $f_\pm = (1/2)(f_1 \pm f_2)$, so that $f_1 = f_+ + f_-$, $f_2 = f_+ - f_-$ and

$(fg)_\pm = f_+g_\pm + f_-g_\mp$. If (v_i, φ_i, B_{2i}), $i = 1, 2$, are two solutions of the auxiliary system (2.16)–(2.18), then (v_-, φ_-, B_{2-}) satisfies the system

$$i\partial_t v_- = H_+ v_- + H_- v_+ \tag{4.23}$$

$$\partial_t \varphi_- = t^{-1} g_- + \check{B}_{1L-} \tag{4.24}$$

$$B_{2-} = tF_2\big(P(2\,\mathrm{Im}\,\bar{v}_+ \nabla_{K_+} v_- - K_-(|v_+|^2 + |v_-|^2)\big) \tag{4.25}$$

where

$$H_+ = -(1/2)\Delta_{K_+} + (1/2)K_-^2 + \check{B}_{S+}, \tag{4.26}$$

$$H_- = iK_- \cdot \nabla_{K_+} + (i/2)(\nabla \cdot s_-) + \check{B}_{S-}, \tag{4.27}$$

$$B_{1-} = (1/2)\big(B_1(v_1) - B_1(v_2)\big) = -F_1(2P\,\mathrm{Re}\, x\bar{v}_+ v_-), \tag{4.28}$$

$\check{B}_{S\pm}$ and $\check{B}_{L\pm}$ are defined by similar formulas, and g_- and $K_\pm$ are obtained from $g_i = g(v_i)$ and

$$K_i = B_0 + B_1(v_i) + B_{2i} + \nabla\varphi_i. \tag{4.29}$$

For $0 < \tau \le 1$, $I = (0, \tau]$ and $h \in \mathscr{C}(I, \mathbb{R}^+)$, we introduce the assumption

(A_-h) $\langle x \rangle v_- \in L^\infty(I, L^2)$ and

$$\|\langle x \rangle v_-(t)\|_2 \le Ch(t) \tag{4.30}$$

for all $t \in I$.

Lemma 4.2. *Let $0 < \tau \le 1$, $I = (0, \tau]$, $\alpha \ge 0$, and let $h \in \mathscr{C}(I, \mathbb{R}^+)$ satisfy*

$$\int_0^\tau dt t^{-3/2} L^\alpha h(t) < \infty. \tag{4.31}$$

(1) *Let v_i, $i = 1, 2$ satisfy $(A_+\alpha)$ with v_- satisfying (A_-h). Then $B_{1-} \in L^\infty(I, \dot{H}^1)$, $\check{B}_{1-} \in L^\infty_{\mathrm{loc}}(I, \dot{H}^1)$, $g_- \in L^\infty_{\mathrm{loc}}(I, \ddot{H}^3)$, and the following estimates hold for all $t \in I$:*

$$\|\nabla B_{1-}\|_2 \le CI_0(\|v_-\|_2 L^\alpha), \tag{4.32}$$

$$\|\nabla \check{B}_{1-}\|_2 \le Ct^{-1} I_{-1}(\|\langle x \rangle v_-\|_2 L^\alpha), \tag{4.33}$$

$$\|\nabla^{k+1} g_-\|_2 \le C(\|v_-\|_2 + \delta_{k,2}\|\nabla v_-\|_2) L^\alpha \quad \textit{for } 0 \le k \le 2. \tag{4.34}$$

Let in addition φ_i, $i = 1, 2$, *satisfy* (2.17) *with* $v = v_i$. *Then* $\partial_t \varphi_- \in L^\infty_{\text{loc}}(I, \ddot{H}^3)$ *and the following estimates hold for all* $t \in I$

$$\|\nabla^{k+1}\partial_t \varphi_-\|_2 = \|\nabla^k \partial_t s_-\|_2 \le C\{(\|v_-\|_2 + \delta_{k,2}\|\nabla v_-\|_2)t^{-1}L^\alpha + t^{-1-k/2} I_{-1}(\|\langle x\rangle v_-\|_2 L^\alpha)\} \quad \text{for } 0 \le k \le 2. \tag{4.35}$$

(2) *Let* B_0 *satisfy*

$$\|\nabla^k (t\partial_t)^j B_0\|_\infty \vee \|\nabla^k \check{B}_0\|_\infty \le Ct^{-k} \quad \textit{for } 0 \le j + k \le 1 \tag{4.36}$$

for all $t \in I$. *Let* (v_i, φ_i, B_{2i}), $i = 1, 2$, *be two solutions of the system* (2.16)–(2.18) *such that* v_i *satisfy* $(A_+\alpha)$, *such that* $B_{2i} \in L^\infty_{\text{loc}}(I, \dot{H}^1)$ *with*

$$\|\nabla B_{2i}(t)\|_2 \le CL^{2\alpha} \tag{4.37}$$

for all $t \in I$, *and such that* $\nabla\varphi_i(t_0) \in \ddot{H}^2$ *with* $\nabla\varphi_-(t_0) \in L^2$ *for some* $t_0 \in I$, *so that* $s_- = \nabla\varphi_- \in \mathscr{C}(I, H^2)$ *by* (4.35). *Then the following estimates hold:*

$$|\partial_t \|v_-\|_2| \le C\{\|\nabla B_-\|_2 L^{1+3\alpha} + \|\nabla s_-\|_2 L^\alpha + \|s_-\|_2 L^{1+3\alpha} + \|\nabla \check{B}_{1-}\|_2 t^{1/2} L^\alpha + \|\nabla \check{B}_{2-}\|_2 L^\alpha\} \equiv E(t), \tag{4.38}$$

$$|\partial_t \|x v_-\|_2| \le \|\nabla_{K_+} v_-\|_2 + E(t), \tag{4.39}$$

$$|\partial_t \|\nabla_{K_+} v_-\|_2| \le C\{(\|v_-\|_2 + \|v_-\|_3)t^{-1}L^{2\alpha} + \|s_-\|_2 t^{-1} L^\alpha + \|\nabla B_-\|_2 t^{-1} L^\alpha + \|\nabla s_-\|_2 L^{1+3\alpha} + \|\nabla\nabla\cdot s_-\|_2 L^\alpha + \|\nabla \check{B}_{1-}\|_2 L^\alpha + \|\nabla \check{B}_{2-}\|_2 L^{1+3\alpha}\}, \tag{4.40}$$

$$\|\nabla B_{2-}\|_2 \le t I_1\big(\|v_-\|_2 L^{1+3\alpha} + (\|s_-\|_2 + \|\nabla B_-\|_2)L^{2\alpha}\big), \tag{4.41}$$

$$\|\nabla \check{B}_{2-}\|_2 \le I_0\big(\|v_-\|_2 L^{1+3\alpha} + (\|s_-\|_2 + \|\nabla B_-\|_2)L^{2\alpha}\big). \tag{4.42}$$

Remark 4.3. The assumption that v_- satisfies (A_-h) with h satisfying (4.31) serves to ensure the finiteness of the RHS of (4.33) and is never used otherwise. Similarly the assumption that $\nabla\varphi_-(t_0) \in L^2$ serves only to ensure tha $s_- \in \mathscr{C}(I, L^2)$.

Proof. Part (1). We first derive the estimates (4.32)–(4.35). It follows from (4.28) (3.11) (3.13) and $(A_+\alpha)$ that

$$\|\nabla B_{1-}\|_2 \le 2I_0(\|x\bar{v}_+ v_-\|_2) \le CI_0(\|v_-\|_2 L^\alpha),$$
$$\|\nabla \check{B}_{1-}\|_2 \le 2t^{-1} I_{-1}(\|\langle x\rangle^2 \bar{v}_+ v_-\|_2) \le Ct^{-1} I_{-1}(\|\langle x\rangle v_-\|_2 L^\alpha),$$

while

$$\|\nabla^{k+1} g_-\|_2 = 2\|\nabla^{k-1}\bar{v}_+ v_-\|_2$$

from which (4.34) follows by the use of $(A_+\alpha)$. (4.35) follows from (2.17) (3.2) (4.33) (4.34).

In order to complete the proof, we need to estimate a lower norm of B_{1-}, $\check{B}_{1-}$ and g. As in the proof of Lemma 4.1, Part (1), we estimate the L^4 norm of those quantities by using (4.17), the HLS inequality and $(A_+\alpha)$.

Part (2). We first note that from (2.18) and Lemma 4.1, Part (3), especially (4.14)–(4.16), it follows that $B_{2+} \in L^\infty_{\text{loc}}(I, H^2)$, $\check{B}_{2+} \in L^\infty_{\text{loc}}(I, \ddot{H}^2)$, $\partial_t B_{2+} \in L^\infty_{\text{loc}}(I, H^1)$ and that the following estimate holds for all $t \in I$

$$\|B_{2+}; H^2\| \vee t\|\check{B}_{2+}; \ddot{H}^2\| \vee \|t\partial_t B_{2+}; H^1\| \leq CL^{2\alpha}. \tag{4.43}$$

Together with (4.36) and with Lemma 4.1, Part (1), especially (4.6) (4.11), this implies that $K_+ \in L^\infty_{\text{loc}}(I, \ddot{H}^2)$ and that K_+ satisfies the estimate

$$\|K_+\|_\infty \leq C\|K_+; \ddot{H}^2\| \leq CL^{1+2\alpha}. \tag{4.44}$$

We next estimate $\|v_-\|_2$. From (4.23) (4.27) (3.1) we obtain

$$\begin{aligned}
|\partial_t \|v_-\|_2| &\leq \|H_- v_+\|_2 \\
&\leq C\{\|\nabla B_-\|_2(\|\nabla v_+\|_3 + \|K_+\|_\infty \|v_+\|_3) \\
&\qquad + \|\nabla s_-\|_2(\|\nabla v_+\|_3 + \|v_+\|_\infty) \\
&\qquad + (\|s_-\|_2\|K_+\|_\infty + t^{1/2}\|\nabla\check{B}_{1-}\|_2)\|v_+\|_\infty \\
&\qquad + \|\nabla\check{B}_{2-}\|_2\|v_+\|_3\},
\end{aligned} \tag{4.45}$$

from which (4.38) follows by the use of $(A_+\alpha)$ and (4.44).

We next estimate $\|xv_-\|_2$. From (4.23) and the commutation relation

$$[x, H_+] = \nabla_{K_+}$$

we obtain that

$$|\partial_t \|xv_-\|_2| \leq \|\nabla_{K_+} v_-\|_2 + \|xH_- v_+\|_2$$

from which (4.39) follows by estimating the last norm in the same way as in (4.45), with the additional factor x everywhere absorbed by v_+. We next estimate $\|\nabla_{K_+} v_-\|_2$. Taking the covariant gradient of (4.23) yields

$$\begin{aligned} i\partial_t \nabla_{K_+} v_- = & -(1/2)\nabla_{K_+}\Delta_{K_+} v_- + \left((1/2)K_-^2 + \check{B}_{S+}\right)\nabla_{K_+} v_- \\ & + (\partial_t K_+ + K_-\nabla K_- + \nabla \check{B}_{S+})v_- \\ & + iK_- \cdot \nabla_{K_+}^2 v_+ + i(\nabla K_-)\cdot \nabla_{K_+} v_+ \\ & + (i/2)(\nabla \cdot s_-)\nabla_{K_+} v_+ + (i/2)(\nabla\nabla \cdot s_-)v_+ \\ & + \check{B}_{S-}\nabla_{K_+} v_+ + (\nabla \check{B}_{S-})v_+ \end{aligned} \tag{4.46}$$

from which we estimate

$$\begin{aligned} |\partial_t \|\nabla_{K_+} v_-\|_2| \le & \|(\partial_t K_+ + \nabla \check{B}_{S+})v_-\|_2 + \|K_- \cdot \nabla_{K_+}^2 v_+\|_2 \\ & + \|\nabla K_-\|_2(\|\nabla_{K_+} v_+\|_\infty + \|K_- v_-\|_\infty) \\ & + \|\nabla\nabla \cdot s_-\|_2 \|v_+\|_\infty \\ & + \|\nabla \check{B}_{1-}\|_2 (t^{1/2}\|\nabla_{K_+} v_+\|_\infty + \|v_+\|_\infty) \\ & + \|\nabla \check{B}_{2-}\|_2 (\|\nabla_{K_+} v_+\|_3 + \|v_+\|_\infty). \end{aligned} \tag{4.47}$$

We next estimate the first two terms in the right hand side of (4.47). We estimate

$$\begin{aligned} \|(\partial_t K_+ + \nabla \check{B}_{S+})v_-\|_2 \le & \|\partial_t (s_+ + B_0 + B_{1+}) \\ & + \nabla(\check{B}_0 + \check{B}_{1+})\|_\infty \|v_-\|_2 \\ & + \|\partial_t B_{2+} + \nabla \check{B}_{2+}\|_6 \|v_-\|_3 \\ \le & C(\|v_-\|_2 + \|v_-\|_3)t^{-1}L^{2\alpha} \end{aligned} \tag{4.48}$$

where we have used (4.7) (4.9) (4.10) (4.36) (4.43), and

$$\begin{aligned} & \|K_- \nabla_{K_+}^2 v_+\|_2 \\ & \quad \le \|s_-\|_3 (\|\nabla^2 v_+\|_6 + \|\nabla(s_+ + B_{2+})\|_6 \|v_+\|_\infty) \\ & \qquad + \|s_-\|_2 (\|K_+\|_\infty \|\nabla v_+\|_\infty + (\|\nabla(B_0 + B_{1+})\|_\infty + \|K_+\|_\infty^2)\|v_+\|_\infty) \\ & \qquad + \|B_-\|_6 (\|\nabla^2 v_+\|_3 + \|K_+\|_\infty \|\nabla v_+\|_3 + \|\nabla(B_0 + B_{1+})\|_\infty \|v_+\|_3 \\ & \qquad + \|\nabla(s_+ + B_{2+})\|_6 \|v_+\|_6 + \|K_+\|_\infty^2 \|v_+\|_3) \\ & \quad \le C\{\|s_-\|_3 L^{1+3\alpha} + (\|s_-\|_2 + \|\nabla B_-\|_2)t^{-1}L^\alpha\} \end{aligned} \tag{4.49}$$

where we have used $(A_+\alpha)$ (4.6) (4.11) (4.36) (4.43) (4.44).

Substituting (4.48) (4.49) into (4.47) and estimating the remaining terms of (4.47) by the use of $(A_+\alpha)$ and (4.44) yields (4.40).

We finally estimate B_{2-}. From (4.25) and (3.11) (3.13) we obtain

$$\begin{aligned}\|\nabla B_{2-}\|_2 &\le tI_1(\|v_-\|_2\|\nabla_{K_+}v_+\|_\infty + \|s_-\|_2\|v_+\|_\infty^2 + \|B_-\|_6\|v_+\|_6^2)\\ \|\nabla \check{B}_{2-}\|_2 &\le I_0(\|\langle x\rangle v_-\|_2\|\nabla_{K_+}v_+\|_\infty + \|s_-\|_2\|v_+\|_\infty\|\langle x\rangle v_+\|_\infty\\ &\quad + \|B_-\|_6\|v_+\|_6\|\langle x\rangle v_+\|_6)\end{aligned}$$

from which (4.41) (4.42) follow by the use of $(A_+\alpha)$ and (4.44). □

We now state the uniqueness result for the system (2.16)–(2.18).

Proposition 4.1. *Let* $0 < \tau \le 1$, *let* $I = (0, \tau]$, $\alpha \ge 0$ *and let* $h \in \mathscr{C}(I, \mathbb{R}^+)$ *be such that* $\bar{h}(t) = t^{-1}(1 - \ell nt)^\alpha h(t)$ *be non decreasing and satisfy*

$$\int_0^t dt' t'^{-1}\bar{h}(t') \le c\bar{h}(t) \tag{4.50}$$

for some $c > 0$ *and for all* $t \in I$. *Let* B_0 *satisfy* (4.36) *for all* $t \in I$. *Let* (v_i, φ_i, B_{2i}), $i = 1, 2$, *be two solutions of the system* (2.16)–(2.18) *such that* v_i *satisfies* $(A_+\alpha)$, *such that* $B_{2i} \in L^\infty_{\text{loc}}(I, \dot{H}^1)$ *satisfy* (4.37) *for all* $t \in I$, *and such that* $\nabla\varphi_i(t_0) \in \ddot{H}^2$ *for some* $t_0 \in I$. *Assume in addition that* $\varphi_-(0) = 0$ *and that* v_- *satisfy* (A_-h).

Then $(v_1, \varphi_1, B_{21}) = (v_2, \varphi_2, B_{22})$.

Proof. Note first that (4.50) implies (4.31) so that Lemma 4.2 can be applied. From (4.35) with $k = 0$ and mild assumptions on v_-, it follows that $\varphi_-(t)$ has a limit in $\dot{H}^1$ as $t \to 0$, thereby giving a meaning to the assumption $\varphi_-(0) = 0$. Actually it follows from (4.35) (4.50) and (A_-h) that the limit exists in $\ddot{H}^3$.

We first prove the proposition for τ sufficiently small by using Lemma 4.2. We define

$$y_0 = \|\langle x\rangle v_-\|_2, \qquad y_1 = \|\nabla_{K_+}v_-\|_2, \qquad Y_0 = \sup_{t\in I} h(t)^{-1}y_0(t).$$

From Lemma 4.2, especially (4.32) (4.33) (4.35) (4.50) and from (3.17), we obtain

$$\|\nabla B_{1-}\|_2 \le CY_0L^\alpha h, \tag{4.51}$$

$$\|\nabla \check{B}_{1-}\|_2 \le CY_0t^{-1}L^\alpha h, \tag{4.52}$$

$$\|\nabla^{k+1}\varphi_-\|_2 = \|\nabla^k s_-\|_2 \le C\left\{Y_0t^{-k/2}L^\alpha h + \delta_{k,2}\int_0^t dt' t'^{-1}L'^\alpha y_1(t')\right\} \tag{4.53}$$

for $0 \le k \le 2$ and for all $t \in I$, with $L' = 1 - \ell n t'$. The time integral of the last term in (4.53) converges because of the estimate

$$\|\nabla v_-\|_2^2 \le \|v_-\|_2 \|\Delta v_-\|_2 \le C Y_0 L^\alpha h$$

and we have replaced the ordinary derivative by the covariant one in that integral, thereby producing an innocuous term with $Y_0 L^{1+3\alpha} h$. On the other hand from (4.41) (4.51) (4.53) (3.17) we obtain

$$\|\nabla B_{2-}\|_2 \le C\{ Y_0 t L^{1+3\alpha} h + t I_1(\|\nabla B_{2-}\|_2 L^{2\alpha})\}. \tag{4.54}$$

From the assumptions on B_{2i}, it follows that $B_{2-} \in L^\infty_{\rm loc}(I, \dot{H}^1)$ with

$$\|\nabla B_{2-}(t)\|_2 \le C L^{2\alpha}.$$

Using that fact, one obtains easily from (4.54) that

$$\|\nabla B_{2-}\|_2 \le C Y_0 t L^{1+3\alpha} h \tag{4.55}$$

for all $t \in I$ and for τ sufficiently small. Substituting that result into (4.42) gives

$$\|\nabla \check{B}_{2-}\|_2 \le C Y_0 L^{1+3\alpha} h. \tag{4.56}$$

Substituting (4.51) (4.52) (4.53) (4.55) (4.56) into (4.40) yields

$$|\partial_t y_1| \le C\left\{ Y_0 t^{-1} L^{2\alpha} h + (Y_0 h y_1)^{1/2} t^{-1} L^{2\alpha} + L^\alpha \int_0^t dt' t'^{-1} L'^\alpha y_1(t') \right\} \tag{4.57}$$

which takes the form

$$|\partial_t y| \le f + g y^{1/2} + C \int_0^t dt' t'^{-1} L'^{2\alpha} y(t') \tag{4.58}$$

with

$$y_1 = Y_0 y, \qquad f = C t^{-1} L^{2\alpha} h, \qquad g = C t^{-1} L^{2\alpha} h^{1/2}.$$

We define

$$z(t) = \int_0^t dt' t'^{-1} L'^{2\alpha} y(t')$$

so that $t\partial_t z = L^{2\alpha} y$ and

$$F(t) = \int_0^t dt' f(t').$$

Integrating (4.58) over time with $y(0) = 0$ yields

$$\begin{aligned} y(t) &\leq F(t) + \int_0^t dt' g(t') y(t')^{1/2} + C \int_0^t dt' z(t') \\ &\leq F(t) + Ctz(t) + z(t)^{1/2} \left\{ \int_0^t dt' t' L'^{-2\alpha} g^2(t') \right\}^{1/2}. \end{aligned} \tag{4.59}$$

The last integral is estimated by

$$\int_0^t dt' t' L'^{-2\alpha} g^2(t') \leq CL^{2\alpha} h$$

by (4.50), so that (4.59) yields

$$y(t) \leq F(t) + C\big(tz(t) + \big(z(t)h(t)\big)^{1/2} L^\alpha\big) \tag{4.60}$$

and therefore

$$\partial_t z \leq t^{-1} L^{2\alpha} F(t) + CL^{2\alpha} z + C(zh)^{1/2} t^{-1} L^{3\alpha}. \tag{4.61}$$

Integrating (4.61) over time (see for instance Lemma 2.3 in [3]) we obtain

$$z(t) \leq \exp(CtL^{2\alpha}) \left\{ \int_0^t dt' t'^{-1} L'^{2\alpha} F(t') + \left(\int_0^t dt' t'^{-1} L'^{3\alpha} h(t')^{1/2} \right)^2 \right\}. \tag{4.62}$$

We next estimate

$$\int_0^t dt' t'^{-1} L'^{2\alpha} F(t') \leq CL^{4\alpha} h,$$

$$\left(\int_0^t dt' t'^{-1} L'^{3\alpha} h(t')^{1/2} \right)^2 \leq \left(\int_0^t dt' t'^{-2} L'^{\alpha} h(t') \right) \left(\int_0^t dt' L'^{5\alpha} \right) \leq CL^{6\alpha} h$$

by (4.50) and an elementary computation. Substituting those estimates into (4.62) yields

$$z(t) \leq CL^{6\alpha} h$$

and therefore by (4.60)

$$y_1(t) \le CY_0 L^{4\alpha} h. \tag{4.63}$$

Substituting (4.51) (4.52) (4.53) (4.55) (4.56) (4.63) into (4.38) (4.39) yields

$$|\partial_t y_0| \le CY_0 t^{-1/2} h$$

and therefore by integration over time with $y_0(0) = 0$

$$y_0(t) \le CY_0 t^{1/2} h$$

so that

$$Y_0 \le CY_0 \tau^{1/2}$$

which implies that $Y_0 = 0$ and therefore $v_- = 0$ for τ sufficiently small. Substituting that result into (4.35) (4.55) shows that $\varphi_- = 0$, $B_{2-} = 0$ and hence $(v_1, \varphi_1, B_{21}) = (v_2, \varphi_2, B_{22})$.

The extension of the proof to the case of general τ proceeds by similar but more standard arguments. □

We now turn to the proof of Proposition 1.1.

Proof of Proposition 1.1. The first step consists in rewriting that proposition in an equivalent form in terms of the variables (u_c, B_2) where u_c is defined by (2.4) or (2.5).

Proposition 4.2. *Let* $0 < \tau \le 1$, $I = (0, \tau]$, $\alpha \ge 0$. *Let* A_0 *be a divergence free solution of the free wave equation such that* B_0 *defined by* $(2.6)_0$ *satisfy* (4.36) *for all* $t \in I$. *Let* (u_{ci}, B_{2i}), $i = 1, 2$, *be two solutions of the system* (2.8) (2.9) *such that* u_{ci} *satisfy* $(A_+\alpha)$ *and that* $(B - B_0)_i \equiv B_1(u_{ci}) + B_{2i} \in L^\infty_{\text{loc}}(I, \dot{H}^1)$ *with*

$$\|\nabla(B - B_0)_i(t)\|_2 \le C(1 - \ell n t)^{2\alpha} \tag{4.64}$$

for all $t \in I$. *Assume in addition that* u_{c-} *satisfy* (A_-h) *for some function* $h \in \mathscr{C}(I, \mathbb{R}^+)$ *such that the function*

$$\bar{h}(t) = t^{-1}(1 - \ell n t)^{3+9\alpha} h(t) \tag{4.65}$$

be non decreasing for t sufficiently small and satisfy

$$\int_0^t dt' t'^{-1} \bar{h}(t') \le c\bar{h}(t) \tag{4.66}$$

for all $t \in I$.

Then $(u_{c1}, B_{21}) = (u_{c2}, B_{22})$.

We first show the equivalence of Proposition 1.1 and Proposition 4.2, with $\tau = T^{-1}$ and $h(t) = h_\star(1/t)$. The equivalence of (1.10) for A_0 and (4.36) for B_0 follows from $(2.6)_0$ and from the relations

$$SA_0(t) = t^{-1} D_0(t)(t\partial_t B_0)(1/t)$$
$$(x \cdot A_0)(t) = -t^{-1} D_0(t)\check{B}_0(1/t).$$

The equivalence of the assumption on u_i in Proposition 1.1 with $(A_+\alpha)$ for u_{ci} follows from (2.5), from the fact that $V = FV_\star$ and from the commutation relation

$$xU(-t) = U(-t)(x + it\nabla)$$

which implies that

$$| \, \|U(-t)v; V\| - \|v; V\| \, | \le |t| \, \|v; H^3\|$$

so that $(A_+\alpha)$ for u_{ci} is equivalent to $(A_+\alpha)$ for $\tilde{u}_{ci}$. The equivalence of (1.12) for $A - A_0$ with (4.64) for $(B - B_0)$ follows from (2.6). Finally the equivalence of the assumption (1.13) for u_- with $(A_- h)$ for u_{c-} follows from (2.4). □

We are now left with the task of deriving Proposition 4.2 from Proposition 4.1.

Proof of Proposition 4.2. Let (u_{ci}, B_{2i}) satisfy the assumptions of Proposition 4.2. We need to construct (v_i, φ_i, B_{2i}) satisfying the assumptions of Proposition 4.1. The main step is to construct φ_i from u_{ci}. Now it follows from (2.14) that (2.17) is equivalent to

$$\partial_t \varphi = t^{-1} g(u_c) + \check{B}_{1L}(u_c) \tag{4.67}$$

and we define the phases φ_i by integrating that equation over time with some initial condition $\nabla\varphi_i(t_0) \in \ddot{H}^2$. It follows from Lemma 4.1, Part (1), with v replaced by u_c that φ_i satisfy (4.11) and from Part (2) with (v, u, φ) replaced by $(u_c, v, -\varphi)$ that v_i defined by (2.14) satisfy $(A_+\alpha_1)$ with $\alpha_1 = 3 + 7\alpha$. Furthermore, again by Lemma 4.1, Part (1), $B_1(u_{ci}) = B_1(v_i)$ satisfy the assumptions made on $B - B_0$ in Proposition 4.2, so that the latter are equivalent to the assumptions made on B_2 in Proposition 4.1.

It remains to ensure the assumption $(A_- h_1)$ on v_- for a suitable h_1. This will be done by suitably adjusting the initial conditions for the phases φ_i. If φ_i, $i = 1, 2$, satisfy (4.67), then φ_- is estimated in the same way as in Lemma 4.2 with v replaced by u_c, namely,

$$\begin{aligned}\|\nabla^{k+1}\partial_t\varphi_-\|_2 &\leq C\{(\|u_{c-}\|_2+\delta_{k,2}\|\nabla u_{c-}\|_2)t^{-1}L^\alpha+t^{-1-k/2}I_{-1}(\|\langle x\rangle u_{c-}\|_2L^\alpha)\}\\ &\leq C\{t^{-1}L^\alpha(h+\delta_{k,2}h^{2/3}L^{\alpha/3})+t^{-1-k/2}L^\alpha h\}\end{aligned} \quad (4.68)$$

by using $(A_-\alpha)$ for u_{c-} and estimating

$$\|\nabla u_{c-}\|_2 \leq \|u_{c-}\|_2^{2/3}\|\nabla^3 u_{c-}\|^{1/3} \leq Ch^{2/3}L^{\alpha/3}.$$

We now adjust the phases as follows. We choose arbitrarily φ_1 with $\nabla\varphi_1(t_0)\in\ddot{H}^2$, we define φ_- by integrating (4.68) over time with initial condition $\varphi_-(0)=0$, and we define $\varphi_2=\varphi_1-2\varphi_-$. The integral of (4.68) converges for $0\leq k\leq 2$ by (4.66) and yields

$$\|\nabla^{k+1}\varphi_-\|_2 \leq C\{(1+t^{-k/2})L^\alpha h+\delta_{k,2}L^{\alpha/3}h^{2/3}\}. \quad (4.69)$$

In particular $\nabla\varphi_2(t_0)\in\ddot{H}^2$ and

$$\|\nabla\varphi_-\|_2 \leq CL^\alpha h. \quad (4.70)$$

We can now estimate v_-.

$$\begin{aligned}v_- &= (1/2)\big(u_{c1}\exp(i\varphi_1)-u_{c2}\exp(i\varphi_2)\big)\\ &= (1/2)\exp(i\varphi_1)\big(u_{c1}-u_{c2}+u_{c2}\big(1-\exp(-2i\varphi_-)\big)\end{aligned}$$

so that

$$\|\langle x\rangle v_-\|_2 \leq \|\langle x\rangle u_{c-}\|_2+c\|u_{c2}\|_3\|\nabla\varphi_-\|_2 \leq Ch(1+L^{2\alpha}) \quad (4.71)$$

by $(A_+\alpha)$, (A_-h) and (4.70). Therefore v_- satisfies (A_-h_1) with $h_1=hL^{2\alpha}$. Finally the assumption (4.50) for

$$\bar{h}_1(t)=t^{-1}(1-\ell nt)^{\alpha_1}h_1(t)=t^{-1}(1-\ell nt)^{3+9\alpha}h(t)$$

is equivalent to (4.66).

Proposition 4.2 then follows from Proposition 4.1. □

We conclude this section by showing that Proposition 1.1 applies to the solutions of the system (1.3) (1.5) constructed in II. Because of the equivalence of Proposition 1.1 and Proposition 4.2, it is sufficient to consider the solutions (u_c, B_2) of the system (2.8) (2.9). In II we proved the existence of solutions of that system with prescribed asymptotic behaviour (u_{ca}, B_{2a}) as t tends to zero, with $u_c = v\exp(-i\varphi)$ and $u_{ca} = v_a\exp(-i\varphi_a)$. It follows from Lemma 6.1 in II (note that v, v_a of this paper are w, w_a of II) that $v_a \in L^\infty(I,V)$, $\nabla\varphi_a \in L^\infty_{\rm loc}(I,\ddot{H}^2)$ and $B_{2a} \in L^\infty(I,\dot{H}^1)$ and from Proposition 7.1 in II that the same properties hold for v, $\nabla\varphi$ and B_2. In particular v satisfies (A_+0) so that by Lemma 4.1, Part (2), u_c satisfies (A_+3). Furthermore $v - v_a$ satisfies (A_-h) with

$$h = t^2(1 - \ell nt)^4 \tag{4.72}$$

and $(\varphi - \varphi_a)(0) = 0$. In particular if (v_i, φ_i, B_{2i}), $i = 1, 2$, are two solutions of the system (2.16)–(2.18) associated with the same v_a, then v_- satisfies (A_-h) with h given by (4.72), so that by the same argument as in the proof of Proposition 4.2, but now with u_c and v interchanged, u_{c-} also satisfies (A_-h). Therefore the solutions constructed in II for fixed (u_{ca}, B_{2a}) satisfy the assumptions of Proposition 4.2 with $\alpha = 3$ and h given by (4.72), which proves uniqueness for fixed (u_{ca}, B_{2a}) as chosen in II. Note that the proof uses Proposition 4.1 with $\alpha_1 = 24$ and

$$h_1 = t^2(1 - \ell nt)^6.$$

Remark 4.4. The construction of v from u_c or of u_c from v through (2.14), (2.17) or (4.67) and Lemma 4.1, Part (2) entails some loss in the estimates, typically from $(A_+\alpha)$ to $(A_+\alpha_1)$ with $\alpha_1 = 3 + 7\alpha$. In the existence proof given in II, starting from v satisfying (A_+0), we obtained u_c satisfying only (A_+3), and a direct uniqueness proof for u_c can start only from that weaker assumption. The reconstruction of v with only u_c available to start with then produces another loss and yields only (A_+24) for v. Only that weaker assumption can then be used in the auxiliary uniqueness result of Proposition 4.1, even though v was known to satisfy the better estimate (A_+0) to start with.

Remark 4.5. The loss on α in Lemma 4.1, part (2), can be reduced by using a more accurate estimate than (4.5) in $(A_+\alpha)$. If one assumes instead

$$\|\nabla^k \langle x\rangle^\ell v\|_2 \le CL^{k\alpha} \qquad \text{for } 0 \le \ell \le 1,\ 0 \le k + \ell \le 3,$$

then Lemma 4.1, Part (2), still holds with $\alpha_1 = 1 + 3\alpha$.

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