Portugal. Math. (N.S.)
Vol. 66, Fasc. 1, 2009, 1–12

Portugaliae Mathematica

Existence and uniqueness of periodic solutions for a delay differential equation with piecewise constant arguments

Haihua Liang and Genqiang Wang

(Communicated by Luís Sanchez)

Abstract. We establish existence and uniqueness criteria for the periodic solutions of a first order differential equation with piecewise constant arguments.

Mathematics Subject Classification (2000). 34K13, 34A12.

Keywords. First order differential equation with piecewise constant arguments, periodic solutions, existence and uniqueness.

1. Introduction

It is known that many pattern forming processes in biomedical models of disease are described by delay differential equation with piecewise constant argument, a model developed originally by Busenerg and Cooke (we refer the reader to [2] for more details). The introductions of these equations were motivated by the fact that they represent a hybrid of discrete and continuous dynamical systems, so they combine the properties of both differential and differential-difference equation. In the recent years, these kind of equations have attracted the attention of many authors, and we refer the reader for the bibliographic references given at the end of this paper.

In [1] the authors consider the equation

$$y'(t) + ay(t) + by([t]) = 0, \qquad t \geq 0$$

where $[\cdot]$ denotes the greatest-integer function. There they determine the set of all periodic solutions of this equation.

In [12] the following more general equation is discussed:

$$y'(t) + a(t)y(t) + b(t)y([t]) = 0, \qquad t \geq 0 \tag{1.1}$$

where $a(t)$ and $b(t)$ are continuous real functions with positive integer period ω. Again the authors are able to identify the corresponding set of periodic solutions.

In the present paper we study the following equation

$$x'(t) + f\big(t, x(t-h), x([t-k])\big) = 0 \tag{1.2}$$

where h, k are integers, $f \in C(\mathbb{R}^3, \mathbb{R})$, and $f(t+T, u, v) = f(t, u, v)$ for $(t, u, v) \in \mathbb{R}^3$. Here the period T is assumed to be a positive integer. It is clear that (1.2) is a more general model than (1.1).

The main aim of this paper is to establish sufficient conditions for the existence and uniqueness of T-periodic solutions of (1.2). Our results are new and they also answer, to some extent, the question of what is the set of all periodic solutions of (1.2). An example to illustrate our theorem will be given in Section 3 below.

Definition 1.1. We will say that a function $x(t)$, defined on $\mathbb{R}$, is a solution of (1.2) if it satisfies the following conditions:

(i) $x(t)$ is continuous on $\mathbb{R}$;

(ii) the derivative $x'(t)$ exists at each point $t \in R$, with the possible exception of the points $[t] \in \mathbb{R}$, where one-side derivatives must exist;

(iii) equation (1.2) is satisfied on each interval $[n, n+1) \subset \mathbb{R}$ with integer endpoints.

Let us recall now some basic concepts related to the Mawhin's continuation theorem (for more details, see [7]). Let X and Y be two Banach space and $L : \mathrm{dom}(L) \cap X \to Y$ be a Fredholm map of index zero, i.e., L is a linear mapping such that $\mathrm{im}\, L$ is closed in Y and $\dim \ker L = \mathrm{codim}\, \mathrm{im}\, L < +\infty$. Then there exist continuous projectors $P : X \to \mathrm{dom}\, L$ and $Q : Y \to Y$ such that $\mathrm{im}\, P = \ker L$ and $\mathrm{im}\, L = \ker Q = \mathrm{im}(I - Q)$. It follows that $L|_{\mathrm{dom}\, L \cap \ker P} : \mathrm{dom}\, L \cap \ker P \to \mathrm{im}\, L$ is invertible. We denote the inverse of that map by K_P. Let Ω be an open bounded subset of X and $N : \overline{\Omega} \to Y$ be a continuous mapping. N is called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact. Note that $\mathrm{Im}\, Q$ is isomorphic to $\mathrm{Ker}\, L$, so we fix an isomorphism $J : \mathrm{im}\, Q \to \ker L$.

Theorem 1.1. *Lex X and Y be two Banach space and $L : \mathrm{dom}\, L \cap X \to Y$ be a Fredholm mapping of index zero. Suppose that Ω is open bounded in X and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. Furthermore, suppose that*

(i) *for each $\lambda \in (0, 1)$ and $x \in \partial\Omega$, $Lx \neq \lambda Nx$;*

(ii) *for each $x \in \partial\Omega \cap \ker L$, $QNx \neq 0$, and*

(iii) $\deg(JQN, \Omega \cap \ker L, 0) \neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \mathrm{dom}\, L$.

The following conditions will always be assumed throughout this paper.

(A) there exist constants b_1, b_2 with $b_1 + b_2 < \frac{2}{T}$ such that

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \le b_1|u_1 - u_2| + b_2|v_1 - v_2|;$$

(B) $f(t, u, v)$ is either strict monotonically decreasing with respect to u and v or strict monotonically increasing with respect to u and v.

We will also use the following conditions, where d and M are positive constants.

(C_1) $f(t, u, v) > 0$ for $t \in \mathbb{R}$ and $u, v \ge d$;

(C_2) $f(t, u, v) < 0$ for $t \in \mathbb{R}$ and $u, v \ge d$;

(C_3) $f(t, u, v) > 0$ for $t \in \mathbb{R}$ and $u, v \le -d$;

(C_4) $f(t, u, v) < 0$ for $t \in \mathbb{R}$ and $u, v \le -d$;

(D_1) $f(t, u, v) \ge -M$ for $(t, u, v) \in \mathbb{R}^3$;

(D_2) $f(t, u, v) \le M$ for $(t, u, v) \in \mathbb{R}^3$.

2. The main result

Using the Mawhin's continuation theorem, we will prove

Theorem 2.1. *If any of the following conditions are satisfied, then* (1.2) *has a unique T-periodic solution:*

(1) (C_1), (C_3) *and* (D_1);

(2) (C_2), (C_4) *and* (D_1);

(3) (C_1), (C_3) *and* (D_2);

(4) (C_2), (C_4) *and* (D_2).

We will only give a proof of existence of a unique solution of (1.2) under condition (1). The proofs assuming any of the other conditions are similar.

It is easy to see that $x(t)$ is a T-periodic solution of (1.2) if and only if $x(t)$ is a T-periodic solution of the equation

$$x(t) = x(0) - \int_0^t f\big(s, x(s-h), x([s-k])\big)\, ds. \tag{2.1}$$

Let

$$X_T = \{x \in C(\mathbb{R}, \mathbb{R}) \,|\, x(t+T) = x(t) \text{ for all } t \in \mathbb{R}\}$$

and endowed X with the norm

$$\|x\|_1 = \max_{t \in [0,T]} |x(t)|.$$

Let

$$Y = \{y \in C(\mathbb{R}, \mathbb{R}) \mid y(0) = 0, y(t) = \alpha t + h(t) \text{ for some } \alpha \in \mathbb{R}, h \in X_T\}$$

and endowed Y with the norm

$$\|y\|_2 = |\alpha| + \|h\|_1.$$

It is clear that both $(X_T, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ are Banach spaces. We define mappings $L, N : X_T \to Y$ by

$$Lx(t) = x(t) - x(0),$$
$$Nx(t) = -\int_0^t f\big(s, x(s-h), x([s-k])\big)\, ds.$$

Note that if we set

$$\alpha = -\frac{1}{T}\int_0^T f\big(s, x(s-h), x([s-k])\big)\, ds$$

and

$$h(t) = -\int_0^t f\big(s, x(s-h), x([s-k])\big)\, ds - \alpha t,$$

then one can easily check that $h(t)$ is a T-periodic function of t for any $x \in X_T$. This shows that N is well defined.

On the other hand, we have $\ker L = \{x \in X_T \mid x(t) = x(0) \text{ for all } t \in \mathbb{R}\}$ and $\operatorname{im} L = Y \cap X_T$. So we can also define projections $P : X_T \to X_T$ and $Q : Y \to Y$ as follows:

$$Px(t) = x(0),$$
$$Qy(t) = \alpha t \quad \text{if } y(t) = \alpha t + h(t).$$

Then $X_T = \ker P \oplus \ker L$ and $Y = \operatorname{im} L \oplus \operatorname{im} Q$. It is clear that $\operatorname{im} L = \{y \in X_T \mid y(0) = 0\}$ is closed in Y and $\operatorname{codim} \operatorname{im} Q = 1 = \dim \ker L$. Therefore L is a Fredholm operator of index zero.

Lemma 2.1. *Let L and N be defined as above. If Ω is an open bounded subset of X_T, then N is L-compact on $\overline{\Omega}$.*

Proof. It is easy to see that

$$QNx(t) = -\frac{t}{T}\int_0^T f\big(s, x(s-h), x([s-k])\big)\, ds, \tag{2.2}$$

hence

$$\|QNx(t)\|_2 = \left|\frac{1}{T}\int_0^T f\big(s, x(s-h), x([s-k])\big)\, ds\right|. \tag{2.3}$$

Let us denote the inverse of the map $L|_{\operatorname{dom} L \cap \ker P}\colon \operatorname{dom} L \cap \ker P \to Y$ by K_P. Then

$$\begin{aligned} K_P(I-Q)Nx(t) = {} & -\int_0^t f\big(s, x(s-h), x([s-k])\big)\, ds \\ & + \frac{t}{T}\int_0^T f\big(s, x(s-h), x([s-k])\big)\, ds. \end{aligned} \tag{2.4}$$

From (2.3) we find that $QN(\overline{\Omega})$ is bounded. Since (2.4) holds and N is a completely continuous mapping, the Ascoli–Arzela theorem shows that $K_P(I-Q)N\overline{\Omega}$ is relatively compact, thus N is L-compact on $\overline{\Omega}$. □

Denote now by C_Δ be the set of all $x(t) \in X_T$ such that $x'(t)$ exists at each $t \in R$ with the possible exception of $[t]$, where a one-side derivative exists. Then we have:

Lemma 2.2. *Suppose that $x(t) \in C_\Delta$ and $\xi \in [0, T]$. Then*

$$\|x\|_1 \le |x(\xi)| + \frac{1}{2}\int_0^T |x'(s)|\, ds.$$

Proof. It is clear that both $x'(t)$ and $|x'(t)|$ are integrable in any $[a, b] \subset \mathbb{R}$. Since $x(t)$ is continuous for any $t \in [\xi, \xi + T]$, we have

$$x(t) = x(\xi) + \int_\xi^t x'(s)\, ds$$

and

$$x(t) = x(\xi + T) + \int_{\xi+T}^t x'(s)\, ds = x(\xi) - \int_t^{\xi+T} x'(s)\, ds.$$

These two equations show that for any $t \in [\xi, \xi + T]$,

$$2x(t) = 2x(\xi) + \int_{\xi}^{t} x'(s)\, ds - \int_{t}^{\xi+T} x'(s)\, ds,$$

or

$$x(t) = x(\xi) + \frac{1}{2}\left\{ \int_{\xi}^{t} x'(s)\, ds - \int_{t}^{\xi+T} x'(s)\, ds \right\}.$$

This yields

$$|x(t)| \le |x(\xi)| + \frac{1}{2}\int_{\xi}^{\xi+T} |x'(s)|\, ds.$$

Noting that $x(t) \in X_T$, the above inequality obviously implies that

$$\|x\|_1 \le |x(\xi)| + \frac{1}{2}\int_{0}^{T} |x'(s)|\, ds.$$

□

Now we can show

Lemma 2.3. *Equation* (1.2) *has at most one T-periodic solution.*

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of (1.2). Set $z(t) = x_1(t) - x_2(t)$. Then we have

$$z'(t) + f\big(t, x_1(t-h), x_1([t-k])\big) - f\big(t, x_2(t-h), x_2([t-k])\big) = 0. \quad (2.5)$$

Therefore, $z(t) \in C_\Delta$. Now we have two possibilities:

Case (i): For all $t \in [0, T]$, $z(t) \neq 0$. Without loss of generality, we may assume that $z(t) > 0$, i.e., $x_1(t) > x_2(t)$ for all $t \in [0, T]$. Integrating (2.5) from 0 to T, we find

$$\int_0^T \big[f\big(t, x_1(t-h), x_1([t-k])\big) - f\big(t, x_2(t-h), x_2([t-k])\big)\big]\, dt = 0. \quad (2.6)$$

Combining condition (B) and $x_1(t) > x_2(t)$, either

$$f\big(t, x_1(t-h), x_1([t-k])\big) - f\big(t, x_2(t-h), x_2([t-k])\big) > 0 \qquad \text{for all } t \in [0, T]$$

or

$$f\big(t, x_1(t-h), x_1([t-k])\big) - f\big(t, x_2(t-h), x_2([t-k])\big) < 0 \qquad \text{for all } t \in [0, T]$$

holds. This contradicts (2.6).

Case (ii): There exists some $\xi \in [0, T]$ such that $z(\xi) = 0$. It then follows from Lemma 2.2 that

$$\|z\|_1 \le |z(\xi)| + \frac{1}{2}\int_0^T |z'(s)|\,ds = \frac{1}{2}\int_0^T |z'(s)|\,ds.$$

On the other hand, condition (A) leads to

$$\begin{aligned}|z'(t)| &= \left|f\big(t, x_1(t-h), x_1([t-k])\big) - f\big(t, x_2(t-h), x_2([t-k])\big)\right| \\ &\le b_1|x_1(t-h) - x_2(t-h)| + b_2|x_1([t-k]) - x_2([t-k])| \\ &\le (b_1 + b_2)\|z\|_1.\end{aligned}$$

Integrating both sides of the above inequality from 0 to T, we obtain

$$\int_0^T |z'(t)|\,dt \le (b_1 + b_2)T\|z\|_1 \le \frac{b_1 + b_2}{2}T\int_0^T |z'(t)|\,dt.$$

Since $b_1 + b_2 < \frac{2}{T}$, we conclude that

$$\int_0^T |z'(t)|\,dt = 0.$$

Therefore, we must have

$$z'(t) = 0 \qquad \text{for } t \in [0, T]\backslash\{0, 1, \dots, T\}.$$

Now, since $z(t)$ is continuous, there exists a constant C such that $z(t) = C$, or $x_1(t) = x_2(t) + C$. Thus, we deduce from (2.5) that, for $t \in [0, T]\backslash\{0, 1, \dots, T\}$, we have

$$f\big(t, x_1(t-h), x_1([t-k])\big) - f\big(t, x_2(t-h), x_2([t-k])\big) = 0.$$

Combining condition (B) and $x_1(t) = x_2(t) + C$, we conclude that $C = 0$.

Hence, in any case, $x_1(t) = x_2(t)$ for all $t \in [0, T]$, so equation (1.2) has at most one T-periodic solution. □

Next, in order to apply Theorem 1.1, we consider the following equation:

$$x(t) = x(0) - \lambda\int_0^t f\big(s, x(s-h), x([s-k])\big)\,ds = 0, \tag{2.7}$$

where $\lambda \in (0, 1)$.

The following lemma is clear, so we omit its proof.

Lemma 2.4. *Suppose that $g(t)$ is a real bounded continuous function on $[a,b)$ and $\lim_{t\to b^-} g(t)$ exists. Then there is a point $\xi \in (a,b)$ such that*

$$\int_a^b g(s)\,ds = g(\xi)(b-a).$$

Lemma 2.5. *Suppose that (C_1), (C_3) and (D_1) hold. Then, for any T-periodic solution $x(t)$ of (2.7), we have $|x(t)| \le d + 2TM$ for all $t \in [0,T]$.*

Proof. Let $x(t)$ be a T-periodic solution of (2.7). Then we have

$$\int_0^T f\big(s, x(s-h), x([s-k])\big)\,ds = 0. \tag{2.8}$$

We claim that there exists a $t_1 \in [0,T]$ such that

$$|x(t_1)| < d. \tag{2.9}$$

First note that, by Lemma 2.4, there exist $\xi_i \in (i-1,i)$ $(i = 1,2,\ldots,T)$ such that

$$\begin{aligned} 0 &= \int_0^T f\big(s, x(s-h), x([s-k])\big)\,ds \\ &= \sum_{i=1}^T \int_{i-1}^i f\big(s, x(s-h), x([s-k])\big)\,ds \\ &= \sum_{i=1}^T \big[f\big(\xi_i, x(\xi_i - h), x(i-1-k)\big)\big]. \end{aligned}$$

If $|x(i-1-k)| < d$ for some i, then $|x(\xi_i - h)| < d$, and we are done. Otherwise, by (C_1), (C_3) and the above equation, there must exist η_1 and η_2 such that $x(\eta_1) \ge d$ and $x(\eta_2) \le -d$. Noting that $x(t)$ is continuous on R, the mean value theorem yields $x(\eta_3)$ such that $-d < x(\eta_3) < d$. Since $x(t)$ is periodic, there is some $t_1 \in [0,T]$ such that $|x(t_1)| = |x(\eta_3)| < d$. So the claim follows.

Next, if we write

$$F_+(t) = \max\big\{f\big(t, x(t-h), x([t-k])\big), 0\big\} \tag{2.10}$$

and

$$F_-(t) = \max\big\{-f\big(t, x(t-h), x([t-k])\big), 0\big\}, \tag{2.11}$$

then $F_+(t)$ and $F_-(t)$ are piecewise continuous functions on $\mathbb{R}$, and we have

$$f\big(t, x(t-h), x([t-k])\big) = F_+(t) - F_-(t), \tag{2.12}$$

and

$$\big|f\big(t, x(t-h), x([t-k])\big)\big| = F_+(t) + F_-(t).$$

In view of (D_1) and (2.11), the last equation yields that

$$|F_-(t)| = F_-(t) \le M, \qquad t \in \mathbb{R}.$$

Therefore

$$\int_0^T F_-(s)\,ds \le TM.$$

From (2.8), (2.12) and the last equation, we have

$$\int_0^T F_+(s)\,ds = \int_0^T F_-(s)\,ds \le TM.$$

Hence

$$\int_0^T \big|f\big(t, x(t-h), x([t-k])\big)\big| \le 2TM.$$

The last equation combined with (2.9) shows that for any $t \in [0, T]$,

$$\begin{aligned} |x(t)| &= \left|x(t_1) - \lambda \int_{t_1}^t \big[f\big(s, x(s-h), x([s-k])\big)\big]\,ds\right| \\ &\le |x(t_1)| + \lambda \int_0^T \big|f\big(s, x(s-h), x([s-k])\big)\big|\,ds \\ &\le d + 2MT. \end{aligned}$$

This completes the proof of the lemma. □

Proof of Theorem 2.1. It follows from Lemma 2.3 that (1.2) has at most one T-periodic solution. Thus to prove Theorem 2.1, it suffices to show that (1.2) has at least one T-periodic solution. To do this, we apply Theorem 1.1.

We set

$$\Omega := \{x \in X_T \mid \|x\|_1 < D\},$$

where D is a fixed constant with $D > d + 2MT$. It is easy to see that Ω is an open bounded subset of X_T, L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$. By Lemma 2.5, we have that $D > d + 2MT$ and that

$$Lx \neq \lambda Nx \quad \text{for each } \lambda \in (0,1),\ x \in \partial\Omega.$$

Since $\ker L = \{x \in X_T \mid x(t) = x(0) \text{ for all } t \in \mathbb{R}\}$, we see that a function $x \in \ker L \cap \partial\Omega$ must be constant with either $x(t) \equiv D$ or $x(t) \equiv -D$. If the isomorphism $J : \operatorname{im} Q \to \ker L$ is defined by $J(\alpha t) = \alpha$, $\alpha \in \mathbb{R}$, then by (2.2), (C_1) and (C_3) we get

$$JQN(x) = -\frac{1}{T}\int_0^T f(s,x,x)\,ds \neq 0. \tag{2.13}$$

In particular, we see that

$$-\frac{1}{T}\int_0^T f(s,D,D)\,ds < 0, \qquad -\frac{1}{T}\int_0^T f(s,-D,-D)\,ds > 0,$$

which shows that

$$\deg(JQN, \Omega \cap \ker L, 0) \neq 0.$$

In this way, we conclude from Lemma 2.3 and Theorem 1.1 that (1.2) has a unique T-periodic solution. □

3. An example

We claim that the equation

$$x'(t) + \frac{1}{3+\cos(\pi t)}x([t-k]) + \frac{2}{5\pi}\big(x(t-h)-1\big) + \ln\big(1+2\sin^2(\pi t)\big) = 0, \tag{3.1}$$

where k and h are fixed integers, has a unique 2-periodic solution.

Proof. Note that equation (3.1) amounts to (1.2) with the choice

$$f(t,x,y) = \frac{1}{3+\cos(\pi t)}y + \frac{2}{5\pi}(x-1) + \ln\big(1+2\sin^2(\pi t)\big).$$

Hence, we have $T = 2$, $b_1 = \frac{1}{2}$, $b_2 = \frac{2}{5\pi}$. It is straightforward to check that conditions (1) in Theorem 2.1 hold. Therefore, (3.1) has a unique 2-periodic solution. □

References

[1] A. R. Aftabizadeh, J. Wiener, and J.-M. Xu, Oscillatory and periodic solutions of delay differential equations with piecewise constant argument. *Proc. Amer. Math. Soc.* **99** (1987), 673–679. Zbl 0631.34078 MR 877038

[2] S. Busenberg and K. L. Cooke, Models of vertically transmitted diseases with sequential-continuous dynamics. In *Nonlinear phenomena in mathematical sciences*, Academic Press, New York 1982, 179–187. Zbl 0512.92018

[3] L. A. V. Carvalho and K. L. Cooke, A nonlinear equation with piecewise continuous argument. *Differential Integral Equations* **1** (1988), 359–367. Zbl 0723.34061 MR 929923

[4] K. L. Cooke and J. Wiener, Retarded differential equations with piecewise constant delays. *J. Math. Anal. Appl.* **99** (1984) 265–297. Zbl 0557.34059 MR 0732717

[5] K. L. Cooke and J. Wiener, An equation alternately of retarded and advanced type. *Proc. Amer. Math. Soc.* **99** (1987), 726–732. Zbl 0628.34074 MR 0877047

[6] K. L. Cooke and J. Wiener, A survey of differential equations with piecewise continuous arguments. In *Delay differential equations and dynamical systems*, Lecture Notes in Math. 1475, Springer-Verlag, Berlin 1991, 1–15. Zbl 0737.34045 MR 1132014

[7] R. E. Gaines and J. L. Mawhin, *Coincidence degree, and nonlinear differential equations*. Lecture Notes in Math. 568, Springer-Verlag, Berlin 1977. Zbl 0339.47031 MR 0637067

[8] K. Gopalsamy, M. R. S. Kulenovic, and G. Ladas, On a logistic equation with piecewise constant arguments. *Differential Integral Equations* **4** (1991), 215–223. Zbl 0727.34061 MR 1079622

[9] Y. K. Huang, Oscillations and asymptotic stability of solutions of first order neutral differential equations with piecewise constant argument. *J. Math. Anal. Appl.* **149** (1992), 70–85. Zbl 0704.34078 MR 1054794

[10] L. C. Lin and G. Wang, Oscillatory and asymptotic behaviour of first order nonlinear differential equations with retarded argument $[t]$. *Chinese Sci. Bull.* **36** (1991), 889–891. Zbl 0741.34046 MR 1138294

[11] G. Wang, Existence theorem of periodic solutions for a delay nonlinear differential equation with piecewise constant arguments. *J. Math. Anal. Appl.* **298** (2004), 298–307. Zbl 1071.34071 MR 2086548

[12] G.-Q. Wang and S. S. Cheng, Note on the set of periodic solutions of a delay differential equation with piecewise constant argument. *Internat J. Pure Appl. Math.* **9** (2003), 139–143. Zbl 1057.34524 MR 2019990

[13] J. Wiener, Boundary value problems for partial differential equations with piecewise constant delay. *Internat. J. Math. Math. Sci.* **14** (1991), 363–379. Zbl 0749.35061 MR 1096879

Received February 27, 2007; revised January 10, 2008

H. Liang, G. Wang, Department of Computer Science, Guangdong Polytechnic Normal University, Guangzhou, Guangdong 510665, P. R. China
E-mail: haiihuaa@tom.com