Portugal. Math. (N.S.)
Vol. 66, Fasc. 1, 2009, 13–27

Portugaliae Mathematica

Existence of solutions for a third-order boundary value problem with *p*-Laplacian operator and nonlinear boundary conditions

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(Communicated by Luís Sanchez)

Abstract. In this paper we study the third-order nonlinear boundary value problem

$$\begin{cases} (\phi(u''))'(t) + f\big(t, u(t), u'(t), u''(t)\big) = 0 \quad \text{a.e. } t \in [0,1], \\ u(0) = 0,\ g\big(u'(0), u''(0)\big) = A,\ h\big(u'(1), u''(1)\big) = B, \end{cases}$$

where $A, B \in \mathbb{R}$, $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function, $g, h \in C^0(\mathbb{R}^2, \mathbb{R})$ and $\phi \in C^0(\mathbb{R}, \mathbb{R})$. Using apriori estimates, the Nagumo condition, upper and lower solutions and the Schauder fixed point theorem, we are able to prove existence of solutions of this problem.

Mathematics Subject Classification (2000). 34B10, 34B15.

Keywords. Third-order nonlinear boundary condition, Nagumo condition, Schauder fixed point theorem.

1. Introduction

Third-order boundary value problems (BVPs) have been studied by many authors; see, e.g., the references listed below. However, the boundary conditions are usually assumed to be linear and only a few authors have studied the case of nonlinear boundary conditions.

In this article we will study the existence of solution for the following nonlinear boundary value problem:

$$\begin{cases} (\phi(u''))'(t) + f\big(t, u(t), u'(t), u''(t)\big) = 0 \quad \text{for a.e. } t \in I = [0,1], \\ u(0) = 0,\ g\big(u'(0), u''(0)\big) = A,\ h\big(u'(1), u''(1)\big) = B, \end{cases} \tag{P}$$

where $A, B \in \mathbb{R}$, and the three following conditions are assume to hold:

*Supported by National Natural Sciences Foundation of China (10771065), Natural Sciences Foundation of Heibei Province (A2007001027) and Doctor's Foundation of North China Electric Power University.

(H1) $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous, strictly increasing function, with $\phi(0) = 0$ and $\phi(\mathbb{R}) = \mathbb{R}$.

(H2) $f : I \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function, i.e.,
  (i) for all $(x, y, z) \in \mathbb{R}^3$, the function $t \to f(t, x, y, z)$ is measurable on I,
  (ii) for almost all $t \in I$, the function $(x, y, z) \to f(t, x, y, z)$ is continuous on $\mathbb{R}^3$, and
  (iii) for every $M > 0$ there exists a real-valued function $\psi_M \in L^1(I)$ such that

$$|f(t, x, y, z)| \leq \psi_M(t)$$

  holds for almost all $t \in I$ and for every $(x, y, z) \in \mathbb{R}^3$ with $|x| \leq M$, $|y| \leq M$ and $|z| \leq M$.

(H3) $g : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function which is non-increasing on the second variable; $h : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function which is non-decreasing on the second variable.

Remark 1.1. An important special case occurs when the function ϕ is the p-Laplacian operator, i.e., $\phi(u) = |u|^{p-2}u$, with $p > 1$.

We emphasize that the term $\big(\phi(u'')\big)'$ in (P) is not assumed to be linear in u, so many of the results that hold in the linear case, in general, will fail for problem (P). In fact, these kind of non-linear BVPs require quite different techniques from the linear case, and we propose here a new method. This method differs from the ones already available in the literature, so let us mentioned some of these works and how their results compare to ours.

Rovderová, in [11], has established existence results for the BVP

$$\begin{cases} u'''(t) + f\big(t, u(t), u'(t), u''(t)\big) = 0, \quad 0 < t < 1, \\ u(0) = A,\ u''(0) = \sigma\big(u'(0)\big),\ u'(1) = \tau\big(u(1)\big), \end{cases} \tag{1}$$

where f, $\partial f/\partial u$, $\partial f/\partial u'$, $\partial f/\partial u''$ are all assumed to be continuous functions on $[0, 1] \times \mathbb{R}^3$, and $\sigma(v) \in C^1(\mathbb{R}, \mathbb{R})$, $\tau(v) \in C^0(\mathbb{R}, \mathbb{R})$.

In [8], the authors study the existence of solutions for the same problem under the following very special boundary conditions:

$$u(0) = 0, \qquad au'(0) - bu''(0) = A, \qquad cu'(1) + du''(1) = B, \tag{2}$$

but where f is only assumed to be continuous. Later, Du et al. in [6], following the some set of ideas developed in [8], extended these existence results to the following more general type of boundary conditions:

$$u(0) = 0, \qquad g\big(u'(0), u''(0)\big) = A, \qquad h\big(u'(1), u''(1)\big) = B. \tag{3}$$

In [4], the authors study a more general BVP analogou to our problem (P):

$$\begin{cases} (\phi(u''))'(t) + f(t, u(t), u'(t), u''(t)) = 0 \quad \text{for a.e. } t \in I = [0,1], \\ u(0) = A, \\ L_1(u, u', u'(0), u'(1), u''(0)) = A, \\ L_2(u, u'(0), u'(1), u''(1)) = 0, \end{cases} \tag{4}$$

Our method is also quite different from the one used in [4]. For example, we do not require that $f(t, u, v, w)$ should be non-decreasing in the second variable, an assumptiom which is critical for the method in [4] to work. In fact, our approach, combines the method of lower and upper solutions, the Nagumo condition (to obtain a priori bounds for the second derivative of the solution), and the Schauder fixed point theorem. For this to work we will need to consider a modified form of (P), which makes it possible to use the Schauder fixed point theorem.

2. The main existence result

We will be using some standard notations: $C^0(I)$, $C^k(I)$, $L^k(I)$, $L^\infty(I)$ and $AC(I)$ will denote the classical functions spaces on the interval $I = [0,1]$ of continuous functions, k-times continuously differentiable functions, measurable real-valued functions whose k^{th} power is Lebesgue integrable, measurable functions that are essentially bounded, and absolutely continuous functions, respectively.

We start by introducing two basic definitions:

Definition 2.1. We say that $y \in C^2(I)$ is a *lower solution* for problem (P) if $\phi(y'') \in AC(I)$ and

$$\begin{cases} (\phi(y''))'(t) + f(t, y(t), y'(t), y''(t)) \geq 0 \quad \text{for a.e. } t \in I, \\ y(0) \leq 0,\ g(y'(0), y''(0)) \leq A,\ h(y'(1), y''(1)) \leq B. \end{cases}$$

Moreover, y is called an *upper solution* of (P) if the reversed inequalities hold; if equalities hold, we say that y is a *solution* of (P).

Definition 2.2. We say that a Carathéodory function $f : I \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the *Nagumo condition relative to the pair* α *and* β, where $\alpha, \beta \in C^2(I)$, if $\alpha(t) \leq \beta(t)$ and $\alpha'(t) \leq \beta'(t)$ for $t \in I$, and there exist continuous functions $k \in L^p(I)$ $(1 \leq p \leq \infty)$ and $\theta : [0, \infty) \to (0, \infty)$ such that

$$|f(t, u, v, w)| \leq k(t)\theta(|w|) \qquad \text{for a.e. } (t, u, v, w) \in \Omega,$$

where $\Omega = \{(t, u, v, w) \in I \times \mathbb{R}^3 \mid \alpha(t) \leq u \leq \beta(t), \alpha'(t) \leq v \leq \beta'(t)\}$, and

$$\int_{\phi(\eta)}^{+\infty} \frac{|\phi^{-1}(u)|^{(p-1)/p}}{\theta(|\phi^{-1}(u)|)} du, \qquad \int_{-\infty}^{\phi(-\eta)} \frac{|\phi^{-1}(u)|^{(p-1)/p}}{\theta(|\phi^{-1}(u)|)} du > \mu^{(p-1)/p} \|k\|_p,$$

where

$$\mu = \max_{t \in I} \beta'(t) - \min_{t \in I} \alpha'(t) \quad \text{and} \quad \eta = \max\{|\alpha'(0) - \beta'(1)|, |\alpha'(1) - \beta'(0)|\}.$$

Note that in the previous definition we have used the standard convention

$$\|k\|_p = \begin{cases} \sup_{t \in I} |k(t)| & \text{if } p = \infty, \\ (\int_0^1 |k(t)|^p \, dt)^{1/p} & \text{if } 1 \le p < \infty, \end{cases}$$

where $(p-1)/p \equiv 1$ for $p = \infty$.

The following classical result is critical to our method (see, e.g., [1]).

Lemma 2.1 (Schauder Fixed Point Theorem). *Let K be a closed convex subset of a normed linear space E. Then every compact, continuous map $T : K \to K$ has at least one fixed point.*

Let us assume that hypotheses (H1)–(H3) and the Nagumo condition relative to a lower solution α and an upper solution β are satisfied. We start by constructing a modified BVP equivalent to our problem (P).

First, by Definition 2.2, we can find two real numbers $M_- < 0 < M_+$ such that

$$M_- < -\eta \le \eta < M_+, \qquad M_- < \alpha''(t), \qquad \beta''(t) < M_+ \qquad \text{for all } t \in I \tag{5}$$

and

$$\begin{aligned} \int_{\phi(\eta)}^{\phi(M_+)} \frac{|\phi^{-1}(s)|^{(p-1)/p}}{\theta(|\phi^{-1}(s)|)} ds > \mu^{(p-1)/p} \|k\|_p, \\ \int_{\phi(M_-)}^{\phi(-\eta)} \frac{|\phi^{-1}(s)|^{(p-1)/p}}{\theta(|\phi^{-1}(s)|)} ds > \mu^{(p-1)/p} \|k\|_p. \end{aligned} \tag{6}$$

Second, we define

$$\begin{aligned} \delta_1(t,x) &= \max\{\alpha(t), \min\{x, \beta(t)\}\}, \\ \delta_2(t,x) &= \max\{\alpha'(t), \min\{x, \beta'(t)\}\}, \\ \delta_3(x) &= \max\{M_-, \min\{x, M_+\}\}. \end{aligned}$$

Then $\delta_i(t,x)$ $(i = 1, 2)$ is continuous on $I \times \mathbb{R}$ and $\delta_3(x)$ is continuous on $\mathbb{R}$.

Lemma 2.2. *For any $u \in C^2(I)$, the following two properties hold:*

(i) $\frac{d}{dt}\delta_2(t, u'(t))$ *exists for a.e.* $t \in I$;

(ii) *if* $u_0, u_j \in C^2(I)$ *and* $u_j \to u_0$ *in* $C^2(I)$, *then*

$$\frac{d}{dt}\delta_2(t, u_j'(t)) \to \frac{d}{dt}\delta_2(t, u_0'(t)) \qquad \textit{for a.e. } t \in I.$$

Proof. The proof can be found in [7]. □

Now we consider the following modified version of problem (P)

$$\begin{cases} (\phi(u''))' + F(t,u) = 0 \quad \text{for a.e. } t \in I, \\ u(0) = 0, \\ u'(0) = G(u), \\ u'(1) = H(u), \end{cases} \tag{P*}$$

where $F(t,u) : I \times C^2(I) \to \mathbb{R}$, $G(u) : C^2(I) \to \mathbb{R}$ and $H(u) : C^2(I) \to \mathbb{R}$ are given by

$$F(t,u) = f\left(t, \delta_1(t,u), \delta_2(t,u'), \delta_3\left(\frac{d}{dt}\delta_2(t,u')\right)\right) - \tanh(u' - \delta_2(t,u')), \tag{7}$$

$$G(u) = A - g(\delta_2(0,u'(0)), \delta_3(u''(0))) + \delta_2(0,u'(0)), \tag{8}$$

$$H(u) = B - h(\delta_2(1,u'(1)), \delta_3(u''(1))) + \delta_2(0,u'(1)). \tag{9}$$

For problem (P*), the following two lemmas hold, from which we can conclude that every solution of (P*) in the sector

$$[\alpha, \beta] := \{u \in C^2(I) \,|\, \alpha(t) \le u(t) \le \beta(t), \alpha'(t) \le u'(t) \le \beta'(t), t \in I\}$$

is also a solution of (P).

Lemma 2.3. *Assume that* $f : I \times \mathbb{R}^3 \to \mathbb{R}$ *satisfies*

$$f(t, \alpha(t), x_1, x_2) \le f(t, x_0, x_1, x_2) \le f(t, \beta(t), x_1, x_2) \tag{H4}$$

for all $(t, x_1, x_2) \in I \times \mathbb{R}^2$ *and* $\alpha(t) \le x_0 \le \beta(t)$. *Then, for every solution* u *of* (P*) *one has* $u(t) \in [\alpha, \beta]$, *for all* $t \in I$.

Proof. We shall only prove that $u'(t) \le \beta'(t)$, for all $t \in I$. A similar reasoning shows that $\alpha'(t) \le u'(t)$, for all $t \in I$. Since $\alpha(0) \le 0$, $u(0) = 0$ and $\beta(0) \ge 0$, it follows that $\alpha(t) \le u(t) \le \beta(t)$, for all $t \in I$.

Suppose that our assertion was not true. Then there exists $t_1 \in I$ such that

$$u'(t_1) - \beta'(t_1) = \max_{t \in I}[u'(t) - \beta'(t)] > 0, \qquad u'(t) - \beta'(t) < u'(t_1) - \beta'(t_1) \tag{10}$$

for all $t \in [0, t_1)$.

If $t_1 \in (0,1)$, then $u''(t_1) - \beta''(t_1) = 0$, so by the continuity of $u'(t) - \beta'(t)$ at $t = t_1$ there exists $t_2 \in (0, t_1)$ such that $(u-\beta)'(t) > 0$, for all $t \in [t_2, t_1]$. We note that there must exist some $t_0 \in (t_2, t_1)$ for which $(u-\beta)''(t_0) > 0$ (if not, i.e., $(u-\beta)''(t) \le 0$ for all $t \in (t_2, t_1)$, then $(u-\beta)'(t)$ is decreasing on (t_2, t_1), which contradicts (10)). Let us set $\bar{t} = \sup\{t \in [t_0, t_1), (u-\beta)''(s) > 0, s \in (t_0, t)\}$. Then we have

$$(u-\beta)''(\bar{t}) = 0, \qquad (u-\beta)''(t) > 0, \qquad (u-\beta)'(t) > 0, \qquad t \in (t_0, \bar{t}).$$

We conclude that:

$$\phi(u''(t)) \ge \phi(\beta''(t)), \qquad t \in (t_0, \bar{t}), \qquad \text{and} \qquad \delta_2(t, u'(t)) = \beta'(t), \qquad t \in (t_0, \bar{t}).$$

Therefore, by (H4), we find that

$$\begin{aligned}
0 &\ge [\phi(u''(\bar{t})) - \phi(\beta''(\bar{t}))] - [\phi(u''(t_0)) - \phi(\beta''(t_0))] \\
&= \int_{t_0}^{\bar{t}} (\phi(u'') - \phi(\beta''))'(t)\, dt \\
&= \int_{t_0}^{\bar{t}} [(\phi(u''))'(t) - (\phi(\beta''))'(t)]\, dt \\
&\ge \int_{t_0}^{\bar{t}} \Bigg[-f\left(t, \delta_1(t, u(t)), \delta_2(t, u'(t)), \delta_3\left(\frac{d}{dt}\delta_2(t, u'(t))\right)\right) \\
&\qquad + \tanh(u'(t) - \delta_2(t, u'(t))) + f(t, \beta(t), \beta'(t), \beta''(t)) \Bigg] dt \\
&= \int_{t_0}^{\bar{t}} [-f(t, \delta_1(t, u(t)), \beta'(t), \beta''(t)) \\
&\qquad + \tanh(u'(t) - \delta_2(t, u'(t))) + f(t, \beta(t), \beta'(t), \beta''(t))]\, dt \\
&> \int_{t_0}^{\bar{t}} [-f(t, \delta_1(t, u(t)), \beta'(t), \beta''(t)) + f(t, \beta(t), \beta'(t), \beta''(t))]\, dt \ge 0,
\end{aligned}$$

which is a contradiction. Hence, $t_1 \notin (0,1)$.

Now, if $t_1 = 0$ we have:

$$\max_{t \in I}[u'(t) - \beta'(t)] = u'(0) - \beta'(0) > 0.$$

Then $u''(0) - \beta''(0) \le 0$, and it follows that:

$$\delta_3\big(u''(0)\big) \le \beta''(0).$$

Therefore, by (H3):

$$\begin{aligned}\beta'(0) < u'(0) &= A - g\big(\delta_2\big(0, u'(0)\big), \delta_3\big(u''(0)\big)\big) + \delta_2\big(0, u'(0)\big)\\ &\le A - g\big(\beta'(0), \beta''(0)\big) + \beta'(0)\\ &\le \beta'(0),\end{aligned}$$

which is a contradiction. Thus, we must have $t_1 \neq 0$. The case $t_1 = 1$ is discard in a similar fashion.

Altogether, we have shown that $u'(t) \le \beta'(t)$, for all $t \in I$, so Lemma 2.3 holds. □

Lemma 2.4. *If u is a solution of* (P*), *then $M_- \le u''(t) \le M_+$ for all $t \in I$. Here M_- and M_+ denote the Nagumo constants given by* (5) *and* (6), *and they only depend on α, β, ϕ, θ and k.*

Proof. Let $u \in C^2(I)$ be a solution of (P*). By Lemma 2.3, we have $u \in [\alpha, \beta]$, so that

$$-\big(\phi(u'')\big)' = F(t, u) = f\big(t, u, u', \delta_3(u'')\big) \qquad \text{for a.e. } t \in I.$$

Also, by the mean-value theorem, there exists $t_0 \in (0, 1)$ such that

$$u''(t_0) = u'(1) - u'(0).$$

Then

$$M_- < -\eta \le \alpha'(1) - \beta'(0) \le u''(t_0) \le \beta'(1) - \alpha'(0) \le \eta < M_+.$$

Let us set $\eta_0 = |u''(t_0)|$ and suppose that the conclusion of Lemma 2.4 is not true. Then, there must exist $\bar{t} \in I$ such that $u''(\bar{t}) > M_+$ or $u''(\bar{t}) < M_-$. By the continuity of u'' we can choose $t_1, t_2 \in I$ satisfying one of the following situations:

(i) $u''(t_2) = \eta_0$, $u''(t_1) = M_+$ and $\eta_0 \le u''(t) \le M_+$ for all $t \in (t_2, t_1)$;

(ii) $u''(t_1) = M_+$, $u''(t_2) = \eta_0$ and $\eta_0 \le u''(t) \le M_+$ for all $t \in (t_1, t_2)$;

(iii) $u''(t_2) = -\eta_0$, $u''(t_1) = M_-$ and $M_- \le u''(t) \le -\eta_0$ for all $t \in (t_2, t_1)$;

(iv) $u''(t_1) = M_-$, $u''(t_2) = -\eta_0$ and $M_- \le u''(t) \le -\eta_0$ for all $t \in (t_1, t_2)$.

Assume that (i) holds (the other cases can be excluded by similar arguments). Since $M_- \le \eta_0 \le u''(t) \le M_+$ for all $t \in (t_2, t_1)$, we have

$$-\big(\phi(u'')\big)' = f(t, u, u', u'') \qquad \text{for a.e. } t \in (t_2, t_1),$$

so, by the Nagumo condition,

$$\big|\big(\phi(u'')\big)'(t)\big| = |f(t, u, u', u'')| \le k(t)\theta(|u''|) \qquad \text{for a.e. } t \in (t_2, t_1).$$

Note that $\phi^{-1}(s) \ge 0$ for $s \in [\phi(\eta_0), \phi(M_+)]$. On the other hand, we have $\eta_0 \le \eta$ and thus $\phi(\eta_0) \le \phi(\eta)$, which leads us to

$$\int_{\phi(\eta_0)}^{\phi(M_+)} \frac{\big(\phi^{-1}(s)\big)^{(p-1)/p}}{\theta\big(\phi^{-1}(s)\big)}\, du \ge \int_{\phi(\eta)}^{\phi(M_+)} \frac{\big(\phi^{-1}(s)\big)^{(p-1)/p}}{\theta\big(\phi^{-1}(s)\big)}\, du > \mu^{(p-1)/p}\|k\|_p. \tag{11}$$

Consider now the function $\varphi : [t_2, t_1] \to [\phi(\eta_0), \phi(M_+)]$ defined by

$$\varphi(r) = \phi\big(u''(r)\big) \qquad \text{for } r \in [t_2, t_1].$$

By the very definition of a solution, φ is an absolutely continuous function. After a convenient change of variable, and applying assumption (H2), we find

$$\begin{aligned}
\int_{\phi(\eta_0)}^{\phi(M_+)} \frac{\big(\phi^{-1}(s)\big)^{(p-1)/p}}{\theta\big(\phi^{-1}(s)\big)}\, du &= \int_{t_2}^{t_1} \frac{\big(u''(s)\big)^{(p-1)/p}\big(\phi(u'')\big)'(s)}{\theta(u'')(s)}\, ds \\
&= \int_{t_2}^{t_1} \frac{\big(u''(s)\big)^{(p-1)/p}}{\theta(u'')(s)} \big[-f\big(s, u(s), u'(s), u''(s)\big)\big]\, ds \\
&\le \int_{t_2}^{t_1} k(s)\big(u''(s)\big)^{(p-1)/p}\, ds.
\end{aligned}$$

By Hölder's inequality,

$$\int_{\phi(\eta_0)}^{\phi(M_+)} \frac{\big(\phi^{-1}(s)\big)^{(p-1)/p}}{\theta\big(\phi^{-1}(s)\big)}\, du \le \|k\|_p \mu^{(p-1)/p},$$

which contradicts (11). Thus Lemma 2.4 holds. □

Now we can prove our main result.

Theorem 2.1. *Let α be a lower solution and β be an upper solution for problem* (P) *with $\alpha(t) \le \beta(t)$ and $\alpha'(t) \le \beta'(t)$ for all $t \in I$. Assume further that hypotheses* (H1)–(H4) *are satisfied and that the Nagumo condition relative to α and β holds. Then* (P) *has at least one solution*

$$u \in \{u \in C^2(I) \mid \alpha(t) \le u(t) \le \beta(t), \alpha'(t) \le u'(t) \le \beta'(t), t \in I\}$$

that satisfies $M_- < u'' < M_+$ for all $t \in I$, where M_- and M_+ are two constants depending only on α, β, ϕ, θ and k.

Proof. By Lemma 2.3 and Lemma 2.4, the proof will be completed once we have shown that (P*) admits a solution. In what follows, we will show that (P*) has a solution.

Step 1. For any fixed $u \in C^2(I)$, define $\xi_u(x) : \mathbb{R} \to \mathbb{R}$ as

$$\xi_u(x) = \int_0^1 \phi^{-1}\Big(x - \int_0^s F\big(r, u(r)\big)\, dr\Big)\, ds - \big(H(u) - G(u)\big), \qquad x \in \mathbb{R}, \tag{12}$$

where $F(t,u)$, $H(u)$, $G(u)$ are defined as in (7), (8) and (9), respectively. We claim that there exists a unique τ_u such that $\xi_u(\tau_u) = 0$.

Clearly, ξ_u is continuous and strictly increasing on $\mathbb{R}$. By (H2), there exists $\psi \in L^1(I)$ such that

$$\big|F\big(s, u(s)\big)\big| \le \psi(s) \qquad \text{for a.e. } s \in I \text{ and for all } u \in C^2(I). \tag{13}$$

It follows that

$$\Big|\int_0^t F\big(s, u(s)\big)\, ds\Big| \le \|\psi\|_1 \qquad \text{for all } t \in I \text{ and for all } u \in C^2(I),$$

and this implies that

$$\xi_u\big(\|\psi\|_1 + \phi\big(H(u) - G(u)\big)\big) \ge 0, \qquad \xi_u\big(-\|\psi\|_1 + \phi\big(H(u) - G(u)\big)\big) \le 0.$$

Thus, there exists a unique

$$\tau_u \in \big[-\|\psi\|_1 + \phi\big(H(u) - G(u)\big), \|\psi\|_1 + \phi\big(H(u) - G(u)\big)\big]$$

satisfying $\xi_u(\tau_u) = 0$, i.e.,

$$\int_0^1 \phi^{-1}\Big(\tau_u - \int_0^s F\big(r, u(r)\big)\, dr\Big)\, ds = H(u) - G(u). \tag{14}$$

Define the function $\tau : C^2(I) \to \mathbb{R}$ by $\tau(u) = \tau_u$, where τ_u is the unique solution of (14) corresponding to $u \in C^2(I)$. We claim that $\tau : C^2(I) \to \mathbb{R}$ is uniformly bounded and continuous.

In fact, by the very definition of $H(u)$ and $G(u)$, we obtain that $\big(H(u) - G(u)\big)$ is uniformly bounded in $C^2(I)$. Since $\tau_u \in \big[-\|\psi\|_1 + \phi\big(H(u) - G(u)\big), \|\psi\|_1 + \phi\big(H(u) - G(u)\big)\big]$, this implies that $\tau(u)$ is uniformly bounded. Therefore, there exists $L > 0$ such that

$$|\tau_u| \leq L \qquad \text{for all } u \in C^2(I). \tag{15}$$

As for the continuity of $\tau(u)$, suppose that $u_n \to u_0$ is a convergente sequence in $C^2(I)$. Denote τ_n $(n = 0, 1, 2, \ldots)$, the unique solution of (14) corresponding to u_n $(n = 0, 1, 2, \ldots)$. We claim that

$$\lim_{n \to \infty} \tau_n = \tau_0.$$

If this is not the case, and since $\{\tau_n\}$ is uniformly bounded, there exist two subsequences $\{\tau_{n_{k_1}}\}$ and $\{\tau_{n_{k_2}}\}$ with $\tau_{n_{k_1}} \to c_1$ and $\tau_{n_{k_2}} \to c_2$, but $c_1 \neq c_2$. By the definition of τ_n, we have

$$\int_0^1 \phi^{-1}\Big(\tau_{n_{k_1}} - \int_0^s F\big(r, u_{n_{k_1}}(r)\big)\, dr\Big)\, ds = H(u_{n_{k_1}}) - G(u_{n_{k_1}}). \tag{16}$$

Now, using Lemma 2.2, we have that

$$F\big(t, u_n(t)\big) \to F\big(t, u_0(t)\big) \qquad \text{for a.e. } t \in I. \tag{17}$$

Combining (13), (17), (H1), and applying the Lebesgue's dominated convergence theorem to (16), we conclude that:

$$\begin{aligned}
H(u_0) - G(u_0) &= \lim_{n_{k_1} \to \infty} [H(u_{n_{k_1}}) - G(u_{n_{k_1}})] \\
&= \lim_{n_{k_1} \to \infty} \int_0^1 \phi^{-1}\Big(\tau_{n_{k_1}} - \int_0^s F\big(t, u_{n_{k_1}}(r)\big)\, dr\Big)\, ds \\
&= \int_0^1 \phi^{-1}\Big(c_1 - \lim_{n_{k_1} \to \infty} \int_0^s F\big(r, u_{n_{k_1}}(r)\big)\, dr\Big)\, ds \\
&= \int_0^1 \phi^{-1}\Big(c_1 - \int_0^s \lim_{n_{k_1} \to \infty} F\big(r, u_{n_{k_1}}(r)\big)\, dr\Big)\, ds \\
&= \int_0^1 \phi^{-1}\Big(c_1 - \int_0^s F\big(r, u_0(r)\big)\, dr\Big)\, ds.
\end{aligned}$$

Since τ_0 was the unique solution of (14), we conlcude that $c_1 = \tau_0$. Similarly, $c_2 = \tau_0$, so that $c_1 = c_2$ which is a contradiction. Therefore, $\tau_n \to \tau_0$ for any sequence $u_n \to u_0$ in $C^2(I)$, which means that $\tau : C^2(I) \to \mathbb{R}$ is continuous.

Step 2. Let us define $T : C^2(I) \to C^2(I)$ by

$$(Tu)(t) = tG(u) + \int_0^t \left[\int_0^s \phi^{-1}\left(\tau_u - \int_0^r F(\zeta, u(\zeta))\, d\zeta\right) dr\right] ds, \tag{18}$$

where τ_u is the unique solution of (14) corresponding to $u \in C^2(I)$.

If $u \in C^2(I)$ is a fixed point of T, then differentiating (18) we obtain

$$u'(t) = G(u) + \int_0^t \phi^{-1}\left(\tau_u - \int_0^r F(\zeta, u(\zeta))\, d\zeta\right) dr. \tag{19}$$

Differentiating (19) and using the regularity of $F(t, u)$, shows that that $u \in C^2(I)$, $(\phi(u'')) \in AC(I)$ and u satisfies the differential equation of (P*). The fact that u satisfies the boundary conditions of (P*) follows from (14), (18) and (19) easily. Thus, if u is a fixed point of T, then u is a solution of (P*).

We will now prove that T has a fixed point $u \in C^2(I)$ using the Schauder fixed point theorem.

First, we show that the operator T is continuous in $C^2(I)$. Suppose that $u_n \to u_0$ in $C^2(I)$. Since F is a Carathéodory function, by Lemma 2.2, it follows that

$$F(t, u_n(t)) \to F(t, u_0(t)) \qquad \text{in a.e. } t \in I.$$

Hence, by (13), we see that:

$$\lim_{n\to\infty} \int_0^1 \left|F(t, u_n(t)) - F(t, u_0(t))\right| dt = 0. \tag{20}$$

On the other hand, we have already proved in Step 1 that:

$$\lim_{n\to\infty} \tau_n = \tau_0. \tag{21}$$

Equations (20) and (21) together, tell us that

$$\tau_n - \int_0^t F(s, u_n(s))\, ds \to \tau_0 - \int_0^t F(s, u_0(s))\, ds$$

uniformly on I. Thus, by the uniform continuity of ϕ^{-1}, we conclude that:

$$Tu_n \to Tu \text{ uniformly on } I,$$
$$(Tu_n)' \to (Tu)' \text{ uniformly on } I,$$
$$(Tu_n)'' \to (Tu)'' \text{ uniformly on } I,$$

and hence $T : C^2(I) \to C^2(I)$ is continuous.

Second, we show that $T\big(C^2(I)\big)$ is a relatively compact set in $C^2(I)$. Using (13), (15) and (H1), together with the expression of Tu, we have that

$$\begin{aligned} |(Tu)(t)| &\le Q \qquad \text{for all } t \in I \text{ and all } u \in C^2(I), \\ |(Tu)'(t)| &\le Q \qquad \text{for all } t \in I \text{ and all } u \in C^2(I), \\ |(Tu)''(t)| &\le \phi^{-1}(L + \|\psi\|_1) \qquad \text{for all } t \in I \text{ and all } u \in C^2(I), \end{aligned} \tag{22}$$

where

$$Q = |A| + \max_{\substack{|x| \le \max\{|\alpha'(0)|, |\beta'(0)|\} \\ |y| \le \{|M_-|, |M_+|\}}} |g(x, y)| + \max\{|\alpha'(0)|, |\beta'(0)|\} + \phi^{-1}(L + \|\psi\|_1).$$

Inequalities (22) show that $T\big(C^2(I)\big), \big(T\big(C^2(I)\big)\big)' = \big\{ y' \mid y \in T\big(C^2(I)\big)\big\}$ and $\big(T\big(C^2(I)\big)\big)'' = \big\{ y'' \mid y \in T\big(C^2(I)\big)\big\}$ are uniformly bounded on I. It follows from (22) that $T\big(C^2(I)\big)$ and $\big(T\big(C^2(I)\big)\big)'$ are two equicontinuous subsets of $C^0(I)$. To see that $\big(T\big(C^2(I)\big)\big)''$ is an equicontinuous subset of $C^0(I)$, we use (13): for any $u \in C^2(I)$, we find

$$\big|\big(\phi\big((Tu)''\big)\big)'(t)\big| = \big|-F\big(t, u(t)\big)\big| \le \psi(t) \in L^1(I).$$

This readily implies that that $\phi\big(\big(T\big(C^2(I)\big)\big)''\big)$ is an equicontinuous subset of $C^0(I)$. Thus, $\big(T\big(C^2(I)\big)\big)''$ is an equicontinuous subset of $C^0(I)$. We can now invoke the Ascoli–Arzela Theorem to deduce that $T\big(C^2(I)\big)$ is a relatively compact set in $C^2(I)$, as claimed.

We have verify the conditions of the Schauder Fixed Point Theorem, so we can conclude that the operator T has at least a fixed point $u \in C^2(I)$. This means that (P*) has at least one solution $u \in C^2(I)$.

Step 3. Let $u \in C^2(I)$ be a solution of (P*). We claim that u is also a solution of (P).

By Lemma 2.3,

$$\alpha(t) \le u(t) \le \beta(t) \qquad \text{and} \qquad \alpha'(t) \le u'(t) \le \beta'(t) \qquad \text{for any } t \in I, \tag{23}$$

and, by Lemma 2.4,

$$M_- \le u''(t) \le M_+ \qquad \text{for any } t \in I. \tag{24}$$

Thus, we see that

$$\begin{aligned} F(t,u) &= f\big(t, \delta_1(t,u), \delta_2(t,u')\big), \delta_3\left(\frac{d}{dt}\delta_2(t,u')\right) + \tanh\big(u' - \delta_2(t,u')\big) \\ &= f\big(t, u(t), u'(t), u''(t)\big) \end{aligned}$$

and

$$\begin{aligned} u'(0) &= A - g\big(\delta_2\big(0, u'(0)\big), \delta_3\big(u''(0)\big)\big) + \delta_2\big(0, u'(0)\big) \\ &= A - g\big(u'(0), u''(0)\big) + u'(0). \end{aligned}$$

This shows that $g\big(u'(0), u''(0)\big) = A$. Also,

$$u'(1) = B - h\big(\delta_2\big(1, u'(1)\big), \delta_3\big(u''(1)\big)\big) + \delta_2\big(0, u'(1)\big) = B - h\big(u'(1), u''(1)\big) + u'(1),$$

so we also have that $h\big(u'(1), u''(1)\big) = B$. Therefore, we have

$$\begin{cases} \big(\phi_p(u'')\big)'(t) + f\big(t, u(t), u'(t), u''(t)\big) = 0, \quad 0 < t < 1, \\ u(0) = 0, \ g\big(u'(0), u''(0)\big) = A, \ h\big(u'(1), u''(1)\big) = B, \end{cases}$$

which means that $u \in C^2(I)$ is a solution of (P). Moreover, u is in $C^2(I)$ and satisfies (23) and (24). □

Remark 2.1. Comparing our Theorem 2.1 with the results in [6], we note that our result is more general since:

(i) we allow f to be Carathéodory function, rather than just a continuous function,

(ii) we consider nonlinear differential equation, instead of a linear differential equation, and

(iii) we obtain that the second derivative of the solution for (P) is bounded.

3. An example

Consider the following third-order boundary value problem:

$$\begin{cases} (|u''|^{-1/2}u'')' - \frac{1}{t^{1/4}}[(t-u)^2 + t(4+t^2)u' + (u')^2 \sin(u'')] = 0, \\ x(0) = 0, \ 5\big(x'(0)\big)^2 - \frac{1}{2}x''(0) = 5, \ \big(x'(1)\big)^2 + \big(x''(1)\big)^3 = 1. \end{cases} \tag{25}$$

In order to apply our Theorem 2.1, let us set

$$\phi(s) = \begin{cases} 0, & s = 0, \\ |s|^{-1/2}s, & s \neq 0; \end{cases}$$

$$f(t,x,y,z) = -\frac{1}{t^{1/4}}[(t-x)^2 + t(4+t^2)y + y^2\sin(z)],$$

$$g(y,z) = 5y^2 - \frac{1}{2}z,$$

$$h(y,z) = y^2 + z^3.$$

One cheack easily that $\alpha(t) = -t$ and $\beta(t) = t$ are, respectively, lower and upper solutions of (25). The function $\phi : \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing and satisfies: $\phi(\mathbb{R}) = \mathbb{R}$ and $\phi(0) = 0$. The function $f(t,x,y,z) : I \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function and increasing in x, for $-t \leq x \leq t$. The functions $g(y,z)$, $h(y,z)$ are continuous on $\mathbb{R}^2$, $g(y,z)$ is non-increasing in z and $h(y,z)$ is non-decreasing in z.

Finally, we show that f satisfies Nagumo condition relative to the pair $-t$ and t in Ω, where

$$\Omega = \{(t,x,y,z) \in I \times \mathbb{R}^3 \mid -t \leq x \leq t, -1 \leq y \leq 1, z \in \mathbb{R}\}.$$

In fact, for $(t,x,y,z) \in \Omega$, we have

$$\|f(t,x,y,z)\| \leq \frac{10}{t^{1/4}} = k(t) \in L^2(I),$$

We also find

$$\int_{\phi(\eta)}^{\infty} \frac{|\phi^{-1}(u)|^{(p-1)/p}}{\theta(|\phi^{-1}(u)|)} du = \int_{-\infty}^{\phi(-\eta)} \frac{|\phi^{-1}(u)|^{(p-1)/p}}{\theta(|\phi^{-1}(u)|)} du$$
$$= \int_{\sqrt{2}}^{\infty} u\,du = +\infty > \mu^{(p-1)/p}\|k\|_p = 20.$$

If we choose M_- and M_+ as any constants such that $M_- < -42$ and $M_+ > 42$, then all conditions in Definition 2.2 are satisfied.

Therefore, by Theorem 2.1, the BVP (25) has at least one solution $u(t) \in C^2[0,1]$ with

$$-t \leq u(t) \leq t, \quad -1 \leq u'(t) \leq 1, \quad M_- \leq u''(t) \leq M_+, \quad t \in I.$$

Remark 3.1. In this example, the assumptions in [6] or [8] are not satisfied, so the existence results in those works cannot be applied to this BVP.

Acknowledgement. The authors thank the referee for his/her careful review.

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Received June 11, 2007; revised October 28, 2007

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