Portugal. Math. (N.S.)
Vol. 65, Fasc. 4, 2008, 447–484

Portugaliae Mathematica

Quadratic forms for the Liouville equation $w_{tt} + \lambda^2 a(t) w = 0$ with applications to Kirchhoff equation

Renato Manfrin

(Communicated by João Paulo Dias)

Abstract. We introduce a set of quadratic forms for the solutions of the Liouville equation $w_{tt} + \lambda^2 a(t) w = 0$. From these forms we derive estimates for the wave equation $u_{tt} - a(t)\Delta u = 0$ and then prove the global solvability for the Kirchhoff equation in suitable classes of not necessarily smooth or small initial data.

Mathematics Subject Classification (2000). 35L70; 35L15.

Keywords. Liouville equation, Kirchhoff equation, global solvability.

1. Introduction

An essential step in the proof of the global solvability of the Cauchy problem for Kirchhoff equation is the study of the behaviour of solutions of the linear wave equation with time depending coefficient,

$$u_{tt} - a(t)\Delta u = 0, \qquad (x,t) \in \mathbb{R}^n \times [0,T), \tag{1.1}$$

where $0 < T \le +\infty$, $a(t)$ is strictly positive and sufficiently regular. Transforming (1.1) into an ODE for the partial Fourier transform of u with respect to the variable $x \in \mathbb{R}^n$, we are led to consider the Liouville equation

$$w_{tt} + \lambda^2 a(t) w = 0 \qquad \text{in } [0,T), \tag{1.2}$$

with a parameter $\lambda > 0$. Here, introducing a suitable set of quadratic forms, we estimate the solutions of (1.2). Then we derive estimates for (1.1) and, finally, prove the global solvability of the Kirchhoff equation in suitable classes of non-smooth initial data. We assume that in $[0,T)$

$$a(t) > 0, \qquad a(t) \in C^k \qquad \text{with } 2 \le k \in \mathbb{Z}. \tag{1.3}$$

Then, for $\lambda > 0$ and $w(\lambda, t)$ a complex-valued solution of (1.2), we consider the quadratic forms with time dependent coefficients

$$\begin{aligned}\mathcal{Q}_k(\lambda, t) := & \sum_{0 \le i \le [k/2]-1} \alpha_i(t)\lambda^{-2i}\big(a(t)\lambda^2|w|^2 + |w_t|^2\big) \\ & + \sum_{0 \le i \le [k/2]-1} \beta_i(t)\lambda^{-2i}\,\mathrm{Re}(\overline{w}w_t) + \sum_{0 \le i < k/2-1} \gamma_i(t)\lambda^{-2i-2}|w_t|^2, \end{aligned} \tag{1.4}$$

where $\alpha_i(t)$, $\beta_i(t)$, $\gamma_i(t)$ are real-valued functions on $[0, T)$ satisfying the set of *linear* conditions

$$\gamma_{-1} \equiv 0, \qquad \begin{cases} (a\alpha_i)' - a\beta_i = 0 \\ \alpha_i' + \beta_i = -\gamma_{i-1}', \\ \beta_i' - 2a\gamma_i = 0, \end{cases} \tag{1.5}$$

$0 \le i \le [k/2] - 1$. Denoting with $\{\alpha_i, \beta_i, \gamma_i\}_{0 \le i \le [k/2]-1}$ a generic solution

$$\alpha_0, \beta_0, \gamma_0, \ldots, \alpha_{[k/2]-1}, \beta_{[k/2]-1}, \gamma_{[k/2]-1} \tag{1.6}$$

of system (1.5), we have:

Theorem 1.1. *Assume that $a(t)$ satisfies* (1.3). *Then the following holds:*

(1) *System* (1.5) *is solvable in* $[0, T)$. *If $\{\alpha_i, \beta_i, \gamma_i\}_{0 \le i \le [k/2]-1}$ is a solution, then $\alpha_i(t) \in C^{k-2i}$, $\beta_i(t) \in C^{k-2i-1}$, $\gamma_i(t) \in C^{k-2i-2}$ for $0 \le i \le \left[\frac{k}{2}\right] - 1$.*

(2) *Let $\{\alpha_i, \beta_i, \gamma_i\}_{0 \le i \le [k/2]-1}$ be any solution of* (1.5). *Then, for $\lambda > 0$, we have*

$$\frac{d}{dt}\mathcal{Q}_k(\lambda, t) = \begin{cases} \beta_{[k/2]-1}'\lambda^{-k+2}\,\mathrm{Re}(\overline{w}w_t) & \text{if } k \text{ even}, \\ \gamma_{[k/2]-1}'\lambda^{-k+1}|w_t|^2 & \text{if } k \text{ odd}, \end{cases} \tag{1.7}$$

for every complex-valued solution $w(\lambda, t)$ of (1.2).

To describe the structure of these coefficients, we introduce the function

$$\omega(t) := \frac{1}{2\sqrt{a(t)}}. \tag{1.8}$$

Definition 1.1. Given $j, l \ge 0$ integers, we denote by $\mathcal{P}_l^j$ the set of the polynomials P in the variables $y_0, \ldots, y_j$ of the form

$$P(y_0, \ldots, y_j) = \sum c_{\eta_0, \ldots, \eta_j} y_0^{\eta_0}(y_1)^{\eta_1} \ldots (y_j)^{\eta_j}, \tag{1.9}$$

with $c_{\eta_0,\dots,\eta_j} \in \mathbb{R}$ and $\eta_0, \dots, \eta_j \geq 0$ integers such that $\sum_{0\leq h\leq j} \eta_h \leq l$ and $\sum_{0\leq h\leq j} h\eta_h \leq j$. Besides, we say that $P \in \mathscr{H}_l^j$ if each monomial of P satisfies $\sum_{0\leq h\leq j} \eta_h = l$ and $\sum_{0\leq h\leq j} h\eta_h = j$. Given $P \in \mathscr{P}_l^j$ (or $\mathscr{H}_l^j$), defined as in (1.9), and $\varphi(t) \in C^k$ in $[0, T)$ with $k \geq j$, let $P\varphi = \sum c_{\eta_0,\dots,\eta_j} \varphi^{\eta_0} (\varphi^{(1)})^{\eta_1} \dots (\varphi^{(j)})^{\eta_j}$. Finally, given an integer $m \geq 1$, we denote by $P^{(m)}$ the unique polynomial $P^{(m)} \in \mathscr{P}_l^{j+m}$ (or $\mathscr{H}_l^{j+m}$) such that

$$P^{(m)}\varphi = \frac{d^m}{dt^m}(P\varphi) \qquad \text{for all } \varphi \in C^{j+m}.$$

Then, assuming (1.3), we prove that:

Theorem 1.2. *There exist polynomials* $P_i \in \mathscr{H}_{2i}^{2i}$, $Q_i \in \mathscr{H}_{2i+1}^{2i+1}$ *for* $0 \leq i \leq \left[\frac{k}{2}\right] - 1$, *with* $P_0 \equiv 1$, *such that* $\{\alpha_i, \beta_i, \gamma_i\}_{0\leq i\leq[k/2]-1}$ *satisfies* (1.5) *if and only if*

$$\alpha_i = \omega \sum_{h=0}^{i} c_{i-h} P_h \omega, \qquad \beta_i = \sum_{h=0}^{i} c_{i-h} Q_h \omega, \qquad \gamma_i = 2\omega^2 \sum_{h=0}^{i} c_{i-h} Q_h^{(1)} \omega \tag{1.10}$$

for $0 \leq i \leq \left[\frac{k}{2}\right] - 1$ *with* $c_0, \dots, c_{[k/2]-1} \in \mathbb{R}$. *By direct inspection of the proof, it follows that the polynomials* P_i, Q_i *are independent of* k *and* ω.

From (1.10) we get $\alpha_0(t) = c_0\omega(t)$ where $c_0 \in \mathbb{R}$ is an arbitrary constant. Thus, taking $c_0 = 2$, the first term of $\mathscr{Q}_k(\lambda, t)$ is the energy-function

$$\mathscr{E}(\lambda, t) := \lambda^2 \sqrt{a(t)} |w|^2 + \frac{|w_t|^2}{\sqrt{a(t)}}. \tag{1.11}$$

Using Theorems 1.1 and 1.2, we can estimate $\mathscr{E}(\lambda, t)$.

Definition 1.2. For $1 \leq j \leq k$ and $0 \leq T' < T$ put

$$\Phi_j(T') := \max_{1\leq h\leq j} \max_{0\leq t\leq T'} \omega(t)^{1-1/h} |\omega^{(h)}(t)|^{1/h}. \tag{1.12}$$

Theorem 1.3. *Assume that* $a(t)$ *satisfies* (1.3). *Then for all* $C > 1$ *and* $0 \leq T' < T$ *there exists* $\Lambda = \Lambda(k, C) \geq 1$ *such that for* $\lambda \geq \Lambda\Phi_{k-1}(T')$ *we have*

$$\mathscr{E}(\lambda, t) \leq C\mathscr{E}(\lambda, 0) \exp\left\{\lambda^{-k+1} \int_0^t \Psi_k(\tau)\, d\tau\right\} \qquad \textit{for all } t \in [0, T'], \tag{1.13}$$

where $\Psi_k = 2|Q_{[k/2]-1}^{(1)}\omega|$ *for* k *even;* $\Psi_k = 4\omega^{-1}|(\omega^2 Q_{[k/2]-1}^{(1)}\omega)'|$ *for* k *odd. By inspection of the proof, it follows that* $\Psi_k(t) \leq C_k \omega(t)^{-1} \psi(t)^k$ *where* $\psi(t) = \max_{1\leq h\leq k} \omega(t)^{1-1/h} |\omega^{(h)}(t)|^{1/h}$ *and* $C_k > 0$ *is a constant independent of* $\omega(t)$.

Remark 1.1. Estimate (1.13) improves a similar result obtained by Hirosawa in Corollary 1.1 of [7], using a different method. Namely, assuming that $a(t) \in C^k$, $a_0 \le a(t) \le a_1$ for some $a_0, a_1 > 0$, in [7] the energy $\mathscr{E}(\lambda, t)$ was estimated by reducing the Liouville equation (1.2) to a first order system and then applying a refined diagonalization procedure to this system.

Considering the partial Fourier transform $\hat{u}_{tt} + |\xi|^2 a(t)\hat{u} = 0$ of the wave equation (1.1) and setting $E_\rho(t) := \int_{|\xi|>\rho} \big(a(t)^{1/2}|\xi|^2|\hat{u}|^2 + a(t)^{-1/2}|\hat{u}_t|^2\big)\, d\xi$ for $\rho \ge 0$, we can readily derive the following from Theorem 1.3:

Corollary 1.1. *Assume that* (1.3) *holds. Let* $u \in C^h\big([0,T]; H^{1-h}(\mathbb{R}^n)\big)$ $(h = 0, 1)$ *be a solution of* (1.1). *Then for all* $C > 1$ *and* $0 \le T' < T$ *there exists* $\Lambda = \Lambda(k, C) \ge 1$ *such that for* $\rho \ge \Lambda\Phi_{k-1}(T')$, *we have*

$$E_\rho(t) \le CE_\rho(0)\exp\Big\{\rho^{-k+1}\int_0^t \Psi_k(\tau)\, d\tau\Big\} \quad \textit{for all } t \in [0, T']. \tag{1.14}$$

Finally, we apply the quadratic forms (1.4) to the Kirchhoff equation proving the global solvability of the Cauchy problem:

$$u_{tt} - m\Big(\int |\nabla_x u|^2\, dx\Big)\Delta u = 0, \qquad (x, t) \in \mathbb{R}^n \times [0, \infty), \tag{1.15}$$

$$u(0, x) = u_0(x), \qquad u_t(0, x) = u_1(x). \tag{1.16}$$

Here $m(s)$ is a strictly positive, sufficiently regular function in $[0, \infty)$ and the initial data u_0, u_1 are taken in suitable classes of non-analytic functions. In this way, we extend the class of initial data for the global solvability of (1.15), (1.16) from the analytic classes (see [3], [4], [5], [16]) to suitable star-shaped subsets of $H^s \times H^{s-1}$ for $s \ge \frac{3}{2}$. More precisely, taking account of [8], [9], we consider here the following classes of initial data.

Definition 1.3. Given $k \ge 1$ and $u_0, u_1 \in L^2(\mathbb{R}^n)$, we say that $(u_0, u_1) \in B^k_\Delta$ if there exist $\eta > 0$ and a sequence $\{\rho_j\}_{j\ge1}$ such that $\rho_j > 0$, $\lim_{j\to\infty} \rho_j = +\infty$ and

$$\sup_j \int_{|\xi|>\rho_j} [|\xi|^{k+2}|\hat{u}_0(\xi)|^2 + |\xi|^k|\hat{u}_1(\xi)|^2]\frac{e^{\eta\rho_j^k/|\xi|^{k-1}}}{\rho_j^k}\, d\xi < +\infty. \tag{1.17}$$

Besides, we say that $(u_0, u_1) \in \tilde{B}^1_\Delta$ if for all $N \ge 0$ there exists a sequence of positive numbers $\{\rho_j(N)\}_{j\ge1}$, $\rho_j(N) \to \infty$, such that

$$\sup_j e^{N\rho_j(N)} \int_{|\xi|>\rho_j(N)} [|\xi|^3|\hat{u}_0(\xi)|^2 + |\xi|\, |\hat{u}_1(\xi)|^2]\, d\xi < +\infty. \tag{1.18}$$

Theorem 1.4. *Let $m(s)$ be of class C^k in $[0, +\infty)$, with $k \geq 1$ integer. Besides, assume that*

$$\text{either} \quad m(s) \geq \nu > 0 \quad \text{or} \quad m(s) > 0, \quad \int_0^\infty m(s)\,ds = +\infty. \tag{1.19}$$

Then the following holds:

(1) *If $k = 1$, then the Cauchy problem* (1.15), (1.16) *has a unique global solution* $u(x,t) \in C^h([0,+\infty); H^{3/2-h}(\mathbb{R}^n))$ $(h = 0, 1)$ *for all* $(u_0, u_1) \in \tilde{B}^1_\Delta$.

(2) *If $k \geq 2$, then the Cauchy problem* (1.15), (1.16) *has a unique global solution* $u(x,t) \in C^h([0,+\infty); H^{1+k/2-h}(\mathbb{R}^n))$ $(h = 0, 1)$ *for all* $(u_0, u_1) \in B^k_\Delta$.

Remark 1.2. Theorem 1.4 was already proved in [9] for $k = 2, 3$. Afterwards, the global solvability was proved in [7] for $k \geq 2$, $m(s) = 1 + s$ and $(u_0, u_1) \in B^k_\Delta$, by applying the already mentioned diagonalization procedure to the first order system derived from the linearized equation $u_{tt} - a(t)\Delta u = 0$.

Remark 1.3. When $m(s) = (a + bs)^{-2}$ with $a, b > 0$, Pohožaev [17] proved the global solvability of (1.15), (1.16) for all $(u_0, u_1) \in H^2 \times H^1$. In this case he succeeded to find a second order conservation law. See also [15]. For general $m(s) \in C^1$, $m(s) \geq \delta > 0$, the first result of global solvability was established in one space dimension by Bernstein [4] for real analytic initial data. This result was extended in [16] to the case $n > 1$; in [3], [5], for real analytic data, the global solvability was proved even in the weakly hyperbolic case, i.e., when $m(s) \in C^0$, $m(s) \geq 0$. For small and sufficiently regular (C^∞) data the global solvability was proved by Greenberg and Hu [6]. Finally, let us recall that the global solvability for quasi-analytic data was proved by Nishihara in [11], [12]. □

Remark 1.4. $\tilde{B}^1_\Delta$, B^k_Δ $(k \geq 1)$ do not contain compactly supported functions. It is possible to see this by applying a theorem of Paley and Wiener [13]. In the case $n = 1$, this fact is proved in [8]. Moreover, it is easy to show that for $k \geq 1$ the spaces B^k_Δ satisfy the following properties:

- $B^k_\Delta + B^k_\Delta = H^{1+k/2} \times H^{k/2}$,
- $B^k_\Delta \cap (H^{1+(k+1)/2} \times H^{(k+1)/2}) \subsetneqq B^{k+1}_\Delta$,
- $\mathscr{A}_{L^2} \times \mathscr{A}_{L^2} \subsetneqq B^k_\Delta$;

see [8], [9]. For $\tilde{B}^1_\Delta$ and B^1_Δ we have:

- $\tilde{B}^1_\Delta \subsetneqq B^1_\Delta$, $\tilde{B}^1_\Delta + \tilde{B}^1_\Delta = H^{3/2} \times H^{1/2}$,
- $\mathscr{A}_{L^2} \times \mathscr{A}_{L^2} \not\subset \tilde{B}^1_\Delta$ and $\mathscr{A}_{L^2} \times \mathscr{A}_{L^2} \subsetneqq B^1_\Delta$.

Let us show that $\tilde{B}^1_\Delta + \tilde{B}^1_\Delta = H^{3/2} \times H^{1/2}$. Given an element $(u_0, u_1) \in H^{3/2} \times H^{1/2}$, we take the following sequence: fix $\bar{\rho}_1 = 1$, for $j \geq 1$ we inductively select $\bar{\rho}_{j+1} \geq \bar{\rho}_j + 1$ such that

$$\int_{|\xi| > \bar{\rho}_{j+1}} [|\xi|^3 |\hat{u}_0(\xi)|^2 + |\xi|\,|\hat{u}_1(\xi)|^2] e^{j\bar{\rho}_j}\, d\xi \leq 1. \tag{1.20}$$

Then, considering the characteristic function

$$\chi(\xi) := \begin{cases} 1 & \text{if } \bar{\rho}_{2j} \leq |\xi| \leq \bar{\rho}_{2j+1} \text{ for some } j \geq 1, \\ 0 & \text{otherwise,} \end{cases} \tag{1.21}$$

we define $v_i(x)$, $w_i(x)$ by setting

$$\hat{v}_i(\xi) = \chi(\xi)\hat{u}_i(\xi), \qquad \hat{w}_i(\xi) = \big(1 - \chi(\xi)\big)\hat{u}_i(\xi) \tag{1.22}$$

for $i = 0, 1$. Clearly, we have $(v_0, v_1) + (w_0, w_1) = (u_0, u_1)$. Now using (1.20)–(1.22), it is easy to see that (v_0, v_1) satisfies condition (1.18) of Definition 1.3 for all $N \geq 0$ if we define $\rho_j(N) := \bar{\rho}_{2j+1}$ for $j \geq 1$; (w_0, w_1) satisfies condition (1.18) for all $N \geq 0$ if we take the sequence $\rho_j(N) := \bar{\rho}_{2j}$ for $j \geq 1$.

2. Quadratic forms for Liouville equation

Assume that (1.3) holds with $k \geq 2$ integer. Let $w(\lambda, t)$ be a solution in $[0, T)$ of equation (1.2). Denoting by $\left[\frac{k}{2}\right]$ the greatest integer $\leq \frac{k}{2}$, we define:

Definition 2.1. For $\lambda > 0$ and $0 \leq i \leq \left[\frac{k}{2}\right] - 1$ put

$$\begin{aligned} e_i(\lambda, t) &:= \alpha_i(t)\lambda^{-2i}\big(a(t)\lambda^2|w|^2 + |w_t|^2\big), \\ f_i(\lambda, t) &:= \beta_i(t)\lambda^{-2i}\operatorname{Re}(\overline{w}w_t), \\ g_i(\lambda, t) &:= \gamma_i(t)\lambda^{-2i-2}|w_t|^2, \end{aligned} \tag{2.1}$$

where $\alpha_i(t)$, $\beta_i(t)$, $\gamma_i(t)$ are suitable real-valued functions on $[0, T)$.

To choose α_i, β_i, γ_i, we observe that if $w(\lambda, t)$ is a complex-valued solution of (1.2) and α_i, β_i, γ_i are differentiable, we easily find that

$$\begin{aligned} \frac{d}{dt}(e_i + f_i + g_i)(\lambda, t) = {} & \big[\big(a(t)\alpha_i(t)\big)' - a(t)\beta_i(t)\big]\lambda^{-2i+2}|w|^2 \\ & + [\alpha_i'(t) + \beta_i(t)]\lambda^{-2i}|w_t|^2 \\ & + [\beta_i'(t) - 2a(t)\gamma_i(t)]\lambda^{-2i}\operatorname{Re}(\overline{w}w_t) \\ & + \gamma_i'(t)\lambda^{-2i-2}|w_t|^2. \end{aligned} \tag{2.2}$$

Then, considering the derivative

$$\frac{d}{dt}\sum_{i=0}^{[k/2]-1}(e_i + f_i + g_i), \tag{2.3}$$

we determine α_i, β_i, γ_i by equating successively powers of λ to zero. In detail, we start by requiring that α_0, β_0, γ_0 satisfy the linear system

$$\begin{cases} (a\alpha_0)' - a\beta_0 = 0, \\ \alpha_0' + \beta_0 = 0, \\ \beta_0' - 2a\gamma_0 = 0. \end{cases} \tag{2.4}$$

Solving (2.4), we obtain that $\alpha_0 = \alpha_0(t, c_0)$, $\beta_0 = \beta_0(t, c_0)$, $\gamma_0 = \gamma_0(t, c_0)$ linearly dependent upon an arbitrary constant $c_0 \in \mathbb{R}$. Besides, $\alpha_0 \in C^k$, $\beta_0 \in C^{k-1}$, $\gamma_0 \in C^{k-2}$. When $a(t) \in C^k$ with $k \geq 4$ we continue this process: having determined, for some $1 \leq i \leq \left[\frac{k}{2}\right] - 1$, the functions $\alpha_0(t, c_0), \beta_0(t, c_0), \gamma_0(t, c_0), \dots, \alpha_{i-1}(t, c_0, \dots, c_{i-1}), \beta_{i-1}(c_0, \dots, c_{i-1}), \gamma_{i-1}(t, c_0, \dots, c_{i-1})$ linearly dependent upon i arbitrary constants of integration $c_0, \dots, c_{i-1} \in \mathbb{R}$, we determine the functions α_i, β_i, γ_i by requiring that they satisfy the relations

$$\begin{cases} (a\alpha_i)' - a\beta_i = 0, \\ \alpha_i' + \beta_i = -\gamma_{i-1}', \\ \beta_i' - 2a\gamma_i = 0, \end{cases} \tag{2.5}$$

$1 \leq i \leq [k/2] - 1$. Introducing $\gamma_{i-1} = \gamma_{i-1}(t, c_0, \dots, c_{i-1})$ and solving (2.5), we find the functions $\alpha_i = \alpha_i(t, c_0, \dots, c_i)$, $\beta_i = \beta_i(t, c_0, \dots, c_i)$, $\gamma_i = \gamma_i(t, c_0, \dots, c_i)$ linearly dependent upon the $i+1$ arbitrary constants $c_0, \dots, c_{i-1}$ and $c_i \in \mathbb{R}$. In this way, we recursively obtain the functions α_i, β_i, γ_i for $0 \leq i \leq \left[\frac{k}{2}\right] - 1$. Indeed, it is immediate to observe that if $\gamma_{i-1} \in C^r$ for some integer r, $2 \leq r \leq k$, then $\alpha_i \in C^r$, $\beta_i \in C^{r-1}$ and $\gamma_i \in C^{r-2}$. Thus, starting from $\gamma_0 \in C^{k-2}$, we can recursively define the functions α_i, β_i, γ_i as long as $i \leq \left[\frac{k}{2}\right] - 1$. It turns out that $\alpha_i \in C^{k-2i}$, $\beta_i \in C^{k-2i-1}$, $\gamma_j \in C^{k-2i-2}$ for $0 \leq i \leq \left[\frac{k}{2}\right] - 1$. In particular, if $k \geq 2$ is even, $\gamma_{[k/2]-1}$ is merely continuous; when $k \geq 2$ is odd, $\gamma_{[k/2]-1}$ is continuously differentiable and formula (2.2) holds for $0 \leq i \leq \left[\frac{k}{2}\right] - 1$. This shows that (1) of Theorem 1.1 holds.

Now we can introduce the quadratic forms for the solutions of (1.2). Following the recursive procedure described above, we solve the linear systems (2.4), (2.5) (i.e., system (1.5)) determining the *general* solution

$$(\alpha_i, \beta_i, \gamma_i) = (\alpha_i, \beta_i, \gamma_i)(t, c_0, \dots, c_i) \qquad (0 \leq i \leq [k/2] - 1) \tag{2.6}$$

linearly dependent upon $\left[\frac{k}{2}\right]$ arbitrary constants $c_0, \dots, c_{[k/2]-1} \in \mathbb{R}$.

Definition 2.2. Given a solution $\{\alpha_i, \beta_i, \gamma_i\}_{0 \le i \le [k/2]-1}$ of (1.5), for $\lambda > 0$ we define the quadratic form $\mathcal{Q}_k = \mathcal{Q}_k(\lambda, t, \alpha_i, \beta_i, \gamma_i)$ by setting

$$\mathcal{Q}_k := \sum_{0 \le i \le [k/2]-1} (e_i + f_i) + \sum_{i < k/2 - 1} g_i. \tag{2.7}$$

Remark 2.1. This means that $\mathcal{Q}_2 = e_0 + f_0$, $\mathcal{Q}_k = \sum_{i=0}^{[k/2]-1} (e_i + f_i + g_i)$ for $k \ge 3$ odd, and $\mathcal{Q}_k = \sum_{i=0}^{[k/2]-2} (e_i + f_i + g_i) + e_{[k/2]-1} + f_{[k/2]-1}$ for $k \ge 4$ even.

Using Definition 2.2 and the relations of systems (2.4)–(2.5), we may conclude the proof of Theorem 1.1, i.e., we can show that (2) holds.

Proof of (1.7). Let us prove formula (1.7) for $k \ge 2$ odd. In this case

$$\mathcal{Q}_k = \sum_{i=0}^{[k/2]-1} (e_i + f_i + g_i) \tag{2.8}$$

and α_i, β_i, γ_i, for $0 \le i \le \left[\frac{k}{2}\right] - 1$, are C^1 functions satisfying (2.4)–(2.5). Thus, given $w(\lambda, t)$ a solution of (1.2), we can differentiate $\mathcal{Q}_k(\lambda, t)$ with respect to t. For $k = 3$ we have $\mathcal{Q}_3 = e_0 + f_0 + g_0$ and the statement follows from (2.2) since α_0, β_0, γ_0 satisfy (2.4). Now let us suppose $k > 3$ odd. Since α_i, β_i, γ_i satisfy the systems (2.4) and (2.5), it follows that

$$\begin{aligned} &\frac{d}{dt}(e_0 + f_0 + g_0) = \gamma_0' \lambda^{-2} |w_t|^2, \\ &\frac{d}{dt}(e_i + f_i + g_i) = -\gamma_{i-1}' \lambda^{-2i} |w_t|^2 + \gamma_i' \lambda^{-2i-2} |w_t|^2 \qquad (1 \le i \le [k/2] - 1). \end{aligned} \tag{2.9}$$

Thus, if we sum all the terms in (2.9), we obtain that

$$\frac{d}{dt} \sum_{i=0}^{[k/2]-1} (e_i + f_i + g_i) = \gamma_{[k/2]-1}' \lambda^{-k+1} |w_t|^2 \tag{2.10}$$

because $2\left[\frac{k}{2}\right] = k - 1$ for k odd. The proof of (1.7) for $k \ge 2$ even is similar. □

3. Computation of α_i, β_i, γ_i

Before proving Theorem 1.2 in this section we show that it holds for $k \le 7$ by computing explicitly α_i, β_i, γ_i for $0 \le i \le 2$. In formulae (3.2), (3.6), (3.10) below we also define, implicitly, the polynomials $P_i \in \mathscr{H}_{2i}^{2i}$ and $Q_i \in \mathscr{H}_{2i+1}^{2i+1}$ for $0 \le i \le 2$. Finally, we give the general recursive relations for α_i, β_i, γ_i.

As we shall see in the sequel, it is convenient to rewrite the systems (2.4), (2.5) using the function $\omega(t)$ defined in (1.8). Then system (2.4) becomes

$$\begin{cases} \left(\dfrac{\alpha_0}{\omega}\right)' = 0, \\ \beta_0 = -\alpha_0', \\ \gamma_0 = 2\omega^2\beta_0'. \end{cases} \tag{3.1}$$

Since $\omega \in C^k$, with $k \geq 2$, we can immediately write the *general* solution:

$$\begin{aligned} \alpha_0 &= c_0\omega := \omega c_0 P_0 \omega, \\ \beta_0 &= -c_0\omega' := c_0 Q_0 \omega, \\ \gamma_0 &= -2c_0\omega^2\omega'' = 2\omega^2 c_0 Q_0^{(1)}\omega, \end{aligned} \tag{3.2}$$

where $c_0 \in \mathbb{R}$ is an arbitrary constant. Hence $P_0 \equiv 1$, $Q_0\omega = -\omega'$. Next if $k \geq 4$, we compute $\alpha_1, \beta_1, \gamma_1$. By (2.5) $\alpha_1, \beta_1, \gamma_1$ must satisfy the system

$$\begin{cases} \left(\dfrac{\alpha_1}{\omega}\right)' = -\dfrac{\gamma_0'}{2\omega}, \\ \beta_1 = -\gamma_0' - \alpha_1', \\ \gamma_1 = 2\omega^2\beta_1'. \end{cases} \tag{3.3}$$

From the first row of system (3.3) we find that

$$\alpha_1 = -\frac{\omega}{2}\int \frac{\gamma_0'}{\omega}\,dt. \tag{3.4}$$

Then, introducing the expression $\gamma_0 = -2c_0\omega^2\omega''$ into (3.4), we have

$$\alpha_1 = \omega \int c_0\left(2\omega'\omega'' + \omega\omega'''\right)dt = c_1\omega + c_0\omega\left(\frac{\omega'^2}{2} + \omega\omega''\right), \tag{3.5}$$

where $c_1 \in \mathbb{R}$ is arbitrary. Thus, from (3.2)–(3.5) we obtain the *general* solution

$$\begin{aligned} \alpha_1 &= c_1\omega + c_0\omega\left(\frac{\omega'^2}{2} + \omega\omega''\right) := \omega(c_1 P_0\omega + c_0 P_1\omega) \\ \beta_1 &= -\left[c_1\omega + c_0\omega\left(\frac{\omega'^2}{2} - \omega\omega''\right)\right]' := c_1 Q_0\omega + c_0 Q_1\omega \\ \gamma_1 &= 2\omega^2(c_1 Q_0^{(1)}\omega + c_0 Q_1^{(1)}\omega) \end{aligned} \tag{3.6}$$

Finally, if $k \geq 6$, we can also compute α_2, β_2, γ_2. They must satisfy the system

$$\begin{cases} \left(\dfrac{\alpha_2}{\omega}\right)' = -\dfrac{\gamma_1'}{2\omega}, \\ \beta_2 = -\gamma_1' - \alpha_2', \\ \gamma_2 = 2\omega^2\beta_2'. \end{cases} \tag{3.7}$$

Hence

$$\alpha_2 = -\frac{\omega}{2}\int \frac{\gamma_1'}{\omega}\,dt = \omega\int c_1(2\omega'\omega'' + \omega\omega''')\,dt + \frac{\omega}{2}\int \frac{c_0}{\omega}[\omega^2(\omega\omega'^2 - 2\omega^2\omega'')'']'\,dt. \tag{3.8}$$

Now the computation of the second indefinite integral gives

$$\begin{aligned} &\int \omega^{-1}[\omega^2(\omega\omega'^2 - 2\omega^2\omega'')'']'\,dt \\ &\quad = \omega(\omega\omega'^2 - 2\omega^2\omega'')'' + \omega'(\omega\omega'^2 - 2\omega^2\omega'')' - \int(\omega\omega'^2 - 2\omega^2\omega'')'\omega''\,dt \\ &\quad = [\omega(\omega\omega'^2 - 2\omega^2\omega'')']' - \frac{\omega'^4}{4} + \omega^2\omega''^2 + C. \end{aligned} \tag{3.9}$$

Whence, recalling (3.6) we deduce the *general* solution

$$\begin{aligned} \alpha_2 &= \omega(c_2P_0\omega + c_1P_1\omega) + c_0\omega\left\{\left[\omega\left(\frac{\omega\omega'^2}{2} - \omega^2\omega''\right)'\right]' - \frac{\omega'^4}{8} + \frac{\omega^2\omega''^2}{2}\right\} \\ &:= \omega(c_2P_0\omega + c_1P_1\omega + c_0P_2\omega), \\ \beta_2 &= c_2Q_0\omega + c_1Q_1\omega + c_0\left\{\frac{\omega^2}{2}(\omega\omega'^2 - 2\omega^2\omega'')'' \right. \\ &\qquad \left. - \frac{\omega\omega'}{2}(\omega\omega'^2 - 2\omega^2\omega'')' + \frac{\omega\omega'^4}{8} - \frac{\omega^3\omega''^2}{2}\right\}' \\ &:= c_2Q_0\omega + c_1Q_1\omega + c_0Q_2\omega, \\ \gamma_2 &= 2\omega^2(c_2Q_0^{(1)}\omega + c_1Q_1^{(1)}\omega + c_0Q_2^{(1)}\omega), \end{aligned} \tag{3.10}$$

where $c_2 \in \mathbb{R}$ is an arbitrary constant.

To conclude, let us come to the general case. We rewrite systems (2.4) and (2.5) in the equivalent form

$$\gamma_{-1} \equiv 0, \qquad \begin{cases} \left(\dfrac{\alpha_i}{\omega}\right)' = -\dfrac{\gamma'_{i-1}}{2\omega}, \\ \beta_i = -\gamma'_{i-1} - \alpha'_i, \\ \gamma_i = 2\omega^2\beta'_i \end{cases} \tag{3.11}$$

for $0 \le i \le [k/2] - 1$. Assuming that, for some $1 \le i \le \left[\frac{k}{2}\right] - 1$, the *general* solutions

$$(\alpha_0, \beta_0, \gamma_0), \dots, (\alpha_{i-1}, \beta_{i-1}, \gamma_{i-1}),$$

with α_j, β_j, γ_j linearly dependent upon $j+1$ arbitrary constants $c_0, \dots, c_j \in \mathbb{R}$ for $0 \le j \le i-1$, are determined, we can write (with a slight abuse of notation) the general solution of $(\alpha_i, \beta_i, \gamma_i)$ in the form

$$\begin{cases} \alpha_i = -\dfrac{\omega}{2}\displaystyle\int \dfrac{\gamma'_{i-1}}{\omega}\,dt, \\ \beta_i = -\left(\gamma_{i-1} - \dfrac{\omega}{2}\displaystyle\int \dfrac{\gamma'_{i-1}}{\omega}\,dt\right)', \\ \gamma_i = -2\omega^2\left(\gamma_{i-1} - \dfrac{\omega}{2}\displaystyle\int \dfrac{\gamma'_{i-1}}{\omega}\,dt\right)''. \end{cases} \tag{3.12}$$

Namely, given γ_{i-1}, a *particular* solution $(\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i)$ can be obtained by selecting a primitive, say $\tilde{p}_i$, from the indefinite integral $\int \frac{\gamma'_{i-1}}{\omega}\,dt$ and then setting $\tilde{\alpha}_i = -\frac{\omega}{2}\tilde{p}_i$, $\tilde{\beta}_i = -\left(\gamma_{i-1} - \frac{\omega}{2}\tilde{p}_i\right)'$, $\tilde{\gamma}_i = -2\omega^2\left(\gamma_{i-1} - \frac{\omega}{2}\tilde{p}_i\right)''$.

4. Polynomial structure of α_i, β_i, γ_i

To simplify the recursive formula for the coefficients α_i, β_i, γ_i, we set

$$\Gamma_{-1} \equiv 0, \qquad \Gamma_i := \frac{\gamma_i}{2\omega^2}, \qquad 0 \le i \le [k/2] - 1. \tag{4.1}$$

Integrating by parts, we find the identity between indefinite integrals

$$\int \omega \Gamma'_{i-1}\,dt = \frac{\gamma_{i-1}}{\omega} - \frac{1}{2}\int \frac{\gamma'_{i-1}}{\omega}\,dt. \tag{4.2}$$

Then, substituting (4.1), (4.2) in the last equation of (3.12), we find that the functions Γ_i must satisfy the recursive relations

$$\Gamma_{-1} \equiv 0, \qquad \Gamma_i \in -\left(\omega \int \omega \Gamma'_{i-1}\,dt\right)'' \qquad \text{for } 0 \le i \le [k/2] - 1. \tag{4.3}$$

This means that, starting with $\Gamma_{-1} \equiv 0$, every *particular* sequence of coefficients $\{\alpha_i, \beta_i, \gamma_i\}_{0 \le i \le [k/2]-1}$ satisfying (3.11) can be obtained from the recursive relation (4.3) by selecting, at each step, a *particular* primitive

$$p_i \in \int \omega \Gamma'_{i-1}\, dt \tag{4.4}$$

and then setting

$$\Gamma_i = -(\omega p_i)'', \qquad \begin{cases} \alpha_i = \omega p_i - 2\omega^2 \Gamma_{i-1} \\ \beta_i = -(\omega p_i)' \\ \gamma_i = 2\omega^2 \Gamma_i \end{cases} \tag{4.5}$$

for $0 \le i \le \left[\frac{k}{2}\right] - 1$. We claim that the primitives p_i are always polynomials in ω and its derivatives $\omega^{(h)}$ are of order $1 \le h \le 2i$. Before proving this, recalling Definition 1.1, we note the following simple facts about the polynomials of $\mathscr{P}_l^j$.

Remark 4.1. Given $j_a, l_a, j_b, l_b \ge 0$ and $m \ge 1$ integers, it follows that

$$P \in \mathscr{P}_{l_a}^{j_a},\ P\varphi = 0 \text{ for all } \varphi \in C^{j_a} \Leftrightarrow P \equiv 0,$$
$$P \in \mathscr{H}_{l_a}^{j_a} \Rightarrow P^{(m)} \in \mathscr{H}_{l_a}^{j_a+m}.$$

Besides, given $P_a \in \mathscr{P}_{l_a}^{j_a}$, $P_b \in \mathscr{P}_{l_b}^{j_b}$ let $P_a P_b$ be their product as polynomials in the variables $y_0, \dots, y_{j_a \vee j_b}$. Then we can easily see that

$$P_a \in \mathscr{P}_{l_a}^{j_a},\ P_b \in \mathscr{P}_{l_b}^{j_b} \Rightarrow \begin{cases} P_a P_b \in \mathscr{P}_{l_a+l_b}^{j_a+j_b}, \\ (P_a\varphi)(P_b\varphi) = (P_a P_b)\varphi \quad \text{for all } \varphi \in C^{j_a \vee j_b}, \end{cases} \tag{4.6}$$
$$P_a \in \mathscr{H}_{l_a}^{j_a},\ P_b \in \mathscr{H}_{l_b}^{j_b} \Rightarrow P_a P_b \in \mathscr{H}_{l_b+l_b}^{j_a+j_b}.$$

We are now in position to prove:

Lemma 4.1. *There exists a unique sequence of polynomials* $\bar{P}_{-1}, \bar{P}_0, \dots, \bar{P}_{[k/2]-1}$ *such that* $\bar{P}_{-1} \equiv 0$, $\bar{P}_0 \equiv 1$, $\bar{P}_i \in \mathscr{H}_{2i}^{2i}$ *for* $i \ge 0$, *and*

$$\bar{P}_i \varphi \in -\int \varphi (\varphi \bar{P}_{i-1}\varphi)'''\, dt \tag{4.7}$$

for all $\varphi \in C^k$ *and* $0 \le i \le \left[\frac{k}{2}\right] - 1$.

Proof. Since $\bar{P}_{-1}\varphi \equiv 0$, $\bar{P}_0 \varphi \equiv 1$ for all φ, it is clear that relation (4.7) holds for $i = 0$. Besides, for $\varphi \in C^3$ we have

$$-\int \varphi (\varphi \bar{P}_0 \varphi)'''\, dt = -\int \varphi \varphi'''\, dt = -\varphi\varphi'' + \frac{1}{2}(\varphi')^2 + C, \tag{4.8}$$

where $C \in \mathbb{R}$ is an arbitrary constant. Thus, when $k \geq 4$, we must define $\bar{P}_1$ as the unique polynomial of $\mathscr{H}_2^2$ such that

$$\bar{P}_1\varphi = -\varphi\varphi'' + \frac{1}{2}(\varphi')^2 \tag{4.9}$$

for all $\varphi \in C^k$. Clearly, this means that $\bar{P}_1(y_0, y_1, y_2) := -y_0y_2 + \frac{1}{2}y_1^2$.

Having determined $\bar{P}_1 \in \mathscr{H}_2^2$, we now proceed by induction. Fixed an integer i, $1 \leq i \leq \left[\frac{k}{2}\right] - 2$, let us suppose that the polynomials

$$\bar{P} - 1 \equiv 0, \qquad \bar{P}_0 \equiv 1 \in \mathscr{H}_0^0, \qquad \bar{P}_1 \in \mathscr{H}_2^2, \dots, \bar{P}_i \in \mathscr{H}_{2i}^{2i} \tag{4.10}$$

satisfy condition (4.7), i.e., $(\bar{P}_h\varphi)' = -\varphi(\varphi\bar{P}_{h-1}\varphi)'''$ for $0 \leq h \leq i$. Then we want show that there exists a unique $\bar{P}_{i+1} \in \mathscr{H}_{2i+2}^{2i+2}$ such that

$$\bar{P}_{i+1}\varphi \in -\int \varphi(\varphi\bar{P}_i\varphi)'''\,dt \tag{4.11}$$

for all $\varphi \in C^k$. Integrating by parts, we obtain that

$$\begin{aligned} -\int \varphi(\varphi\bar{P}_i\varphi)'''\,dt &= -\varphi(\varphi\bar{P}_i\varphi)'' + \varphi'(\varphi\bar{P}_i\varphi)' - \varphi''(\varphi\bar{P}_i\varphi) \\ &\quad + \int \varphi'''(\varphi\bar{P}_i\varphi)\,dt. \end{aligned} \tag{4.12}$$

By induction, it is easy to see that

$$\begin{aligned} &-\varphi(\varphi\bar{P}_i\varphi)'' + \varphi'(\varphi\bar{P}_i\varphi)' - \varphi''(\varphi\bar{P}_i\varphi) \\ &\quad = -\varphi(\varphi''\bar{P}_i\varphi + 2\varphi'\bar{P}_i^{(1)}\varphi + \varphi\bar{P}_i^{(2)}\varphi) + \varphi'(\varphi'\bar{P}_i\varphi + \varphi\bar{P}_i^{(1)}\varphi) - \varphi''(\varphi\bar{P}_i\varphi) \\ &\quad =: U_{i+1}\varphi \end{aligned} \tag{4.13}$$

for all $\varphi \in C^k$ and for a unique $U_{i+1} \in \mathscr{H}_{2i+2}^{2i+2}$. Thus we find

$$\begin{aligned} -\int \varphi(\varphi\bar{P}_i\varphi)'''\,dt &= U_{i+1}\varphi + \int (\varphi\varphi''')\bar{P}_i\varphi\,dt \\ &= U_{i+1}\varphi - \int (\bar{P}_1\varphi)'\bar{P}_i\varphi\,dt. \end{aligned} \tag{4.14}$$

Now if $i = 1$, then we conclude that, for all $\varphi \in C^k$,

$$-\int \varphi(\varphi\bar{P}_1\varphi)'''\,dt = U_2\varphi - \frac{1}{2}(\bar{P}_1\varphi)^2 + C, \tag{4.15}$$

with $C \in \mathbb{R}$ an arbitrary constant. Since $\bar{P}_1^2 \in \mathscr{H}_4^4$, it is clear that we must take

$$\bar{P}_2 := U_2 - \frac{1}{2}\bar{P}_1^2 \in \mathscr{H}_4^4. \tag{4.16}$$

If $i \geq 2$, we continue integrating by parts. From (4.14) we have

$$\begin{aligned} -\int \varphi(\varphi\bar{P}_i\varphi)''' \, dt &= U_{i+1}\varphi - \bar{P}_1\varphi\bar{P}_i\varphi + \int \bar{P}_1\varphi(\bar{P}_i\varphi)' \, dt \\ &= U_{i+1}\varphi - (\bar{P}_1\bar{P}_i)\varphi - \int \varphi\bar{P}_1\varphi(\varphi\bar{P}_{i-1}\varphi)''' \, dt. \end{aligned} \tag{4.17}$$

Thus, if $i = 2$, we find that

$$-\int \varphi(\varphi\bar{P}_2\varphi)''' \, dt = U_3\varphi - (\bar{P}_1\bar{P}_2)\varphi - \int \varphi\bar{P}_1\varphi(\varphi\bar{P}_1\varphi)''' \, dt \tag{4.18}$$

and

$$-\int \varphi(\varphi\bar{P}_2\varphi)''' \, dt = U_3\varphi - (\bar{P}_1\bar{P}_2)\varphi - \left[\varphi\bar{P}_1\varphi(\varphi\bar{P}_1\varphi)'' - \frac{1}{2}(\varphi\bar{P}_1\varphi)'^2\right] + C, \tag{4.19}$$

where $C \in \mathbb{R}$ is an arbitrary constant. By induction again, it follows that

$$-(\bar{P}_1\bar{P}_2)\varphi - \varphi\bar{P}_1\varphi(\varphi\bar{P}_1\varphi)'' + \frac{1}{2}(\varphi\bar{P}_1\varphi)'^2 = V_3\varphi \tag{4.20}$$

for all $\varphi \in C^k$ and for a unique $V_3 \in \mathscr{H}_6^6$. Hence, we are led to set

$$\bar{P}_3 := U_3 + V_3. \tag{4.21}$$

Otherwise, if $i \geq 3$, we continue integrating by parts the last integral of (4.17). We obtain that

$$\begin{aligned} -\int \varphi(\varphi\bar{P}_i\varphi)''' \, dt = U_{i+1}\varphi &- (\bar{P}_1\bar{P}_i)\varphi - \varphi\bar{P}_1\varphi(\varphi\bar{P}_{i-1}\varphi)'' \\ &+ (\varphi\bar{P}_1\varphi)'(\varphi\bar{P}_{i-1}\varphi)' - (\varphi\bar{P}_1\varphi)''(\varphi\bar{P}_{i-1}\varphi) \\ &+ \int (\varphi\bar{P}_1\varphi)'''(\varphi\bar{P}_{i-1}\varphi) \, dt. \end{aligned} \tag{4.22}$$

As above, we find that

$$\begin{aligned} U_{i+1}\varphi - (\bar{P}_1\bar{P}_i)\varphi - \varphi\bar{P}_1\varphi(\varphi\bar{P}_{i-1}\varphi)'' + (\varphi\bar{P}_1\varphi)'(\varphi\bar{P}_{i-1}\varphi)' \\ - (\varphi\bar{P}_1\varphi)''(\varphi\bar{P}_{i-1}\varphi) = \tilde{U}_{i+1}\varphi \end{aligned} \tag{4.23}$$

for all $\varphi \in C^k$ and for a unique polynomial $\tilde{U}_{i+1} \in \mathscr{H}_{2i+2}^{2i+2}$. Whence, we deduce that

$$\begin{aligned} -\int \varphi(\varphi \bar{P}_i \varphi)''' \, dt &= \tilde{U}_{i+1}\varphi + \int \varphi(\varphi \bar{P}_1 \varphi)''' \bar{P}_{i-1}\varphi \, dt \\ &= \tilde{U}_{i+1}\varphi - \int (\bar{P}_2\varphi)' \bar{P}_{i-1}\varphi \, dt. \end{aligned} \tag{4.24}$$

Thus, if $i = 3$, we can easily obtain $\bar{P}_4$.

Remark 4.2. Now in view of (4.14) and (4.24) it is natural to expect that

$$-\int \varphi(\varphi \bar{P}_i \varphi)''' \, dt = U_{i+1,h}\varphi - \int (\bar{P}_h\varphi)' \bar{P}_{i-h+1}\varphi \, dt \tag{4.25}$$

for $1 \le h \le i$, with $U_{i+1,h} \in \mathscr{H}_{2i+2}^{2i+2}$ a suitable polynomial. To prove this, let us suppose that (4.25) holds for some h with $1 \le h < i - 1$. Then, using (4.7), (4.10) and integrating by parts, we easily find the equalities

$$\begin{aligned} -\int \varphi(\varphi \bar{P}_i \varphi)''' \, dt &= U_{i+1,h}\varphi - \bar{P}_h\varphi \bar{P}_{i-h+1}\varphi + \int \bar{P}_h\varphi(\bar{P}_{i-h+1}\varphi)' \, dt \\ &= U_{i+1,h}\varphi - (\bar{P}_h\bar{P}_{i-h+1})\varphi - \int \varphi \bar{P}_h\varphi(\varphi \bar{P}_{i-h}\varphi)''' \, dt \\ &= U_{i+1,h+1}\varphi + \int (\varphi \bar{P}_h \varphi)''' \varphi \bar{P}_{i-h}\varphi \, dt \\ &= U_{i+1,h+1}\varphi - \int (\bar{P}_{h+1}\varphi)' \bar{P}_{i-h}\varphi \, dt \end{aligned} \tag{4.26}$$

for a suitable polynomial $U_{i+1,h+1} \in \mathscr{H}_{2i+2}^{2i+2}$. Hence, (4.25) holds for $h + 1$.

Taking into account Remark 4.2, by repeated integration by parts, it is clear that we finally obtain:

(1) if $i = 2h - 1$ for some integer $h \ge 1$, then

$$-\int \varphi(\varphi \bar{P}_i \varphi)''' \, dt = U_{i+1,h}\varphi - \int (\bar{P}_h\varphi)' \bar{P}_h\varphi \, dt; \tag{4.27a}$$

(2) if $i = 2h$ for some integer $h \ge 1$, then

$$-\int \varphi(\varphi \bar{P}_i \varphi)''' \, dt = U_{i+1,h}\varphi - \int (\bar{P}_h\varphi)' \bar{P}_{h+1}\varphi \, dt, \tag{4.27b}$$

with $U_{i+1,h} \in \mathscr{H}_{2i+2}^{2i+2}$. In the first case we immediately deduce that

$$-\int \varphi(\varphi \bar{P}_i \varphi)''' \, dt = U_{i+1,h}\varphi - \frac{1}{2}(\bar{P}_h \varphi)^2 + C, \tag{4.28}$$

with $C \in \mathbb{R}$ an arbitrary constant. Since $\bar{P}_h^2 \in \mathscr{H}_{2i+2}^{2i+2}$, by the inductive hypothesis, we are forced to define the polynomial $\bar{P}_{i+1}$ as

$$\bar{P}_{i+1} := U_{i+1,h} - \frac{1}{2}\bar{P}_h^2. \tag{4.29}$$

In the second case, integrating by parts once again, from (4.27b) we have

$$\begin{aligned} -\int \varphi(\varphi \bar{P}_i \varphi)''' \, dt &= U_{i+1,h}\varphi - (\bar{P}_h \bar{P}_{h+1})\varphi + \int \bar{P}_h \varphi (\bar{P}_{h+1}\varphi)' \, dt \\ &= U_{i+1,h}\varphi - (\bar{P}_h \bar{P}_{h+1})\varphi - \int \varphi \bar{P}_h \varphi (\varphi \bar{P}_h \varphi)''' \, dt. \end{aligned} \tag{4.30}$$

Hence, we finally obtain

$$\begin{aligned} -\int \varphi(\varphi \bar{P}_i \varphi)''' \, dt &= U_{i+1,h}\varphi - (\bar{P}_h \bar{P}_{h+1})\varphi \\ &\quad - \left[\varphi \bar{P}_h \varphi (\varphi \bar{P}_h \varphi)'' - \frac{1}{2}(\varphi \bar{P}_h \varphi)'^2 \right] + C, \end{aligned} \tag{4.31}$$

with $C \in \mathbb{R}$ an arbitrary constant. Since $2h + 1 = i + 1$, by the inductive hypothesis

$$-(\bar{P}_h \bar{P}_{h+1})\varphi - \varphi \bar{P}_h \varphi (\varphi \bar{P}_h \varphi)'' + \frac{1}{2}(\varphi \bar{P}_h \varphi)'^2 = V_{i+1}\varphi \tag{4.32}$$

for a unique $V_{i+1} \in \mathscr{H}_{2i+2}^{2i+2}$. Hence we set

$$\bar{P}_{i+1} := U_{i+1,h} + V_{i+1}. \tag{4.33}$$

This completes the proof. □

Lemma 4.2. *Given $\varphi \in C^k$ with $k \geq 2$, let $W_0, W_1, \ldots, W_{[k/2]-1}$ be a sequence of real-valued functions such that*

$$W_0 \equiv \text{const.}, \qquad W_i \in -\int \varphi(\varphi W_{i-1})''' \, dt \tag{4.34}$$

for $1 \le i \le \left[\frac{k}{2}\right] - 1$. *Then there exists a unique sequence of real constants* $C_0, \dots, C_{[k/2]-1}$ *such that*

$$W_i = \sum_{h=0}^{i} C_{i-h} \bar{P}_h \varphi \quad \textit{for } 0 \le i \le [k/2] - 1. \tag{4.35}$$

Proof. First of all since $W_0 \equiv$ const. and $\bar{P}_0 \equiv 1$, we have

$$W_0 = C_0 = C_0 \bar{P}_0 \varphi, \tag{4.36}$$

for a suitable $C_0 \in \mathbb{R}$. Moreover, assuming $k \ge 4$, we find that

$$-\int \varphi(\varphi W_0)''' \, dt = -C_0 \int \varphi\varphi''' \, dt = -C_0 \left[\varphi\varphi'' - \frac{1}{2}(\varphi')^2\right] + C, \tag{4.37}$$

where $C \in \mathbb{R}$ is an arbitrary constant. Hence, by (4.9) and (4.34), we find

$$W_1 = -C_0\left(\varphi\varphi'' - \frac{1}{2}\varphi'^2\right) + C_1 = C_0 \bar{P}_1 \varphi + C_1 \bar{P}_0 \varphi \tag{4.38}$$

for a suitable $C_1 \in \mathbb{R}$. This means that formula (4.35) holds for $i = 0, 1$. Now we proceed by induction assuming that (4.35) holds for some integer i with $1 \le i \le \left[\frac{k}{2}\right] - 2$. Then, recalling (4.7), we have

$$\begin{aligned} -\int \varphi(\varphi W_i)''' \, dt &= -\int \varphi\left(\varphi \sum_{h=0}^{i} C_{i-h} \bar{P}_h \varphi\right)''' dt \\ &= -\sum_{h=0}^{i} C_{i-h} \int \varphi(\varphi \bar{P}_h \varphi)''' \, dt = \sum_{h=0}^{i} C_{i-h} \bar{P}_{h+1} \varphi + C \end{aligned} \tag{4.39}$$

with $C \in \mathbb{R}$ an arbitrary constant. Thus, from the recursive relation (4.34), we obtain that

$$W_{i+1} = \sum_{h=0}^{i} C_{i-h} \bar{P}_{h+1} \varphi + C_{i+1} \tag{4.40}$$

with $C_{i+1} \in \mathbb{R}$ a suitable constant. Now since $\bar{P}_0 \equiv 1$, we may write

$$W_{i+1} = \sum_{h=0}^{i} C_{i-h} \bar{P}_{h+1} \varphi + C_{i+1} \bar{P}_0 \varphi = \sum_{h=0}^{i+1} C_{i+1-h} \bar{P}_h \varphi. \tag{4.41}$$

This proves (4.35) for $i + 1$. By direct inspection of the proof, we can see that the constants $C_0, \dots, C_{[k/2]-1}$ are uniquely determined. □

Conclusion of the proof of Theorem 1.2

By (4.4)–(4.5) every *particular* sequence $\{\alpha_i, \beta_i, \gamma_i\}_{0 \le i \le [k/2]-1}$ satisfying (3.11) has the form

$$\alpha_i = \omega p_i - 2\omega^2 \Gamma_{i-1}, \qquad \beta_i = -(\omega p_i)', \quad \gamma_i = 2\omega^2 \Gamma_i, \tag{4.42}$$

where $\Gamma_0, \dots, \Gamma_{[k/2]-1}$ is a finite sequence obtained from the recursive relation (4.3) by selecting, at each step, a suitable primitive

$$p_i \in \int \omega \Gamma'_{i-1}\, dt, \qquad 0 \le i \le [k/2] - 1, \tag{4.43}$$

and then setting $\Gamma_i = -(\omega p_i)''$. Having $\Gamma_{-1} \equiv 0$, the primitives p_i satisfy the recursive relation

$$p_0 \equiv \text{const.}, \qquad p_i \in -\int \varphi(\varphi p_{i-1})'''\, dt \tag{4.44}$$

for $1 \le i \le \left[\frac{k}{2}\right] - 1$. Hence, by Lemma 4.2, there exist constants $c_0, \dots, c_{[k/2]-1} \in \mathbb{R}$ such that

$$p_i = \sum_{h=0}^{i} c_{i-h} \bar{P}_h \omega \qquad \text{for } 0 \le i \le [k/2] - 1. \tag{4.45}$$

Then, by the first equation of (4.42), we easily have $\alpha_0 = \omega c_0 \bar{P}_0$ and, for $i \ge 1$,

$$\begin{aligned} \alpha_i &= \omega \sum_{h=0}^{i} c_{i-h} \bar{P}_h \omega + 2\omega^2 \sum_{h=0}^{i-1} c_{i-1-h} (\omega \bar{P}_h \omega)'' \\ &= \omega \sum_{h=0}^{i} c_{i-h} \bar{P}_h \omega + \omega \sum_{h=1}^{i} c_{i-h} [2\omega(\omega \bar{P}_{h-1}\omega)'']. \end{aligned} \tag{4.46}$$

Thus, $\alpha_i = \omega \sum_{h=0}^{i} c_{i-h} P_h \omega$ for $0 \le i \le \left[\frac{k}{2}\right] - 1$ if we define $P_0 := \bar{P}_0 \equiv 1$ and $P_i \in \mathscr{H}_{2i}^{2i}$ such that

$$P_i \varphi := \bar{P}_i \varphi + 2\varphi(\varphi \bar{P}_{i-1} \varphi)'' \tag{4.47}$$

for all $\varphi \in C^k$, when $1 \le i \le \left[\frac{k}{2}\right] - 1$. For the coefficients β_i from (4.45) we have

$$\beta_i = -(\omega p_i)' = -\sum_{h=0}^{i} c_{i-h} (\omega \bar{P}_h \omega)' = \sum_{h=0}^{i} c_{i-h} Q_h \omega, \tag{4.48}$$

where, for $0 \le i \le \left[\frac{k}{2}\right] - 1$, the polynomials Q_i are defined by

$$Q_i\varphi := -(\varphi \bar{P}_i \varphi)' \tag{4.49}$$

for all $\varphi \in C^k$. Since $\bar{P}_i \in \mathscr{H}_{2i}^{2i}$, we clearly have $P_i \in \mathscr{H}_{2i}^{2i}$, $Q_i \in \mathscr{H}_{2i+1}^{2i+1}$. This proves that every solution $\{\alpha_i, \beta_i, \gamma_i\}_{0 \le i \le [k/2]-1}$ of (1.5) satisfies (1.10). Conversely, using the definitions (4.47), (4.49) and Lemma 4.1 it is easy to see that (1.10) gives a solution of (1.5) for every finite sequence $c_0, \dots, c_{[k/2]-1} \in \mathbb{R}$.

5. Proof of Theorem 1.3

We give here the proof of Theorem 1.3 for $k \ge 2$ even. For $k \ge 2$ odd, the proof is similar. To show that (1.13) holds, first of all we choose a particular sequence of coefficients $\{\alpha_i, \beta_i, \gamma_i\}_{0 \le i \le [k/2]-1}$ using formulae (1.10) with constants

$$c_0 = 2, \qquad c_i = 0 \qquad \text{for } i \ge 1. \tag{5.1}$$

It follows that, for $0 \le i \le \left[\frac{k}{2}\right] - 1$,

$$\alpha_i = 2\omega P_i \omega, \qquad \beta_i = 2Q_i\omega, \qquad \gamma_i = 4\omega^2 Q_i^{(1)}\omega \tag{5.2}$$

where $P_i \in \mathscr{H}_{2i}^{2i}$ and $Q_i \in \mathscr{H}_{2i+1}^{2i+1}$. Since $P_0 \equiv 1$, we have $\alpha_0 = 2\omega$. Hence, as remarked before the Definition (1.11),

$$e_0(\lambda, t) = \lambda^2 \sqrt{a(t)}|w|^2 + \frac{|w_t|^2}{\sqrt{a(t)}} = \mathscr{E}(\lambda, t). \tag{5.3}$$

Then we apply the identity (1.7) with $k \ge 2$ even. For $\lambda > 0$ we find that

$$\mathscr{E}(\lambda, t) = \mathscr{E}(\lambda, 0) - [\mathscr{R}_k]_0^t + \int_0^t \lambda^{-k+2} \beta'_{[k/2]-1} \operatorname{Re}(\bar{w}w_t)\, d\tau \tag{5.4}$$

for all $t \in [0, T)$, where

$$\mathscr{R}_k := \mathscr{Q}_k - e_0. \tag{5.5}$$

To continue, we now estimate the terms e_i, f_i, g_i of $\mathscr{R}_k$. For this purpose, by (5.2), it suffices to estimate the norms of $P_i\omega$, $Q_i\omega$, $Q_i^{(1)}\omega$.

i) *Estimates of $P_i\omega$ for $1 \le i \le [k/2] - 1$.*

Since $P_i \in \mathscr{H}_{2i}^{2i}$, we have

$$P_i\omega = \sum c^i_{\eta_0,\dots,\eta_{2i}} \omega^{\eta_0} (\omega^{(1)})^{\eta_1} \dots (\omega^{(2i)})^{\eta_{2i}}, \tag{5.6}$$

with $c^i_{\eta_0,\dots,\eta_{2i}} \in \mathbb{R}$ and $\eta_0,\dots,\eta_{2i} \geq 0$ integers such that $\sum_{0\leq h\leq 2i}\eta_h = 2i$ and $\sum_{0\leq h\leq 2i} h\eta_h = 2i$. Noting that $\eta_0 = \sum_{h=1}^{2i}(h-1)\eta_h$, we can write

$$P_i\omega = \sum c^i_{\eta_0,\dots,\eta_{2i}} \prod_{h=1}^{2i} (\omega^{h-1}\omega^{(h)})^{\eta_h}. \tag{5.7}$$

Besides, given $0 \leq T' < T$ and $1 \leq h \leq 2i$, from Definition 1.2 we deduce that

$$\omega(t)^{(h-1)\eta_h}|\omega^{(h)}(t)|^{\eta_h} \leq \Phi_{2i}(T')^{h\eta_h} \qquad \text{for } t \in [0, T']. \tag{5.8}$$

Hence, for $1 \leq i \leq [k/2] - 1$ and k even, we easily obtain that

$$|P_i\omega| \leq \Phi_{k-2}(T')^{2i} \sum |c^i_{\eta_0,\dots,\eta_{2i}}| \leq E\Phi_{k-2}(T')^{2i} \qquad \text{for } t \in [0, T'] \tag{5.9}$$

with $E > 0$ a suitable constant independent of ω.

ii) *Estimates of* $Q_i\omega$ *for* $0 \leq i \leq [k/2] - 1$.

In this case we have $Q_i \in \mathscr{H}^{2i+1}_{2i+1}$, with $1 \leq 2i + 1 \leq k - 1$. By continuing the reasoning used above, for $0 \leq i \leq [k/2] - 1$ and k even, we obtain that

$$|Q_i\omega| \leq F\Phi_{k-1}(T')^{2i+1} \qquad \text{in } [0, T'], \tag{5.10}$$

with $F > 0$ a suitable constant independent of ω.

iii) *Estimates of* $Q_i^{(1)}\omega$ *for* $0 \leq i < \frac{k}{2} - 1$.

We note that $Q_i^{(1)} \in \mathscr{H}^{2i+2}_{2i+1}$ with $2 \leq 2i + 2 \leq k - 2$ because k is even. Thus, we obtain that

$$|Q_i^{(1)}\omega| \leq \omega^{-1} G\Phi_{k-2}(T')^{2i+2} \qquad \text{in } [0, T'], \tag{5.11}$$

with $G > 0$ a suitable constant independent of ω.

Summarizing the estimates (5.9)–(5.11), the terms e_i, f_i, g_i of $\mathscr{R}_k$ satisfy the inequalities

$$\begin{aligned} |e_i(\lambda, t)| &= \lambda^{-2i}|P_i\omega| e_0(\lambda, t) \leq E\big(\lambda^{-1}\Phi_{k-2}(T')\big)^{2i}\mathscr{E}(\lambda, t), \\ |f_i(\lambda, t)| &= 2\lambda^{-2i}|Q_i\omega|\,|\mathrm{Re}(\overline{w}w_t)| \leq F\big(\lambda^{-1}\Phi_{k-1}(T')\big)^{2i+1}\mathscr{E}(\lambda, t), \\ |g_i(\lambda, t)| &= 4\lambda^{-2i-2}\omega^2|Q_i^{(1)}\omega|\,|w_t|^2 \leq 2G\big(\lambda^{-1}\Phi_{k-2}(T')\big)^{2i+2}\mathscr{E}(\lambda, t) \end{aligned} \tag{5.12}$$

for $\lambda > 0$ and $t \in [0, T']$. Therefore, assuming

$$\lambda \geq L\Phi_{k-1}(T') \tag{5.13}$$

for some $L \geq 1$ and setting $R = 2\max\{E, F, G\}$, we have

$$|\mathscr{R}_k(\lambda, t)| \leq R \sum_{i=1}^{k-1} \big(\lambda^{-1}\Phi_{k-1}(T')\big)^i \mathscr{E}(\lambda, t) \leq \frac{kR}{L}\mathscr{E}(\lambda, t). \tag{5.14}$$

Introducing this estimate into formula (5.4), for $t \in [0, T']$ we have

$$\mathscr{E}(\lambda, t) \leq \frac{L + kR}{L - kR}\mathscr{E}(\lambda, 0) + \frac{L}{L - kR}\frac{\lambda^{-k+1}}{2}\int_0^t |\beta'_{[k/2]-1}|\mathscr{E}(\lambda, \tau)\, d\tau, \tag{5.15}$$

provided that $L > \max\{kR, 1\}$ and $\lambda \geq L\Phi_{k-1}(T')$. Now it is easy to obtain the estimate (1.11). Given $C > 1$ we can find $\Lambda(k, C) > \max\{kR, 1\}$ such that

$$\frac{L + kR}{L - kR} \leq \min\{2, C\}, \tag{5.16}$$

provided that $L \geq \Lambda(k, C)$. Hence, for $\lambda \geq \Lambda(k, C)\Phi_{k-1}(T')$ and $t \in [0, T']$, we find that

$$\mathscr{E}(\lambda, t) \leq C\mathscr{E}(\lambda, 0) + \lambda^{-k+1}\int_0^t |\beta'_{[k/2]-1}(\tau)|\mathscr{E}(\lambda, \tau)\, d\tau. \tag{5.17}$$

Finally, by (5.2), $\beta_{[k/2]-1} = 2Q_{[k/2]-1}\omega$. Thus, applying Gronwall's lemma, we obtain the estimate (1.13), which proves Theorem 1.3.

We conclude this section with a simple application of Theorem 1.3.

Corollary 5.1. *Assume that* (1.3) *holds with* $k = 2$ *and* $0 < T < +\infty$. *Then, for all* $\varepsilon \in (0, T]$ *and for all* $C > 1$, *there exists* $\rho = \rho(\varepsilon, C) > 0$ *such that for* $\lambda \geq \rho(\varepsilon, C)$

$$\frac{\mathscr{E}(\lambda, 0)}{C} \leq \mathscr{E}(\lambda, t) \leq C\mathscr{E}(\lambda, 0) \quad \textit{for all } t \in [0, T - \varepsilon]. \tag{5.18}$$

Proof. To prove the second inequality of (5.18), i.e., $\mathscr{E}(\lambda, t) \leq C\mathscr{E}(\lambda, 0)$ in the interval $[0, T - \varepsilon]$, we apply (1.13) with $k = 2$. More precisely, we set

$$\alpha_0 = 2\omega P_0\omega, \qquad \beta_0 = 2Q_0\omega. \tag{5.19}$$

Then, for every $C > 1$, taking

$$\lambda \geq \Lambda(2, \sqrt{C})\Phi_1(T - \varepsilon), \tag{5.20}$$

we have

$$\mathscr{E}(\lambda, t) \leq \sqrt{C}\mathscr{E}(\lambda, 0)\exp\left\{2\frac{(T - \varepsilon)}{\lambda}\max_{t \in [T - \varepsilon]} |Q_0^{(1)}\omega|\right\} \tag{5.21}$$

for all $t \in [0, T-\varepsilon]$. Then, to obtain the second inequality of (5.18), it is enough to take λ sufficiently large, i.e., such that $\lambda \geq \Lambda(2, \sqrt{C})\Phi_1(T-\varepsilon)$ and

$$\frac{(T-\varepsilon)}{\lambda} \max_{t \in [T-\varepsilon]} |Q_0^{(1)} \omega| \leq \frac{\ln C}{4}. \tag{5.22}$$

Finally, for fixed $t \in (0, T-\varepsilon]$, to prove the first inequality of (5.18) it suffices to apply the second one to the function $\tilde{w}(\lambda, \tau) = w(\lambda, t-\tau)$ for $\tau \in [0, t]$. □

6. Preliminary estimates for Kirchhoff equation

Let us consider the Kirchhoff equation (1.15). By partial Fourier transform, we are led to the ordinary problem:

$$\hat{u}_{tt} + m\big(s(t)\big)|\xi|^2 \hat{u} = 0 \qquad \text{with } s(t) := \int |\xi|^2 |\hat{u}|^2 \, d\xi, \tag{6.1}$$

$$\hat{u}(0, \xi) = \hat{u}_0(\xi), \qquad \hat{u}_t(0, \xi) = \hat{u}_1(\xi) \tag{6.2}$$

for $\xi \in \mathbb{R}^n$, $t \in [0, +\infty)$. In this section, we suppose that

$$u(x, t) \in C^h\big([0, T); H^{1+k/2-h}(\mathbb{R}^n)\big) \qquad (h = 0, 1) \tag{6.3}$$

is a given solution of (1.15) with $k \geq 0$ and $0 < T \leq +\infty$. Then, defining a suitable micro-energy, we prove some a priori estimates. To begin with, we state:

Lemma 6.1. *Assume that* (6.3) *holds with $k \geq 1$ integer, $m(s) \in C^{k-2}([0, +\infty))$ if $k \geq 2$. Given $g \in C^\ell([0, +\infty))$ with $1 \leq \ell \leq k$ integer, then $g\big(s(t)\big) \in C^\ell([0, T))$ and its ℓ-th order derivative is a finite sum of products of the form*

$$g^{(h)}\big(s(t)\big) \cdot P \cdot \prod \left(\int |\xi|^{2k_i} \operatorname{Re}(\bar{\hat{u}} \hat{u}_t) \, d\xi \right)^{p_i} \cdot \prod \left(\int |\xi|^{2l_i} |\hat{u}_t|^2 \, d\xi \right)^{q_i} \cdot \prod \left(\int |\xi|^{2m_i} |\hat{u}|^2 \, d\xi \right)^{r_i}, \tag{6.4}$$

where $1 \leq h \leq \ell$, P is a polynomial in $m \equiv m^{(0)}, \ldots, m^{(\ell-2)}$ for $\ell \geq 2$, while $P \equiv 1$ for $\ell = 1$; finally, $k_i, l_i \geq 1$, $m_i \geq 2$ and $p_i, q_i, r_i \geq 0$ are integers such that

$$\sum_i (2k_i - 1)p_i + \sum_i 2l_i q_i + \sum_i 2(m_i - 1)r_i = \ell. \tag{6.5}$$

Proof. First of all observe that

$$\hat{u}(\xi, t) \in C^h\big([0, T); L^2(\mathbb{R}^n_\xi, d\xi_h)\big) \qquad \text{for } h = 0, 1, \tag{6.6}$$

where $d\xi_h$ is the measure defined by $d\xi_h = (1+|\xi|^2)^{1+k/2-h}\,d\xi$. From this it immediately follows that $s(t) \in C^1$, because $k \geq 1$, and

$$\frac{d}{dt} g\big(s(t)\big) = 2g^{(1)}\big(s(t)\big) \int |\xi|^2 \operatorname{Re}(\bar{\hat{u}}\hat{u}_t)\,d\xi \tag{6.7}$$

if $g \in C^1([0,+\infty))$. Hence, for $\ell = k = 1$ the statement is true. Then we proceed by induction. Assuming $k \geq 2$, let us suppose that Lemma 6.1 holds for some integer ℓ with $1 \leq \ell \leq k-1$. Given $g \in C^{\ell+1}([0,+\infty))$, we must show that each factor in (6.4) is *at least* of class C^1 and that deriving with respect to t the product (6.4) we obtain is a finite sum of products of the same form. Namely we get

$$g^{(h')} \cdot \tilde{P} \cdot \prod \Big(\int |\xi|^{2k_i'} \operatorname{Re}(\bar{\hat{u}}\hat{u}_t)\,d\xi \Big)^{p_i'} \cdot \prod \Big(\int |\xi|^{2l_i'} |\hat{u}_t|^2\,d\xi \Big)^{q_i'} \cdot \prod \Big(\int |\xi|^{2m_i'} |\hat{u}|^2\,d\xi \Big)^{r_i'}, \tag{6.8}$$

where $1 \leq h' \leq \ell+1$, $\tilde{P}$ is suitable a polynomial in $m \equiv m^{(0)}, \ldots, m^{(\ell-1)}$; the integers $k_i', l_i' \geq 1$, $m_i' \geq 2$ and $p_i', q_i', r_i' \geq 0$ satisfy the relation

$$\sum_i (2k_i'-1)p_i' + \sum_i 2l_i'q_i' + \sum_i 2(m_i'-1)r_i' = \ell+1. \tag{6.9}$$

We start by noting that condition (6.5) with $\ell \leq k-1$ implies that $2k_i \leq k$, $2l_i \leq k-1$, and $2m_i \leq k+1$. Besides, since we assume $m(s) \in C^{k-2}$ for $k \geq 2$, it follows that (6.6) holds also for $h = 2$. Then, provided that (6.5) holds with $\ell \leq k-1$, it is easy to deduce that

$$\Big(\int |\xi|^{2k_i} \operatorname{Re}(\bar{\hat{u}}\hat{u}_t)\,d\xi \Big)' = \int |\xi|^{2k_i} |\hat{u}_t|^2\,d\xi + \int |\xi|^{2k_i} \operatorname{Re}(\bar{\hat{u}}\hat{u}_{tt})\,d\xi, \tag{6.10a}$$

$$\Big(\int |\xi|^{2l_i} |\hat{u}_t|^2\,d\xi \Big)' = 2\int |\xi|^{2l_i} \operatorname{Re}(\bar{\hat{u}}_{tt}\hat{u}_t), \tag{6.10b}$$

$$\Big(\int |\xi|^{2m_i} |\hat{u}|^2\,d\xi \Big)' = 2\int |\xi|^{2m_i} \operatorname{Re}(\hat{u}\hat{u}_t)\,d\xi \tag{6.10c}$$

where the right-hand sides of (6.10a)–(6.10c) are continuous functions on $[0,T)$. Hence, (6.4) represents a C^1 function. Moreover, differentiating (6.4) and replacing $\hat{u}_{tt}$ by

$$-m\big(s(t)\big)|\xi|^2\hat{u}, \tag{6.11}$$

we find a finite sum of the following products:

(1) products containing the element $2g^{(h+1)}\int|\xi|^2\,\mathrm{Re}(\bar{\hat{u}}\hat{u}_t)\,d\xi$ instead of $g^{(h)}$, with $1\le h\le\ell$ a suitable integer;

(2) products with $2\frac{\partial P}{\partial m^{(h)}}m^{(h+1)}\int|\xi|^2\,\mathrm{Re}(\bar{\hat{u}}\hat{u}_t)\,d\xi$ instead of P, with $0\le h\le\ell-2$ a suitable integer;

(3) products with a factor of type $\int|\xi|^{2k_i}(|\hat{u}_t|^2-m|\xi|^2|\hat{u}|^2)\,d\xi$ instead of a factor of the form $\int|\xi|^{2k_i}\,\mathrm{Re}(\bar{\hat{u}}\hat{u}_t)\,d\xi$;

(4) products with a factor of type $-2m\int|\xi|^{2l_i+2}\,\mathrm{Re}(\bar{\hat{u}}\hat{u}_t)\,d\xi$ instead of a term $\int|\xi|^{2l_i}|\hat{u}_t|^2\,d\xi$;

(5) products with a factor $2\int|\xi|^{2m_i}\,\mathrm{Re}(\bar{\hat{u}}\hat{u}_t)\,d\xi$ instead of $\int|\xi|^{2m_i}|\hat{u}|^2\,d\xi$.

In either case, after performing these substitutions, we obtain products of the form (6.8). Besides, it can be easily verified that the corresponding exponents k_i', l_i', m_i' and p_i', q_i', r_i' satisfy (6.9). This completes the proof. □

To continue, let us recall that the Kirchhoff equation (1.15) is of *variational* type. Assuming that (6.3) holds with $k\ge1$, we have the equality

$$\|\hat{u}_t\|_{L^2}^2+M(\|\,|\xi|\hat{u}\|_{L^2}^2)=\|\hat{u}_1\|_{L^2}^2+M(\|\,|\xi|\hat{u}_0\|_{L^2}^2):=K \tag{6.12}$$

for all $t\in[0,T)$, where $M(s):=\int_0^s m(y)\,dy$. From (6.12), in both cases of (1.19) we derive that

$$\delta\le m(s)\le\mu,\qquad s\in[0,M^{-1}(K)], \tag{6.13}$$

for suitable $\delta,\mu>0$. Thus, assuming from now on that (1.19) holds, we have the following *a-priori* estimates:

$$\begin{cases}\displaystyle\int|\hat{u}_t|^2\,d\xi+\delta\int|\xi|^2|\hat{u}|^2\,d\xi\le K,\\ 0<\delta\le m\big(s(t)\big)\le\mu\end{cases} \tag{6.14}$$

for $t\in[0,T)$. To obtain more refined estimates, we introduce the following micro-energies:

Definition 6.1. For $k\ge0$, $\rho>0$, we define

$$\mathscr{E}_k(\xi,t):=\sqrt{m\big(s(t)\big)}|\xi|^{k+2}|\hat{u}|^2+\frac{|\xi|^k|\hat{u}_t|^2}{\sqrt{m\big(s(t)\big)}}, \tag{6.15}$$

$$\mathscr{E}_k^\rho(t):=\frac{1}{\rho^k}\int_{|\xi|>\rho}\mathscr{E}_k(\xi,t)\,d\xi. \tag{6.16}$$

From (6.14) and the definition above, we have:

Lemma 6.2. *Assume that* (6.3) *holds with* $k \geq 1$. *Then for all* $t \in [0, T)$ *and for all* $\rho \geq 0$ *the Fourier transform* $\hat{u}(\xi, t)$ *satisfies the a priori estimates*

$$\int |\xi|^l |\hat{u}|\,|\hat{u}_t|\,d\xi \leq \frac{\rho^{l-1}K}{2\sqrt{\delta}} + \frac{1}{2}\int_{|\xi|>\rho} \frac{\mathscr{E}_k(\xi,t)}{|\xi|^{k+1-l}}\,d\xi, \qquad l \geq 1, \tag{6.17}$$

$$\int |\xi|^l |\hat{u}_t|^2\,d\xi \leq \rho^l K + \sqrt{\mu}\int_{|\xi|>\rho} \frac{\mathscr{E}_k(\xi,t)}{|\xi|^{k-l}}\,d\xi, \qquad l \geq 0, \tag{6.18}$$

$$\int |\xi|^l |\hat{u}|^2\,d\xi \leq \frac{\rho^{l-2}K}{\delta} + \frac{1}{\sqrt{\delta}}\int_{|\xi|>\rho} \frac{\mathscr{E}_k(\xi,t)}{|\xi|^{k+2-l}}\,d\xi, \qquad l \geq 2. \tag{6.19}$$

Proof. Let us prove (6.17). To begin with, for $0 \leq t < T$ the following inequalities hold:

a) for all $l \geq 1$ and $\rho \geq 0$ we have

$$|\xi|^l |\hat{u}|\,|\hat{u}_t| \leq \frac{\rho^{l-1}}{2\sqrt{\delta}}(|\hat{u}_t|^2 + \delta|\xi|^2|\hat{u}|^2) \qquad \text{for } |\xi| \leq \rho; \tag{6.20}$$

b) from the definition of $\mathscr{E}_k(\xi, t)$ we have

$$|\xi|^l |\hat{u}(\xi,t)|\,|\hat{u}_t(\xi,t)| \leq \frac{1}{2}\frac{\mathscr{E}_k(\xi,t)}{|\xi|^{k+1-l}} \qquad \text{for } |\xi| > 0. \tag{6.21}$$

Then, applying a) and (6.14) for $|\xi| \leq \rho$, and b) for $|\xi| > \rho$, we can estimate the left-hand side of (6.17). For all $\rho \geq 0$ and $t \in [0, T)$ we easily have

$$\begin{aligned}\int |\xi|^l |\hat{u}|\,|\hat{u}_t|\,d\xi &= \int_{|\xi|\leq\rho} |\xi|^l |\hat{u}|\,|\hat{u}_t|\,d\xi + \int_{|\xi|>\rho} |\xi|^l |\hat{u}|\,|\hat{u}_t|\,d\xi \\ &\leq \frac{\rho^{l-1}K}{2\sqrt{\delta}} + \frac{1}{2}\int_{|\xi|>\rho} \frac{\mathscr{E}_k(\xi,t)}{|\xi|^{k+1-l}}\,d\xi,\end{aligned} \tag{6.22}$$

provided that $l \geq 1$. By similar arguments we deduce (6.18) and (6.19). □

Having proved Lemmas 6.1 and 6.2, assuming (6.14) we are in a position to estimate the sup-norm of the ℓ-th order derivatives of $g(s(t))$ in $[0, T)$.

Lemma 6.3. *Assume* (6.3) *with* $k \geq 1$ *integer,* $m(s) \in C^{k-2}$ *if* $k \geq 2$. *Then, given* $g \in C^\ell([0, +\infty))$ *with* $1 \leq \ell \leq k$, *for every* $\rho > 0$ *we have*

$$\left|\frac{d^\ell}{dt^\ell} g(s(t))\right| \leq C_\ell \rho^\ell \left[\left(K + \mathscr{E}_k^\rho(t)\right) + \left(K + \mathscr{E}_k^\rho(t)\right)^\ell\right], \tag{6.23}$$

where $C_\ell = C(K, \ell, \delta, \mu, m, g)$ *is a suitable positive constant.*

Proof. By Lemma 6.2, for every $\rho > 0$ we have

$$\int |\xi|^l |\hat{u}|\,|\hat{u}_t|\,d\xi \le C\rho^{l-1}\big(K + \mathscr{E}_k^\rho(t)\big), \qquad 1 \le l \le k+1, \tag{6.24}$$

$$\int |\xi|^l |\hat{u}_t|^2\,d\xi \le C\rho^{l}\big(K + \mathscr{E}_k^\rho(t)\big), \qquad 0 \le l \le k, \tag{6.25}$$

$$\int |\xi|^l |\hat{u}|^2\,d\xi \le C\rho^{l-2}\big(K + \mathscr{E}_k^\rho(t)\big), \qquad 2 \le l \le k+2, \tag{6.26}$$

where $C = C(\delta, \mu) > 0$ is a suitable constant. Using these a priori bounds, we can now estimate the general term (6.4) of the ℓ-th derivative of $g\big(s(t)\big)$ when $1 \le \ell \le k$. In fact, from (6.24) we easily see that

$$\Big|\prod_i \Big(\int |\xi|^{2k_i}\,\mathrm{Re}(\bar{\hat{u}}\hat{u}_t)\,d\xi\Big)^{p_i}\Big| \le C^{\sum p_i}\rho^{\sum(2k_i-1)p_i}\big(K + \mathscr{E}_k^\rho(t)\big)^{\sum p_i}, \tag{6.27}$$

because we know that $k_i \ge 1$ and $2k_i - 1 \le \ell$, hence $2 \le 2k_i \le k+1$. Using (6.25), (6.26) we find similar estimates for the other types of products in (6.4). From this, recalling (6.14), we deduce that a generic term of the form (6.4) can be estimated by

$$\tilde{C}\rho^{\ell}\big[C\big(K + \mathscr{E}_k^\rho(t)\big)\big]^{\sum p_i + \sum q_i + \sum r_i} \tag{6.28}$$

where

$$\tilde{C} = \max_{0 \le s \le K/\delta}\big|g^{(h)}(s)P\big(m(s), \ldots, m^{(\ell-2)}(s)\big)\big|. \tag{6.29}$$

Finally, noting that $1 \le \sum p_i + \sum q_i + \sum r_i \le \ell$, we readily deduce (6.23). □

7. Micro-quadratic forms for Kirchhoff equation

Setting $\lambda = |\xi|$, $w = \hat{u}$ and $a(t) = m\big(s(t)\big)$, we are now in a position to apply Theorems 1.1 and 1.2 to the Kirchhoff equation, namely to the ODE with parameter

$$\hat{u}_{tt} + m\big(s(t)\big)|\xi|^2\hat{u} = 0. \tag{7.1}$$

To this end, in the following we assume (6.3) and $m(s) \in C^k$ with $k \ge 2$ an integer. Besides, we suppose that (6.14) holds. Then, for $|\xi| > 0$ we consider the micro-quadratic forms

$$\mathscr{Q}_k(\xi, t) := \sum_{0 \le i \le [k/2]-1} (e_i + f_i) + \sum_{i < k/2-1} g_i, \tag{7.2}$$

where

$$
\begin{aligned}
e_i(\xi,t) &= \alpha_i(t)|\xi|^{-2i}\big[m\big(s(t)\big)|\xi|^2|\hat{u}|^2 + |\hat{u}_t|^2\big],\\
f_i(\xi,t) &= \beta_i(t)|\xi|^{-2i}\operatorname{Re}(\bar{\hat{u}}\hat{u}_t),\\
g_i(\xi,t) &= \gamma_i(t)|\xi|^{-2i-2}|\hat{u}_t|^2.
\end{aligned}
\tag{7.3}
$$

According to Theorem 1.2, setting $c_0 = 2$ and $c_i = 0$ for $1 \le i \le \left[\frac{k}{2}\right] - 1$, we find

$$
\alpha_i = 2\omega P_i\omega, \qquad \beta_i = 2Q_i\omega, \qquad \gamma_i = 4\omega^2 Q_i^{(1)}\omega, \tag{7.4}
$$

where $P_i \in \mathscr{H}_{2i}^{2i}$, $Q_i \in \mathscr{H}_{2i+1}^{2i+1}$ for $0 \le i \le \left[\frac{k}{2}\right] - 1$ and

$$
\omega(t) := \frac{1}{2\sqrt{m\big(s(t)\big)}}. \tag{7.5}
$$

From now on we assume that (7.4) and (7.5) hold. In order to use the forms (7.2) in the proof of the global solvability of Kirchhoff equation, we need to estimate the coefficients α_i, β_i, γ_i. To simplify the following statements, we introduce the functions

$$
\phi_j(r) := r + r^j. \tag{7.6}
$$

Lemma 7.1. *Assume* (6.3) *and* $m(s) \in C^k$ *for some integer* $k \ge 2$. *Then, for any* $\rho > 0$, *we have the following estimates:*

(1) $|\alpha_i| \le C\rho^{2i}\phi_{2i}\big(K + \mathscr{E}_k^\rho(t)\big)$, $1 \le i \le \left[\frac{k}{2}\right] - 1$,

(2) $|\beta_i| \le C\rho^{2i+1}\phi_{2i+1}\big(K + \mathscr{E}_k^\rho(t)\big)$, $0 \le i \le \left[\frac{k}{2}\right] - 1$,

(3) $|\gamma_i| \le C\rho^{2i+2}\phi_{2i+2}\big(K + \mathscr{E}_k^\rho(t)\big)$, $0 \le i \le \left[\frac{k}{2}\right] - 1$,

where $C = C(K, k, \delta, \mu, m)$ *is a suitable positive constant.*

Proof. Let us prove the estimate (1). By Theorem 1.2 and (7.4), for $1 \le i \le \left[\frac{k}{2}\right] - 1$ the functions α_i are polynomials of the form

$$
\alpha_i = \omega \sum \alpha^i_{\eta_0,\dots,\eta_{2i}}\omega^{\eta_0}(\omega^{(1)})^{\eta_1}\dots(\omega^{(2i)})^{\eta_{2i}}, \tag{7.7}
$$

where $\alpha^i_{\eta_0,\dots,\eta_{2i}} \in \mathbb{R}$, $\eta_0,\dots,\eta_{2i} \ge 0$ are integers such that $\sum \eta_h = 2i$, $\sum h\eta_h = 2i$. By Lemma 6.3, we know that $\omega^{(\ell)}$ satisfies the estimate

$$
|\omega^{(\ell)}(t)| \le C_\ell\rho^\ell\phi_\ell\big(K + \mathscr{E}_k^\rho(t)\big), \qquad 1 \le \ell \le k, \tag{7.8}
$$

where $C_\ell = C(K, \ell, \delta, \mu, m)$ is a suitable positive constant. Then,

$$|\alpha_i| \le \sum |\alpha^i_{\eta_0,\dots,\eta_{2i}}| \omega^{\eta_0+1} \prod_{h=1}^{2i} \big(C_h \rho^h \phi_h\big(K + \mathscr{E}^\rho_k(t)\big)\big)^{\eta_h}. \tag{7.9}$$

Hence, noting that

$$\phi_j(r)\phi_l(r) \le C(j,l)\phi_{j+l}(r) \qquad \text{for all } j, l \ge 1 \text{ and all } r \ge 0, \tag{7.10}$$

from (6.14), (7.5), (7.9) we immediately see that (1) holds. The proof of the estimates (2), (3) for β_i and γ_i is similar. □

Finally, the terms β'_i and γ'_i satisfy the following:

Lemma 7.2. *Assume* (6.3) *and* $m(s) \in C^k$ *for some integer* $k \ge 2$. *Then, for any* $\rho > 0$, *the following estimates hold:*

i) $|\beta'_i| \le C\rho^{2i+2}\phi_{2i+2}\big(K + \mathscr{E}^\rho_k(t)\big)$, $0 \le i \le \left[\frac{k}{2}\right] - 1$,

ii) $|\gamma'_i| \le C\rho^{2i+3}\phi_{2i+3}\big(K + \mathscr{E}^\rho_k(t)\big)$, $0 \le i < \frac{k}{2} - 1$,

where $C = C(K, k, \delta, \mu, m)$ *is a suitable positive constant.*

Proof. Let us prove the estimate i) for β'_i. By Theorem 1.2 and (7.4), we have

$$\beta_i = \sum \beta^i_{\eta_0,\dots,\eta_{2i+1}} \omega^{\eta_0} (\omega^{(1)})^{\eta_1} \dots (\omega^{(2i+1)})^{\eta_{2i+1}} \tag{7.11}$$

where $\beta^i_{\eta_0,\dots,\eta_{2i+1}} \in \mathbb{R}$, $\eta_0, \dots, \eta_{2i+1} \ge 0$ are integers such that $\sum \eta_h = 2i + 1$, $\sum h\eta_h = 2i + 1$. Since $2i + 2 \le k$, we see that each term in (7.11) is continuously differentiable. This means that β'_i has the form

$$\beta'_i = \sum \tilde{\beta}^i_{\eta_0,\dots,\eta_{2i+2}} \omega^{\eta_0} (\omega^{(1)})^{\eta_1} \dots (\omega^{(2i+2)})^{\eta_{2i+2}}, \tag{7.12}$$

where $\tilde{\beta}^i_{\eta_0,\dots,\eta_{2i+2}} \in \mathbb{R}$ are suitable constants, $\eta_0, \dots, \eta_{2i+2} \ge 0$ are integers such that $\sum \eta_h = 2i + 1$ and $\sum h\eta_h = 2i + 2$. Now, applying (7.8) and (7.10) as above, we can easily verify that i) holds. Finally, to prove ii), it suffices to observe that

$$\gamma_i = \omega^2 \sum_\eta \gamma^i_{\eta_0,\dots,\eta_{2i+2}} \omega^{\eta_0} (\omega^{(1)})^{\eta_1} \dots (\omega^{(2i+2)})^{\eta_{2i+2}}, \tag{7.13}$$

where $\gamma^i_{\eta_0,\dots,\eta_{2i+2}} \in \mathbb{R}$, $\eta_0, \dots, \eta_{2i+2} \ge 0$ are integers such that $\sum \eta_h = 2i + 1$, $\sum h\eta_h = 2i + 2$. Having $2i \le k - 3$, it follows that each term in the expression

(7.13) of γ_i is continuously differentiable. Then, differentiating (7.13) and applying the estimate (7.8) as above, we obtain ii). $\square$

Remark 7.1. Note that $\left[\frac{k}{2}\right] = \frac{k-1}{2}$ for k odd. Hence, in this case, we can estimate the quantity $\gamma'_{[k/2]-1}$.

We now come to the estimates of the micro-energy $\mathscr{E}_k(\xi,t)$, with $k \geq 2$. By the choice made in (7.4) we have

$$\mathscr{E}_k(\xi,t) = |\xi|^k e_0(\xi,t). \tag{7.14}$$

Then, setting

$$\mathscr{R}_k(\xi,t) := \mathscr{Q}_k(\xi,t) - e_0(\xi,t) \tag{7.15}$$

and applying Theorem 1.1, for $|\xi| > 0$ and $0 \leq t_0 \leq t < T$ we obtain the identities:

(1) for $k \geq 2$ even,

$$\mathscr{E}_k(\xi,t) = \mathscr{E}_k(\xi,t_0) - |\xi|^k [\mathscr{R}_k]_{t_0}^t + \int_{t_0}^t |\xi|^2 \beta'_{[k/2]-1} \operatorname{Re}(\bar{\hat{u}}\hat{u}_t)\, d\tau; \tag{7.16}$$

(2) for $k \geq 2$ odd,

$$\mathscr{E}_k(\xi,t) = \mathscr{E}_k(\xi,t_0) - |\xi|^k [\mathscr{R}_k]_{t_0}^t + \int_{t_0}^t |\xi| \gamma'_{[k/2]-1} |\hat{u}_t|^2\, d\tau. \tag{7.17}$$

Now we estimate the right-hand side of (7.16), (7.17). Considering the terms in $|\xi|^k \mathscr{R}_k(\xi,t)$, and applying (6.14) and Lemma 7.1 we have

$$\begin{aligned}
|\xi|^k |e_i| &= |\xi|^{-2i} |\alpha_i| \sqrt{m} \left(\sqrt{m} |\xi|^{k+2} |\hat{u}|^2 + \frac{|\xi|^k |\hat{u}_t|^2}{\sqrt{m}} \right) \\
&= |\xi|^{-2i} |\alpha_i| \sqrt{m} \mathscr{E}_k(\xi,t) \\
&\leq C \frac{\rho^{2i}}{|\xi|^{2i}} \sqrt{m} \phi_{2i}\big(K + \mathscr{E}_k^\rho(t)\big) \mathscr{E}_k(\xi,t) \qquad (7.18)
\end{aligned}$$

for $|\xi|, \rho > 0$ and $1 \leq i \leq \left[\frac{k}{2}\right] - 1$; besides

$$\begin{aligned}
|\xi|^k |f_i| &= |\xi|^{k-2i} |\beta_i|\, |\operatorname{Re}(\bar{\hat{u}}\hat{u}_t)| \\
&\leq \frac{1}{2|\xi|^{2i+1}} |\beta_i| \mathscr{E}_k(\xi,t) \\
&\leq C \frac{\rho^{2i+1}}{2|\xi|^{2i+1}} \phi_{2i+1}\big(K + \mathscr{E}_k^\rho(t)\big) \mathscr{E}_k(\xi,t), \qquad (7.19)
\end{aligned}$$

$$
\begin{aligned}
|\xi|^k|g_i| &= |\xi|^{k-2i-2}|\gamma_i|\,|\hat{u}_t|^2 \\
&\le \frac{\sqrt{m}}{|\xi|^{2i+2}}|\gamma_i|\mathscr{E}_k(\xi,t) \\
&\le C\frac{\rho^{2i+2}}{|\xi|^{2i+2}}\phi_{2i+2}\big(K+\mathscr{E}_k^{\rho}(t)\big)\mathscr{E}_k(\xi,t)
\end{aligned}
\tag{7.20}
$$

for $|\xi|, \rho > 0$ and $0 \le i \le \left[\frac{k}{2}\right] - 1$. Finally we have to consider $|\xi|^2\beta'_{[k/2]-1}\operatorname{Re}(\bar{\hat{u}}\hat{u}_t)$ for k even, and $|\xi|\gamma'_{[k/2]-1}|\hat{u}_t|^2$ for k odd. From Lemma 7.2 we obtain

$$
\begin{aligned}
|\xi|^2|\beta'_{[k/2]-1}|\,|\operatorname{Re}(\bar{\hat{u}}\hat{u}_t)| &\le \frac{1}{2|\xi|^{k-1}}|\beta'_{[k/2]-1}|\mathscr{E}_k(\xi,t) \\
&\le C\frac{\rho^k}{|\xi|^{k-1}}\phi_k\big(K+\mathscr{E}_k^{\rho}(t)\big)\mathscr{E}_k(\xi,t)
\end{aligned}
\tag{7.21}
$$

for $|\xi|, \rho > 0$ and k even, and

$$
\begin{aligned}
|\xi|\,|\gamma'_{[k/2]-1}|\,|\hat{u}_t|^2 &\le \frac{\sqrt{m}}{|\xi|^{k-1}}|\gamma'_{[k/2]-1}|\mathscr{E}_k(\xi,t) \\
&\le C\frac{\rho^k}{|\xi|^{k-1}}\phi_k\big(K+\mathscr{E}_k^{\rho}(t)\big)\mathscr{E}_k(\xi,t)
\end{aligned}
\tag{7.22}
$$

for $|\xi|, \rho > 0$ and k odd. Now noting that for $1 \le i \le j$,

$$
\phi_i(r) \le 2\phi_j(r) \qquad \text{for all } r \ge 0,
\tag{7.23}
$$

and taking into account Definition (7.15) of $\mathscr{R}_k(\xi,t)$, we can summarize the estimates above as follows. For a suitable constant

$$
\mathscr{C}_k = \mathscr{C}_k(K,k,\delta,\mu,m) > 0
\tag{7.24}
$$

we have:

i) for $|\xi| \ge \rho > 0$ and for all $k \ge 2$ the quantity $|\xi|^k\mathscr{R}_k(\xi,t)$ satisfies

$$
|\xi|^k|\mathscr{R}_k(\xi,t)| \le \mathscr{C}_k\frac{\rho}{|\xi|}\phi_{k-1}\big(K+\mathscr{E}_k^{\rho}(t)\big)\mathscr{E}_k(\xi,t);
\tag{7.25}
$$

ii) for $|\xi|, \rho > 0$ the quantities $|\xi|^2|\beta'_{[k/2]-1}\operatorname{Re}(\bar{\hat{u}}\hat{u}_t)|$ and $|\xi|\,|\gamma'_{[k/2]-1}|\,|\hat{u}_t|^2$, for $k \ge 2$ even and for $k \ge 2$ odd respectively, are bounded by

$$
\mathscr{C}_k\frac{\rho^k}{|\xi|^{k-1}}\phi_k\big(K+\mathscr{E}_k^{\rho}(t)\big)\mathscr{E}_k(\xi,t).
\tag{7.26}
$$

Assuming (6.14) and introducing the estimates (7.25), (7.26) in (7.16) and (7.17), we have proved the following result.

Proposition 7.1. *Assume* (6.3) *and* $m(s) \in C^k$ *for some integer* $k \geq 2$. *Then, for any* $\rho > 0$, $|\xi| \geq \Lambda\rho$ *with* $\Lambda \geq 1$, $0 \leq t_0 \leq t < T$, *the following estimate holds:*

$$\mathscr{E}_k(\xi, t) D_k^\rho(t) \leq \mathscr{E}_k(\xi, t_0)\big(2 - D_k^\rho(t_0)\big) + \frac{\mathscr{C}_k \rho^k}{|\xi|^{k-1}} \int_{t_0}^t \phi_k\big(K + \mathscr{E}_k^\rho(\tau)\big)\mathscr{E}_k(\xi, \tau)\, d\tau, \tag{7.27}$$

where $D_k^\rho(t) := 1 - \frac{\mathscr{C}_k}{\Lambda}\phi_{k-1}\big(K + \mathscr{E}_k^\rho(t)\big)$.

8. Proof of Theorem 1.4

As is known, assuming (1.19) and $m(s) \in C^1$, problem (1.15), (1.16) is locally well posed in $H^s \times H^{s-1}$ for $s \geq \frac{3}{2}$. See [1], [2], [10], [18], [19], [20]. More precisely, given $(u_0, u_1) \in H^s \times H^{s-1}$ with $s \geq \frac{3}{2}$, it was proved that there exists a unique *local* solution $u(x, t) \in C^h\big([0, T); H^{s-h}(\mathbb{R}^n)\big)$ $(h = 0, 1)$ with T bounded from below by

$$T \geq \mathscr{T}(\|u_0\|_{H^{3/2}}, \|u_1\|_{H^{1/2}}), \tag{8.1}$$

where $\mathscr{T} : [0, \infty) \times [0, \infty) \to (0, \infty)$ is strictly positive and continuous. From this result, assuming (1.19), $m(s) \in C^1$ and recalling Definition 6.1, we readily deduce the following fact:

Proposition 8.1. *Given* $(u_0, u_1) \in H^s \times H^{s-1}$ *with* $s \geq \frac{3}{2}$, *let* $[0, T)$ *be the maximal interval of existence of the solution* $u(x, t) \in C^h\big([0, T); H^{s-h}(\mathbb{R}^n)\big)$ $(h = 0, 1)$ *of problem* (1.15), (1.16). *Then*

$$T < +\infty \Rightarrow \lim_{t \to T-0} \int \mathscr{E}_{2s-2}(\xi, t)\, d\xi = +\infty. \tag{8.2}$$

Taking account these considerations and having

$$\tilde{B}_\Delta^1 \subset H^{3/2} \times H^{1/2}, \qquad B_\Delta^k \subset H^{1+k/2} \times H^{k/2} \qquad \text{for } k \geq 1, \tag{8.3}$$

in the case of Theorem 1.4 we will prove the global solvability by showing that

$$(u_0, u_1) \in B_\Delta^k \ (\tilde{B}_\Delta^1 \text{ if } k = 1) \Rightarrow \sup_{t \in [0, T)} \int \mathscr{E}_k(\xi, t)\, d\xi < +\infty, \tag{8.4}$$

independently of $T \in (0, +\infty)$.

Case $k \geq 2$. Assume that $(u_0, u_1) \in B^k_\Delta$. As remarked above, problem (1.15), (1.16) has a unique local solution $u(x,t) \in C^h\big([0,T); H^{k/2+1-h}(\mathbb{R}^n)\big)$ $(h = 0, 1)$ for some $T > 0$. Besides, by Definition 1.3 there exist a sequence $\{\rho_j\}_{j \geq 1}$, $\rho_j \to +\infty$, and constants $\eta, J > 0$ such that

$$\int_{|\xi|>\rho_j} \mathscr{E}_k(\xi, 0) \frac{e^{\eta\rho_j^k/|\xi|^{k-1}}}{\rho_j^k} d\xi < J \tag{8.5}$$

for all $j \geq 1$. In the following, using the estimates of Section 7, we will prove (8.4). To begin with, let us consider the inequality (7.27) with $t_0 = T - \varepsilon \leq t < T$, where $\varepsilon > 0$ will be fixed below. Assuming in Definition (6.12) $K > 0$ (if $K = 0$ then $u(x,t) \equiv 0$), we set

$$\Lambda := 4\mathscr{C}_k \phi_{k-1}(2K) + 1. \tag{8.6}$$

Besides, taking account of Definition 6.1, we may write

$$\mathscr{E}^\rho_k(t) = \mathscr{G}^{\Lambda\rho}_\rho(t) + \Lambda^k \mathscr{E}^{\Lambda\rho}_k(t), \tag{8.7}$$

where

$$\mathscr{G}^{\Lambda\rho}_\rho(t) := \frac{1}{\rho^k} \int_{\rho<|\xi|\leq\Lambda\rho} \mathscr{E}_k(\xi, t)\, d\xi. \tag{8.8}$$

Then, from (8.6), (8.7), we have

$$D^\rho_k(t) = 1 - \frac{\mathscr{C}_k}{\Lambda} \phi_{k-1}\big(K + \mathscr{E}^\rho_k(t)\big) > \frac{1}{2} \tag{8.9}$$

for $t \in [T - \varepsilon, T)$ and $\rho > 0$ such that the quantities $\mathscr{G}^{\Lambda\rho}_\rho(t)$, $\Lambda^k \mathscr{E}^{\Lambda\rho}_k(t)$ satisfy

$$\mathscr{G}^{\Lambda\rho}_\rho(t) < \frac{K}{2}, \tag{8.10}$$

$$\Lambda^k \mathscr{E}^{\Lambda\rho}_k(t) < \frac{K}{2}. \tag{8.11}$$

Now we choose $\varepsilon > 0$ and $\rho_a \geq 1$ such that

$$\frac{\rho\mu_1}{2\delta} \left(\frac{K}{\sqrt{\delta}} + K \right) \varepsilon + \ln\left(\frac{4J+1}{K} \right) \leq \frac{\eta\rho}{\Lambda^{k-1}} \tag{8.12}$$

for all $\rho \geq \rho_a$, and

$$\mathscr{C}_k \phi_k(2K) \leq \frac{\eta}{4\varepsilon}, \tag{8.13}$$

where $\mu_1 := \max|m'(s)|$ for $s \in [0, K/\delta]$. Besides, noting that $\mathscr{E}_k^\rho(T-\varepsilon) \to 0$ as $\rho \to +\infty$, we may also suppose that

$$\mathscr{G}_\rho^{\Lambda\rho}(T-\varepsilon) \le \frac{K}{4}, \qquad \Lambda^k \mathscr{E}_k^{\Lambda\rho}(T-\varepsilon) \le \frac{K}{4} \tag{8.14}$$

for all $\rho \ge \rho_a$. Due to (8.14), for every $\rho \ge \rho_a$ the conditions (8.10), (8.11) are satisfied in some maximal right neighborhood of $T-\varepsilon$, say $[T-\varepsilon, \tilde{T}(\rho))$, with

$$\tilde{T}(\rho) := \sup\{\tau : T-\varepsilon \le \tau < T, \text{(8.10), (8.11) hold for all } t \in [T-\varepsilon, \tau)\}. \tag{8.15}$$

Clearly, for all $\rho \ge \rho_a$ we have $T-\varepsilon < \tilde{T}(\rho) \le T$. In the sequel we will prove that, if $\mathscr{E}_k(\xi, 0)$ satisfies (8.5), then $\tilde{T}(\rho_j) = T$ provided ρ_j is sufficiently large. To this end, we now estimate separately $\mathscr{G}_\rho^{\Lambda\rho}(t)$ and $\Lambda^k \mathscr{E}_k^{\Lambda\rho}(t)$.

Estimate of $\mathscr{G}_\rho^{\Lambda\rho}(t)$. We observe that $\frac{d}{dt}\mathscr{E}_k(\xi, t)$ satisfies the elementary inequality

$$\left|\frac{d}{dt}\mathscr{E}_k(\xi, t)\right| \le \frac{1}{2}\frac{|m'(s)|}{m(s)}|s'(t)|\mathscr{E}_k(\xi, t) \le \frac{\mu_1}{2\delta}|s'(t)|\mathscr{E}_k(\xi, t). \tag{8.16}$$

Since $\mathscr{E}_k = |\xi|^k e_0$, from Corollary 5.1 we know that

$$\frac{\mathscr{E}_k(\xi, 0)}{2} \le \mathscr{E}_k(\xi, t) \le 2\mathscr{E}_k(\xi, 0) \qquad \text{in } [0, T-\varepsilon], \tag{8.17}$$

provided that $|\xi| \ge \rho(\varepsilon, 2)$. Hence, for $|\xi| \ge \rho(\varepsilon, 2)$ we have

$$\mathscr{E}_k(\xi, T-\varepsilon) \le 2\mathscr{E}_k(\xi, 0), \tag{8.18}$$

and by (8.16) and Gromwall's Lemma it follows that

$$\mathscr{E}_k(\xi, t) \le 2\mathscr{E}_k(\xi, 0)\exp\left\{\frac{\mu_1}{2\delta}\int_{T-\varepsilon}^t |s'(\tau)|\,d\tau\right\} \tag{8.19}$$

for all $t \in [T-\varepsilon, T)$. Now, by Definition (8.15), for $\rho \ge \rho_a$ we have $\mathscr{E}_k^\rho(t) \le K$ in $[T-\varepsilon, \tilde{T}(\rho))$. Whence, using (6.17) with $l = 2$, for $\rho \ge \rho_a$ we find that

$$\int_{T-\varepsilon}^t |s'(\tau)|\,d\tau \le \rho\left(\frac{K}{\sqrt{\delta}} + K\right)\varepsilon \qquad \text{for } t \in [T-\varepsilon, \tilde{T}(\rho)). \tag{8.20}$$

Thus, setting $\rho_b := \max\{\rho_a, \rho(\varepsilon, 2)\}$ and taking $\rho = \rho_j$ with $\rho_j \ge \rho_b$, from (8.5), (8.12), (8.19), (8.20), we have

$$\begin{aligned}\mathscr{G}_{\rho_j}^{\Lambda\rho_j}(t) &\le \frac{1}{\rho_j^k}\int_{\rho_j<|\xi|\le\Lambda\rho_j} 2\mathscr{E}_k(\xi,0)\exp\left\{\frac{\rho_j\mu_1}{2\delta}\left(\frac{K}{\sqrt{\delta}}+K\right)\varepsilon\right\}d\xi\\ &\le \frac{K}{4J+1}\int_{\rho_j<|\xi|\le\Lambda\rho_j} 2\mathscr{E}_k(\xi,0)\frac{e^{\eta\rho_j^k/|\xi|^{k-1}}}{\rho_j^k}d\xi\\ &\le K\frac{2J}{4J+1}<\frac{K}{2}\end{aligned}\tag{8.21}$$

for all $t \in [T-\varepsilon, \tilde{T}(\rho_j))$. The estimate (8.21) means that, taking $\rho = \rho_j$ with $\rho_j \ge \rho_b$, condition (8.10) is always satisfied as long as (8.11) holds. Thus, it remains to prove that for $\rho = \rho_j$ sufficiently large (8.11) holds for all $t \in [T-\varepsilon, T)$.

Estimate of $\Lambda^k\mathscr{E}_k^{\Lambda\rho}(t)$. Let us consider (7.27) with Λ given by (8.6), $t_0 = T-\varepsilon \le t < T$ and $\rho = \rho_j \ge \rho_b$. Then, as long as the conditions (8.10), (8.11) are satisfied for $t \ge T-\varepsilon$, due to (8.9) and (8.13) we find the inequality

$$\mathscr{E}_k(\xi,t) \le 3\mathscr{E}_k(\xi,T-\varepsilon) + \frac{\eta}{2\varepsilon}\frac{\rho_j^k}{|\xi|^{k-1}}\int_{T-\varepsilon}^t \mathscr{E}_k(\xi,\tau)\,d\tau \tag{8.22}$$

for all $|\xi| \ge \Lambda\rho_j$. Hence, using (8.18) and applying Gronwall's Lemma to (8.22), for $\rho_j \ge \rho_b$ and $t \in [T-\varepsilon, \tilde{T}(\rho_j))$ we find the estimate

$$\mathscr{E}_k(\xi,t) \le 3\mathscr{E}_k(\xi,T-\varepsilon)\exp\left\{\frac{\eta\rho_j^k}{2|\xi|^{k-1}}\frac{t-T+\varepsilon}{\varepsilon}\right\} \quad \text{for } |\xi| \ge \Lambda\rho_j. \tag{8.23}$$

Thus, in order to show that (8.11) holds for all $t \in [T-\varepsilon, T)$ it will be sufficient to choose $\rho_c \ge \rho_b$ such that

$$\int_{|\xi|>\Lambda\rho_j} \mathscr{E}_k(\xi,T-\varepsilon)\frac{e^{\eta\rho_j^k/2|\xi|^{k-1}}}{\rho_j^k}d\xi < \frac{K}{6} \quad \text{for } \rho_j \ge \rho_c. \tag{8.24}$$

To this end, let us take $\gamma > 0$ such that $e^{-\gamma\eta/2}J \le \frac{K}{24}$. Then, setting

$$A_j = \left\{|\xi| > \Lambda\rho_j, \frac{\rho_j^k}{|\xi|^{k-1}} \ge \gamma\right\} \quad \text{and} \quad B_j = \left\{|\xi| > \Lambda\rho_j, \frac{\rho_j^k}{|\xi|^{k-1}} < \gamma\right\}, \tag{8.25}$$

due to (8.18) and (8.25) for $\rho_j \ge \rho_b$ we have

$$\begin{aligned}&\int_{|\xi|>\Lambda\rho_j} \mathscr{E}_k(\xi,T-\varepsilon)\frac{e^{\eta\rho_j^k/2|\xi|^{k-1}}}{\rho_j^k}d\xi\\ &\qquad \le \int_{A_j} 2\mathscr{E}_k(\xi,0)\frac{e^{\eta\rho_j^k/2|\xi|^{k-1}}}{\rho_j^k}d\xi + \int_{B_j} 2\mathscr{E}_k(\xi,0)\frac{e^{\eta\rho_j^k/2|\xi|^{k-1}}}{\rho_j^k}d\xi\end{aligned}$$

$$\le e^{-\gamma\eta/2}\int_{A_j} 2\mathscr{E}_k(\xi,0)\frac{e^{\eta\rho_j^k/|\xi|^{k-1}}}{\rho_j^k}\,d\xi+\int_{B_j} 2\mathscr{E}_k(\xi,0)\frac{e^{\gamma\eta/2}}{\rho_j^k}\,d\xi$$

$$\le \frac{K}{12}+\int_{|\xi|>\Lambda\rho_j} 2\mathscr{E}_k(\xi,0)\frac{e^{\gamma\eta/2}}{\rho_j^k}\,d\xi. \tag{8.26}$$

The last integral in (8.26) tends to 0 as $\rho_j \to +\infty$. Hence, it is clear that condition (8.24) is satisfied, provided that we take $\rho_c \ge \rho_b$ sufficiently large.

This means that for $j_0 \ge 1$ such that $\rho_{j_0} \ge \rho_c$, we have $\tilde{T}(\rho_{j_0}) = T$, i.e., (8.10) and (8.11) are satisfied in the whole interval $[T-\varepsilon, T)$ by taking $\rho = \rho_{j_0}$. In particular, by (8.7) we have $\mathscr{E}_k^{\rho_{j_0}}(t) \le K$ in $[T-\varepsilon, T)$. Thus, using the a priori estimate (6.14), we finally obtain that

$$\int \mathscr{E}_k(\xi,t)\,d\xi \le \rho_{j_0}^k K\left(\frac{1}{\sqrt{\delta}}+\frac{\sqrt{\mu}}{\delta}+1\right), \tag{8.27}$$

for all $t \in [T-\varepsilon, T)$. This concludes the proof of Theorem 1.4 in the case $k \ge 2$.

Case $k = 1$. Let us suppose that $(u_0, u_1) \in \tilde{B}_\Delta^1$. As remarked above, problem (1.15), (1.16) has a unique local solution $u(x,t) \in C^h([0,T]; H^{3/2-h}(\mathbb{R}^n))$ $(h = 0,1)$ for some $T > 0$. Assuming in (6.12) $K > 0$, we set

$$N := 2\frac{K\mu_1}{\delta^{3/2}}T, \tag{8.28}$$

where $\mu_1 := \max|m'(s)|$ for $s \in [0, K/\delta]$. By the Definition 1.3 of $\tilde{B}_\Delta^1$ there exists a sequence $\{\rho_j(N)\}_{j\ge 1}$, $\rho_j(N) \to +\infty$, such that

$$\sup_{j\ge 1} e^{N\rho_j(N)}\int_{|\xi|>\rho_j(N)} \mathscr{E}_1(\xi,0)\,d\xi < +\infty. \tag{8.29}$$

Now we consider the micro-energy $\mathscr{E}_1(\xi,t)$. By (8.16), with $k = 1$, we find that

$$\left|\frac{d}{dt}\mathscr{E}_1(\xi,t)\right| \le \frac{\mu_1}{2\delta}|s'(t)|\mathscr{E}_1(\xi,t), \tag{8.30}$$

where, by (6.17) with $k = 1$, $l = 2$,

$$|s'(t)| \le \frac{\rho K}{\sqrt{\delta}}+\int_{|\xi|>\rho} \mathscr{E}_1(\xi,t)\,d\xi \tag{8.31}$$

for all $\rho \ge 0$. Hence,

$$\frac{d}{dt}\mathscr{E}_1 \le \frac{\rho\mu_1}{2\delta}\left(\frac{K}{\sqrt{\delta}}+\mathscr{E}_1^\rho(t)\right)\mathscr{E}_1 \tag{8.32}$$

for all $\rho > 0$. Now, having $K > 0$, we define

$$\hat{T}(\rho) := \sup\left\{0 \le \tau < T : \mathscr{E}_1^{\rho}(t) \le \frac{K}{\sqrt{\delta}} \text{ for all } t \in [0, \tau)\right\}. \tag{8.33}$$

It is clear that $\hat{T}(\rho) > 0$ provided that $\rho > 0$ is large enough. Moreover, in $[0, \hat{T}(\rho))$ we have the estimate $\mathscr{E}_1(\xi, t) \le \mathscr{E}_1(\xi, 0) \exp\left\{\frac{K\mu_1}{\delta^{3/2}}\rho t\right\}$. From this, by (8.28), we obtain that

$$\mathscr{E}_1^{\rho}(t) \le \frac{e^{(N/2)\rho}}{\rho} \int_{|\xi|>\rho} \mathscr{E}_1(\xi, 0)\, d\xi \quad \text{ in } [0, \hat{T}(\rho)) \tag{8.34}$$

for all $\rho > 0$ large enough. Finally, let us observe that (8.29) implies that

$$\lim_{j\to+\infty} e^{(N/2)\rho_j(N)} \int_{|\xi|>\rho_j(N)} \mathscr{E}_1(\xi, 0)\, d\xi = 0 \tag{8.35}$$

because $\rho_j(N) \to +\infty$. Therefore, there exists an integer $j_0 \ge 1$ such that

$$\frac{e^{(N/2)\rho_j(N)}}{\rho_j(N)} \int_{|\xi|>\rho_j(N)} \mathscr{E}_1(\xi, 0)\, d\xi \le \frac{K}{2\sqrt{\delta}} \tag{8.36}$$

for all $j \ge j_0$. This means that, taking $\rho = \rho_j(N)$ with $j \ge j_0$, we have

$$\mathscr{E}_1^{\rho_j(N)}(t) \le \frac{K}{2\sqrt{\delta}} \tag{8.37}$$

for all $t \in \left[0, \hat{T}\left(\rho_j(N)\right)\right)$. From Definition (8.33) of $\hat{T}(\rho)$ it follows that $\hat{T}(\rho_j) = T$ when $j \ge j_0$. Hence, we obtain that

$$\begin{aligned}\int \mathscr{E}_1(\xi, t)\, d\xi &= \int_{|\xi|\le\rho_j(N)} \mathscr{E}_1(\xi, t)\, d\xi + \int_{|\xi|>\rho_j(N)} \mathscr{E}_1(\xi, t)\, d\xi \\ &\le \rho_j(N) K\left(\frac{2}{\sqrt{\delta}} + \frac{\sqrt{\mu}}{\delta}\right)\end{aligned} \tag{8.38}$$

in $[0, T)$ for all $j \ge j_0$. This completes the proof.

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Received May 22, 2007; revised July 27, 2007

R. Manfrin, DCA, Università IUAV di Venezia, Tolentini S. Croce 191, 30135 Venezia, Italy

E-mail: manfrin@iuav.it