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Existence of a renormalized solution for a class of nonlinear parabolic equations in Orlicz spaces

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Abstract. An existence result for a solution of a class of nonlinear parabolic equations in Orlicz spaces is established. The data belongs to L^1, no growth assumption is made on the nonlinearities and the N-function does not satisfy the Δ_2 condition.

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1. Introduction

In this work we are concerned with the problem of existence of a renormalized solution for a class of nonlinear parabolic equations of the type:

$$\frac{\partial b(u)}{\partial t} - \operatorname{div}\bigl(a(x,t,u,\nabla u)\bigr) - \operatorname{div}\bigl(\Phi(u)\bigr) = f \text{ in } \Omega \times (0,T), \tag{1.1}$$

$$b(u)(t=0) = b(u_0) \text{ in } \Omega, \tag{1.2}$$

$$u = 0 \text{ on } \partial\Omega \times (0,T). \tag{1.3}$$

Here Ω is a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, T is a positive real number, and $Q = \Omega \times (0,T)$. The function b is assumed to be a strictly increasing C^1-function. When the data f and $b(u_0)$ lie in $L^1(Q)$ and $L^1(\Omega)$, respectively, then $Au = -\operatorname{div}\bigl(a(x,t,u,\nabla u)\bigr)$ is a Leray–Lions operator defined on $W_0^{1,x}L_M(\Omega)$, where M is an appropriate N-function (see assumptions (3.2)–(3.5) in Section 3). The function Φ is assumed to be continuous on $\mathbb{R}$.

The difficulties that arise in problem (1.1)–(1.3) are due to the following facts: the data f and $b(u_0)$ only belong to L^1, the function $\Phi(u)$ does not belong to $\bigl(L^1_{\text{loc}}((0,T)\times\Omega)\bigr)^N$ (because the function Φ is just assumed to be continuous on $\mathbb{R}$) and the N-function M does not satisfy the Δ_2 condition (see (2.1) of Section 2). Therefore, proving existence of a weak solution (i.e., in the distribution meaning) seems to be a hard task. To overcome this difficulty, in this paper we will apply the framework of renormalized solutions. This notion was introduced by Lions and Di Perna [16] in their study of the Boltzmann equation (see also P.-L. Lions [26] for a few applications to fluid mechanics models). This notion was then adapted to an elliptic version of (1.1), (1.2), (1.3) by Boccardo, J.-L. Diaz, D. Giachetti, F. Murat [11], and by F. Murat [27]. At the same time, the equivalent notion of entropy solution was developed independently by Bénilan and al. [2] for the study of nonlinear elliptic problems.

For the parabolic equation (1.1)–(1.3) the existence and uniqueness of a renormalized solution has been proved by D. Blanchard, F. Murat and H. Redwane [6] (see also A. Porretta [28]) in the case where $b(u) = u$ and f is replaced by $f + \operatorname{div}(g)$, with $g \in L^{p'}(Q)^N$. The case where the operator $Au = -\operatorname{div}\bigl(a(x,t,u,\nabla u)\bigr)$ is a Leray–Lions which is coercive and grows like $|\nabla u|^{p-1}$ with respect to ∇u (but which is not restricted by any growth condition with respect to u), where b is a strictly increasing function of u (that can possibly blow up for some finite r_0) and $a(x,t,s,\xi)$ is independent of s and linear with respect to ξ, existence and uniqueness has been established by D. Blanchard and H. Redwane [9]. The case where b is a maximal monotone graph on $\mathbb{R}$ and $a(x,t,s,\xi)$ is independent of t, existence and uniqueness has been established by D. Blanchard and A. Porretta [8] (see also D. Blanchard [4], D. Blanchard and F. Murat [5], H. Redwane [30], J. Carrillo [12], J. Carrillo and P. Wittbold [13], [14]).

Let us remark that equations (1.1)–(1.3) find natural applications in physical sciences. Non-standard examples of N-functions which occur in the mechanics of solids and fluids include $M_1(t) = t\log(1+t)$, $M_2(t) = \int_0^t s^{1-\alpha}\bigl(\operatorname{arcsinh}(s)\bigr)^{\alpha}\,ds$ where $0 \le \alpha \le 1$ and $M_3(t) = t\log\bigl(1+\log(1+t)\bigr)$ (see M. Fuchs and L. Gongbao [19] and M. Fuchs and G. Seregin ([20]–[31]). Note that that $M_1(t)$ and $M_3(t)$ do not satisfy the Δ_2-condition.

As an application of our results, we prove the existence of a renormalized solution of the problem

$$e^{\alpha_1 u}\frac{\partial u}{\partial t} - \operatorname{div}\bigl((1+|u|)\nabla u \log^{2\beta}(e+|\nabla u|)\bigr) - \operatorname{div}(e^{\alpha_2 u}) = f \text{ in } \Omega \times (0,T), \tag{1.4}$$

$$u(t=0) = u_0 \text{ in } \Omega, \tag{1.5}$$

$$u = 0 \text{ on } \partial\Omega \times (0,T), \tag{1.6}$$

where $\alpha_1 \ge 0$, $\alpha_2 \in \mathbb{R}$ and $0 < \beta \le \frac{1}{2}$.

The plan of the paper is as follows. In Section 2 we give some preliminaries and the definition of N-function and Orlicz–Sobolev space. In Section 3 we make precise all the assumptions on b, Φ, f and u_0. In Section 4 we give the definition of a renormalized solution of (1.1)–(1.3). In Section 5 we establish the existence of such a solution (Theorem 5.1).

2. Preliminaries

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e., M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$. Equivalently, M admits the representation: $M(t) = \int_0^t a(s)\,ds$ where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t) \to \infty$ as $t \to \infty$. The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \bar{a}(s)\,ds$, where $\bar{a} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\bar{a}(t) = \sup\{s \,|\, a(s) \le t\}$.

The N-function M is said to satisfy the Δ_2 condition if, for some $k > 0$,

$$M(2t) \le kM(t) \qquad \text{for all } t \ge 0. \tag{2.1}$$

When this inequality holds only for $t \ge t_0 > 0$, M is said to satisfy the Δ_2-condition near infinity.

Let P and M be two N-functions. $P \ll M$ means that P grows essentially less rapidly than M, i.e., for each $\varepsilon > 0$,

$$\frac{P(t)}{M(\varepsilon t)} \to 0 \qquad \text{as } t \to \infty. \tag{2.2}$$

This is the case if and only if

$$\frac{M^{-1}(t)}{P^{-1}(t)} \to 0 \qquad \text{as } t \to \infty. \tag{2.3}$$

We will extend these N-functions into even functions on all $\mathbb{R}$. Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $\mathscr{L}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that:

$$\int_\Omega M\big(u(x)\big)\,dx < +\infty \qquad \left(\text{resp. } \int_\Omega M\left(\frac{u(x)}{\lambda}\right) dx < +\infty \text{ for some } \lambda > 0\right). \tag{2.4}$$

Note that $L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 \,\middle|\, \int_\Omega M\left(\frac{u(x)}{\lambda}\right) dx \le 1\right\} \tag{2.5}$$

and $\mathscr{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2-condition for all t or for t large, according to whether or not Ω has infinite measure.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)\,dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$. The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2 condition for all t or for t large, according to whether or not Ω has infinite measure.

We now turn to the Orlicz–Sobolev space. $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). This is a Banach space under the norm

$$\|u\|_{1,M,\Omega} = \sum_{|\alpha|\le 1} \|\nabla^\alpha u\|_{M,\Omega}. \tag{2.6}$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathscr{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathscr{D}(\Omega)$ in $W^1L_M(\Omega)$. We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$, $\int_\Omega M(\frac{\nabla^\alpha u_n - \nabla^\alpha u}{\lambda})\,dx \to 0$ for all $|\alpha| \le 1$. This implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. If M satisfies the Δ_2 condition on $\mathbb{R}^+$ (near infinity only when Ω has finite measure), then modular convergence coincides with norm convergence.

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property, then the space $\mathscr{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (cf. [21]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined. For more details see [1], [23].

For $K > 0$, we define the truncation at height K, $T_K : \mathbb{R} \to \mathbb{R}$ by

$$T_K(s) = \min\big(K, \max(s, -K)\big) = \begin{cases} s & \text{if } |s| \le K, \\ \frac{Ks}{|s|} & \text{if } |s| > K. \end{cases} \tag{2.7}$$

The following abstract lemmas will be applied to the truncation operators.

Lemma 2.1 (cf. [21]). *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly lipschitzian, with $F(0) = 0$. Let M be an N-function and let $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$).*

Then $F(u) \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set of discontinuity points D of F' is finite, then

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega \mid u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega \mid u(x) \in D\}. \end{cases}$$

Lemma 2.2 (cf. [21]). *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly lipschitzian, with $F(0) = 0$. We suppose that the set of discontinuity points of F' is finite. Let M be an N-function. Then the mapping $F : W^1 L_M(\Omega) \to W^1 L_M(\Omega)$ is sequentially continuous with respect to the weak-* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$.*

Let Ω be a bounded open subset of $\mathbb{R}^N$, $T > 0$ and set $Q = \Omega \times (0, T)$. Let M be an N-function. For each $\alpha \in \mathbb{N}^N$, denote by ∇_x^α the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{N}^N$. The inhomogeneous Orlicz–Sobolev spaces are defined as

$$\begin{aligned} W^{1,x} L_M(Q) &= \{u \in L_M(Q) \mid \nabla_x^\alpha u \in L_M(Q) \text{ for all } |\alpha| \le 1\}, \\ W^{1,x} E_M(Q) &= \{u \in E_M(Q) \mid \nabla_x^\alpha u \in E_M(Q) \text{ for all } |\alpha| \le 1\}. \end{aligned} \tag{2.8}$$

The last space is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha| \le 1} \|\nabla_x^\alpha u\|_{M,Q}. \tag{2.9}$$

We can easily show that they form a complementary system when Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_M(Q)$ which have as many copies as there is α-order derivatives, $|\alpha| \le 1$. We shall also consider the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. If $u \in W^{1,x} L_M(Q)$ then the function $t \to u(t) = u(t, \cdot)$ is defined on $(0, T)$ with values in $W^1 L_M(\Omega)$. If, further, $u \in W^{1,x} E_M(Q)$ then the concerned function is a $W^1 E_M(\Omega)$-valued and is strongly measurable. Furthermore the following imbedding holds: $W^{1,x} E_M(Q) \subset L^1\big(0, T; W^1 E_M(\Omega)\big)$. The space $W^{1,x} L_M(Q)$ is not in general separable. If $u \in W^{1,x} L_M(Q)$, we cannot conclude that the function $u(t)$ is measurable on $(0, T)$. However, the scalar function $t \mapsto \|u(t)\|_{M,\Omega}$ is in $L^1(0, T)$. The space $W_0^{1,x} E_M(Q)$ is defined as the (norm) closure in $W^{1,x} E_M(Q)$ of $\mathscr{D}(Q)$. We can easily show as in [22] that when Ω has the segment property, then each element u of the closure of $\mathscr{D}(Q)$ with respect of the weak-* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ is a limit, in $W^{1,x} L_M(Q)$, of some subsequence $(u_i) \subset \mathscr{D}(Q)$ for the modular convergence; i.e., there exists $\lambda > 0$ such that for all $|\alpha| \le 1$,

$$\int_Q M\left(\frac{\nabla_x^\alpha u_i - \nabla_x^\alpha u}{\lambda}\right) dx\, dt \to 0 \qquad \text{as } i \to \infty. \tag{2.10}$$

This implies that (u_i) converges to u in $W^{1,x} L_M(Q)$ for the weak topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. Consequently,

$$\overline{\mathscr{D}(Q)}^{\sigma(\Pi L_M, \Pi E_{\overline{M}})} = \overline{\mathscr{D}(Q)}^{\sigma(\Pi L_M, \Pi L_{\overline{M}})}. \tag{2.11}$$

This space will be denoted by $W_0^{1,x}L_M(Q)$. Furthermore, $W_0^{1,x}L_M(Q) \cap \Pi E_M = W_0^{1,x}E_M(Q)$. Poincaré's inequality also holds in $W_0^{1,x}L_M(Q)$, i.e., there is a constant $C > 0$ such that for all $u \in W_0^{1,x}L_M(Q)$ one has,

$$\sum_{|\alpha| \le 1} \|\nabla_x^\alpha u\|_{M,Q} \le C \sum_{|\alpha|=1} \|\nabla_x^\alpha u\|_{M,Q}. \tag{2.12}$$

Thus both sides of the last inequality are equivalent norms on $W_0^{1,x}L_M(Q)$. We have then the following complementary system

$$\begin{pmatrix} W_0^{1,x}L_M(Q) & F \\ W_0^{1,x}E_M(Q) & F_0 \end{pmatrix}, \tag{2.13}$$

with F being the dual space of $W_0^{1,x}E_M(Q)$. It is also, except for an isomorphism, the quotient of $\Pi L_{\overline{M}}$ by the polar set $W_0^{1,x}E_M(Q)^\perp$, and will be denoted by $F = W^{-1,x}L_{\overline{M}}(Q)$. It is shown that

$$W^{-1,x}L_{\overline{M}}(Q) = \Big\{ f = \sum_{|\alpha| \le 1} \nabla_x^\alpha f_\alpha \mid f_\alpha \in L_{\overline{M}}(Q) \Big\}. \tag{2.14}$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha| \le 1} \|f_\alpha\|_{\overline{M},Q}, \tag{2.15}$$

where the infinum is taken on all possible decompositions

$$f = \sum_{|\alpha| \le 1} \nabla_x^\alpha f_\alpha, \qquad f_\alpha \in L_{\overline{M}}(Q). \tag{2.16}$$

The space F_0 is then given by

$$F_0 = \Big\{ f = \sum_{|\alpha| \le 1} \nabla_x^\alpha f_\alpha \mid f_\alpha \in E_{\overline{M}}(Q) \Big\} \tag{2.17}$$

and is denoted by $F_0 = W^{-1,x}E_{\overline{M}}(Q)$.

Remark 2.3. We can easily check, using Lemma 2.1, that each uniformly lipschitzian mapping F, with $F(0) = 0$, acts in inhomogeneous Orlicz–Sobolev spaces of order 1: $W^{1,x}L_M(Q)$ and $W_0^{1,x}L_M(Q)$.

In the sequel we have to use the following results which concern mollification with respect to time and space variable and some trace results.

Thus we define for all $\mu > 0$ and all $(x,t) \in Q$:

$$u_\mu(x,t) = \mu \int_{-\infty}^{t} \tilde{u}(x,s)\exp\big(\mu(s-t)\big)\,ds \qquad \text{where } \tilde{u}(x,s) = u(x,s)\chi_{(0,T)}. \quad (2.18)$$

Lemma 2.4 (cf. [17]). (1) *If $u \in L_M(Q)$ then $u_\mu \to u$ as $\mu \to +\infty$ in $L_M(Q)$ for the modular convergence.*

(2) *If $u \in W^{1,x}L_M(Q)$ then $u_\mu \to u$ as $\mu \to +\infty$ in $W^{1,x}L_M(Q)$ for the modular convergence.*

(3) *If $u \in W^{1,x}L_M(Q)$ then $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu) \in W^{1,x}L_M(Q)$.*

We will use the following technical lemmas.

Lemma 2.5 (cf. [17]). *Let M be an N-function. Let (u_n) be a sequence of $W^{1,x}L_M(Q)$ such that $u_n \rightharpoonup$ weakly in $W^{1,x}L_M(Q)$ for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\frac{\partial u_n}{\partial t} = h_n + k_n$ in $D'(Q)$ with h_n is bounded in $W^{-1,x}L_{\overline{M}}(Q)$ and k_n is bounded in the space $L^1(Q)$. Then $u_n \to u$ strongly in $L^1_{\mathrm{loc}}(Q)$.*

If further, $u_n \in W_0^{1,x}L_M(Q)$ then $u_n \to u$ strongly in $L^1(Q)$.

Lemma 2.6 (cf. [18]). *Let Ω be a bounded open subset of $\mathbb{R}^N$ with the segment property. Then*

$$\left\{ u \in W_0^{1,x}L_{\overline{M}}(Q) \,\middle|\, \frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q) + L^1(Q) \right\} \subset C\big([0,T], L^1(\Omega)\big).$$

Lemma 2.7 (cf. [3]). *Let Q be an open bounded subset of $\mathbb{R}^N$ which satisfies the segment property. If $u \in W_0^1 L_M(\Omega)$, then*

$$\int_Q \operatorname{div} u \, dx\, dt = 0.$$

3. Assumptions and statement of main results

Throughout this paper, we assume that the following assumptions hold true:

Ω is a bounded open set on $\mathbb{R}^N$ $(N \geq 2)$, $T > 0$ is given and we set $Q = \Omega \times (0,T)$,

$$b : \mathbb{R} \to \mathbb{R} \text{ is a strictly increasing } C^1\text{-function with } b(0) = 0. \quad (3.1)$$

Let M and P be two N-function such that $P \ll M$. Consider a second order partial differential operator $A : D(A) \subset W^{1,x}L_M(Q) \to W^{-1,x}L_{\overline{M}}(Q)$ in divergence form,

$$A(u) = -\operatorname{div}\big(a(x,t,u,\nabla u)\big),$$

where

$$a : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \text{ is a Carathéodory function satisfying} \tag{3.2}$$

for almost every $(x,t) \in Q$ and for every $s \in \mathbb{R}$, $\xi \neq \xi^* \in \mathbb{R}^N$,

$$|a(x,t,s,\xi)| \leq \beta[c(x,t) + k_1\overline{M}^{-1}P(k|s|) + \overline{M}^{-1}M(k|\xi|)], \tag{3.3}$$

$$[a(x,t,s,\xi) - a(x,t,s,\xi^*)][\xi - \xi^*] > 0, \tag{3.4}$$

$$a(x,t,s,\xi)\xi \geq \alpha M(|\xi|), \tag{3.5}$$

where $c(x,t) \in E_{\overline{M}}(Q)$, $c \geq 0$ and $\alpha, \beta, k > 0$ are a given real numbers. Suppose that

$$\Phi : \mathbb{R} \to \mathbb{R}^N \text{ is a continuous function,} \tag{3.6}$$

$$f \text{ is an element of } L^1(Q), \tag{3.7}$$

$$u_0 \text{ is an element of } L^1(\Omega) \text{ such that } b(u_0) \in L^1(\Omega). \tag{3.8}$$

Remark 3.1. As already mentioned in the introduction, problem (1.1)–(1.3) does not admit a weak solution under the assumptions (3.1)–(3.8) (even when $b(u) = u$) since the growths of $\Phi(u)$ is not controlled with respect to u so that the term $-\operatorname{div}\big(\Phi(u)\big)$ is not in general defined as a distribution, even when u belongs to $W_0^{1,x}L_M(Q)$.

Let prove the following lemma which will be needed later.

Lemma 3.2. *With the assumptions* (3.2)–(3.5) *let* (z_n) *be a sequence in* $W_0^{1,x}L_M(Q)$ *such that*

$$z_n \rightharpoonup z \text{ in } W_0^{1,x}L_M(Q) \text{ for } \sigma\big(\Pi L_M(Q), \Pi E_{\overline{M}}(Q)\big), \tag{3.9}$$

$$\big(a(x,t,z_n,\nabla z_n)\big)_n \text{ is bounded in } \big(L_{\overline{M}}(Q)\big)^N, \tag{3.10}$$

$$\int_Q [a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]\,dx\,dt \to 0 \tag{3.11}$$

as n and s tend to $+\infty$, *and where* χ_s *is the characteristic function of*

$$Q_s = \{(x,t) \in Q \,|\, |\nabla z| \leq s\}.$$

Then

$$\nabla z_n \to \nabla z \ \textit{a.e. in } Q, \tag{3.12}$$

$$\lim_{n\to\infty} \int_Q a(x,t,z_n,\nabla z_n)\nabla z_n \,dx\,dt = \int_Q a(x,t,z,\nabla z)\nabla z \,dx\,dt, \tag{3.13}$$

$$M(|\nabla z_n|) \to M(|\nabla z|) \ \textit{in } L^1(Q). \tag{3.14}$$

Proof. Fix $r > 0$. Let $s > r$ and $Q_r = \{(x,t) \in Q \,|\, |\nabla z| \le r\}$. We have

$$\begin{aligned} 0 &\le \int_{Q_r} [a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z)][\nabla z_n - \nabla z] \,dx\,dt \\ &\le \int_{Q_s} [a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z)][\nabla z_n - \nabla z] \,dx\,dt \\ &= \int_{Q_s} [a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s] \,dx\,dt \\ &\le \int_Q [a(x,z_n,\nabla z_n) - a(x,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s] \,dx\,dt. \end{aligned} \tag{3.15}$$

Together with (3.11) this implies that

$$\lim_{n\to\infty} \int_{Q_r} [a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z)][\nabla z_n - \nabla z] \,dx\,dt = 0. \tag{3.16}$$

Following the same argument as in [21], one can show that

$$\nabla z_n \to \nabla z \text{ a.e. in } Q. \tag{3.17}$$

On the one hand we have

$$\begin{aligned} \int_Q a(x,t,z_n,\nabla z_n)\nabla z_n \,dx &= \int_Q [a(x,t,z_n,\nabla z_n) - a(x,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s] \,dx\,dt \\ &\quad + \int_Q a(x,t,z_n,\nabla z\chi_s)(\nabla z_n - \nabla z\chi_s) \,dx\,dt \\ &\quad + \int_Q a(x,t,z_n,\nabla z_n)\nabla z\chi_s \,dx\,dt. \end{aligned} \tag{3.18}$$

Since $\big(a(x,t,z_n,\nabla z_n)\big)_n$ is bounded in $\big(L_{\overline{M}}(Q)\big)^N$, using (3.17) we obtain that

$$a(x,z_n,\nabla z_n) \rightharpoonup a(x,t,z,\nabla z) \text{ weakly in } \big(L_{\overline{M}}(Q)\big)^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M), \tag{3.19}$$

which implies that

$$\int_Q a(x,t,z_n,\nabla z_n)\nabla z\chi_s\,dx \to \int_Q a(x,t,z,\nabla z)\nabla z\chi_s\,dx\,dt \tag{3.20}$$

as $n \to \infty$. Letting also $s \to \infty$, one has

$$\int_Q a(x,t,z,\nabla z)\nabla z\chi_s\,dx \to \int_Q a(x,t,z,\nabla z)\nabla z\,dx\,dt. \tag{3.21}$$

On the other hand it is easy to see that the second term on the right-hand side of (3.18) tends to 0 as $n \to \infty$. Consequently, from (3.9), (3.20) and (3.21) we have

$$\lim_{n\to\infty}\int_Q a(x,t,z_n,\nabla z_n)\nabla z_n\,dx\,dt = \int_Q a(x,t,z,\nabla z)\nabla z\,dx\,dt. \tag{3.22}$$

By virtue of (3.5) and Vitali's theorem, one can deduce that

$$M(|\nabla z_n|) \to M(|\nabla z|) \text{ in } L^1(Q). \qquad \square$$

Remark 3.3. It should be interest to note that the condition (3.10) is not necessary in the case where the N-function M satisfies the Δ_2-condition.

4. Definition of a renormalized solution

As already mentioned in the introduction, problem (1.1)–(1.3) does not admit a weak solution under assumptions (3.1)–(3.8) since the growths of $\Phi(u)$ is not controlled with respect to u (so that these fields are not in general defined as distributions, even when u belongs $W_0^{1,x}L_M(Q)$).

The definition of a renormalized solution for problem (1.1)–(1.3) can be stated as follows.

Definition 4.1. A measurable function u defined on Q is a renormalized solution of problem (1.1)–(1.3) if

$$T_K(u) \in W_0^{1,x}L_M(Q) \text{ for all } K \ge 0 \text{ and } b(u) \in L^\infty\big(0,T;L^1(\Omega)\big), \tag{4.1}$$

$$\int_{\{(t,x)\in Q\,|\,m\le|u(x,t)|\le m+1\}} a(x,t,u,\nabla u)\nabla u\,dx\,dt \to 0 \quad \text{as } m \to +\infty, \tag{4.2}$$

and if, for every function S in $W^{1,\infty}(\mathbb{R})$ with compact support, we have

$$\frac{\partial B_S(u)}{\partial t} - \operatorname{div}\big(S(u)a(x,t,u,\nabla u)\big) + S'(u)a(x,t,u,\nabla u)\nabla u$$
$$- \operatorname{div}\big(S(u)\Phi(u)\big) + S'(u)\Phi(u)\nabla u = fS(u) \text{ in } D'(Q), \tag{4.3}$$

where $B_S(z) = \int_0^z b'(r)S(r)\,dr$ and

$$B_S(u)(t=0) = B_S(u_0) \text{ in } \Omega. \tag{4.4}$$

The following remarks are concerned with a few comments on Definition 4.1.

Remark 4.2. Equation (4.3) is formally obtained through pointwise multiplication of equation (1.1) by $S(u)$. Note that due to (4.1) each term in (4.3) has a meaning in $L^1(Q) + W^{-1,x}L_{\overline{M}}(Q)$.

Indeed, if K is such that $\operatorname{supp} S \subset [-K, K]$, the following identifications are made in (4.3):

- $B_S(u)$ belongs to $W_0^1 L_M(Q)$ since S is a bounded function and $\nabla B_S(u) = S(u)b'\big(T_K(u)\big)\nabla T_K(u)$. The functions S and $b' \circ T_K$ are bounded on $\mathbb{R}$ so that (4.1) implies that $\nabla B_S(u) \in \big(L_M(Q)\big)^N$.
- $S(u)a(x,t,u,\nabla u)$ identifies with $S(u)a\big(x,t,T_K(u),\nabla T_K(u)\big)$ a.e. in Q. Since $|T_K(u)| \le K$ a.e. in Q and $S(u) \in L^\infty(Q)$, we obtain from (3.3), (4.1) that

$$S(u)a\big(x,t,T_K(u),\nabla T_K(u)\big) \in \big(L_{\overline{M}}(Q)\big)^N.$$

- $S'(u)a(x,t,u,\nabla u)\nabla u$ identifies with $S'(u)a\big(x,t,T_K(u),\nabla T_K(u)\big)\nabla T_K(u)$, and in view of (3.1) and (4.1) one has

$$S'(u)a\big(x,t,T_K(u),\nabla T_K(u)\big)\nabla T_K(u) \in L^1(Q).$$

- $S(u)\Phi(u)$ and $S'(u)\Phi(u)\nabla u$ identify with

$$S(u)\Phi\big(T_K(u)\big) \quad \text{and} \quad S'(u)\Phi\big(T_K(u)\big)\nabla T_K(u),$$

  respectively. Due to the properties of S and (3.6), the functions S, S' and $\Phi \circ T_K$ are bounded on $\mathbb{R}$ so that (4.1) implies that $S(u)\Phi\big(T_K(u)\big) \in \big(L^\infty(Q)\big)^N$ and $S'(u)\Phi\big(T_K(u)\big)\nabla T_K(u) \in \big(L_M(Q)\big)^N$.

The above considerations show that equation (4.3) holds in $D'(Q)$ and $\frac{\partial B_S(u)}{\partial t}$ belongs to $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$ and $B_S(u) \in W^{1,x}L_M(Q) \cap L^\infty(Q)$. It follows that $B_S(u)$ belongs to $C^0\big([0,T]; L^1(\Omega)\big)$ so that the initial condition (4.4) makes sense.

5. Existence result

This section is devoted to establish the following existence theorem.

Theorem 5.1. *Under assumption* (3.1)–(3.8) *there exists at at least a renormalized solution of Problem* (1.1)–(1.3).

Proof. The proof is divided into 5 steps. In step 1, we introduce an approximate problem. In Step 2, we establish a few a priori estimates which allow us to prove that the approximate solutions u_n converge to u, $b(u)$ belongs to $L^\infty(0,T;L^1(\Omega))$ and u satisfies (4.1). In step 3, we define a time regularization of the field $T_K(u)$ and we establish Lemma 5.7, which allows us to control the parabolic contribution that arises in the monotonicity method when passing to the limit. In this step we also prove an energy estimate (Lemma 5.6). Step 4 is devoted to prove that u satisfies (4.2). At last, step 5 is devoted to prove that u satisfies (4.3) and (4.4) of Definition 4.1.

Step 1. Let us introduce the following regularization of the data:

$$b_n(r) = T_n\big(b(r)\big) + \frac{1}{n}r \qquad \text{for } n \in \mathbb{N}^*, \tag{5.1}$$

$$\Phi_n \text{ is a Lipschitz continuous bounded function from } \mathbb{R} \text{ into } \mathbb{R}^N, \tag{5.2}$$

such that Φ_n uniformly converges to Φ on any compact subset of $\mathbb{R}$ as n tends to $+\infty$,

$$f_n \in L^2(Q) : \|f_n\|_{L^1} \le \|f\|_{L^1} \text{ and } f_n \to f \text{ in } L^1(Q) \text{ as } n \text{ tends to } +\infty, \tag{5.3}$$

$$u_{0n} \in C_0^\infty(\Omega) : \|b_n(u_{0n})\|_{L^1} \le \|b(u_0)\|_{L^1} \text{ and } b_n(u_{0n}) \to b(u_0) \text{ in } L^1(\Omega) \tag{5.4}$$

as n tends to $+\infty$.

Let us now consider the following regularized problem:

$$\frac{\partial b_n(u_n)}{\partial t} - \operatorname{div}\big(a(x,t,u_n,\nabla u_n)\big) - \operatorname{div}\big(\Phi_n(u_n)\big) = f_n \text{ in } Q, \tag{5.5}$$

$$u_n = 0 \text{ on } (0,T) \times \partial\Omega, \tag{5.6}$$

$$b_n(u_n)(t=0) = b_n(u_{0n}) \text{ in } \Omega. \tag{5.7}$$

As a consequence, proving existence of a weak solution $u_n \in W_0^{1,x}L_M(Q)$ of (5.5)–(5.7) is an easy task (see e.g. [25], [29]).

Step 2. The estimates derived in this step rely on standard techniques for problems of the type (5.5)–(5.7).

Lemma 5.2. *Assume that* (3.1)–(3.8) *hold true and let* u_n *be a solution of the approximate problem* (5.5)–(5.7). *Then for all* $K > 0$, *we have*

$$\|T_K(u_n)\|_{W_0^{1,x}L_M(Q)} \le K(\|f\|_{L^1(Q)} + \|b(u_0)\|_{L^1(\Omega)}) \equiv CK \quad \textit{for all } n,$$

where C *is a constant independent of* n, *and*

$$\int_\Omega B_K^n(u_n)(\tau)\,dx \le K(\|f\|_{L^1(Q)} + \|b(u_0)\|_{L^1(\Omega)}) \equiv CK \quad \textit{for all } n$$

for almost any τ *in* $(0, T)$ *and where* $B_K^n(r) = \int_0^r T_K(s)b_n'(s)\,ds$.

Proof. We take $T_K(u_n)_{\chi(0,\tau)}$ as test function in (5.5). For every $\tau \in (0, T)$ we obtain that

$$\left\langle \frac{\partial b_n(u_n)}{\partial t}, T_K(u_n)_{\chi(0,\tau)} \right\rangle + \int_{Q_\tau} a\big(x, t, T_K(u_n), \nabla T_K(u_n)\big)\nabla T_K(u_n)\,dx\,dt \\ + \int_{Q_\tau} \Phi_n(u_n)\nabla T_K(u_n)\,dx\,dt = \int_{Q_\tau} f_n T_K(u_n)\,dx\,dt, \tag{5.8}$$

which implies that

$$\int_\Omega B_K^n(u_n)(\tau)\,dx + \int_{Q_\tau} a\big(x, t, T_K(u_n), \nabla T_K(u_n)\big)\nabla T_K(u_n)\,dx\,dt \\ + \int_{Q_\tau} \Phi_n(u_n)\nabla T_K(u_n)\,dx\,dt = \int_{Q_\tau} f_n T_K(u_n)\,dx\,dt + \int_\Omega B_K^n(u_0^n)\,dx, \tag{5.9}$$

where $B_K^n(r) = \int_0^r T_K(s)b_n'(s)\,ds$.

The Lipschitz character of Φ_n and Stokes' formula together with the boundary condition (5.6) give

$$\int_{Q_\tau} \Phi_n(u_n)\nabla T_K(u_n)\,dx\,dt = 0. \tag{5.10}$$

Due to the definition of B_K^n we have

$$0 \le \int_\Omega B_K^n(u_0^n)\,dx \le K\int_\Omega |b_n(u_0^n)|\,dx \le K\|b(u_0)\|_{L^1(\Omega)}. \tag{5.11}$$

Using (5.10), (5.11) and $B_K^n(u_n) \ge 0$, it follows from (5.9) that

$$\int_Q a\big(x,t,T_K(u_n),\nabla T_K(u_n)\big)\nabla T_K(u_n)\,dx\,dt \le K(\|f_n\|_{L^1(Q)} + \|b_n(u_0^n)\|_{L^1(\Omega)}) \le CK, \tag{5.12}$$

which implies by virtue of (3.5), (5.3) and (5.4) that

$$\int_Q M\big(\nabla T_K(u_n)\big)\,dx\,dt \le K(\|f\|_{L^1(Q)} + \|b(u_0)\|_{L^1(\Omega)}) \equiv CK. \tag{5.13}$$

We deduce from that above inequality (5.9) and (5.11) that

$$\int_\Omega B_K^n(u_n)(\tau)\,dx \le (\|f\|_{L^1(Q)} + \|b(u_0)\|_{L^1(\Omega)}) \equiv CK \tag{5.14}$$

for almost any τ in $(0,T)$. □

Lemma 5.3. *Let u_n be a solution of* (5.5)–(5.7). *Then*

$$\lim_{K\to\infty} \text{meas}\{(x,t)\in Q \,|\, |u_n| > K\} = 0 \text{ uniformly with respect to } n.$$

Proof. Due to Lemma 5.7 of [21], there exist positive constants δ, λ such that

$$\int_Q M(v)\,dx\,dt \le \delta \int_Q M(\lambda|\nabla v|)\,dx\,dt \qquad \text{for all } v \in W_0^{1,x}L_M(Q). \tag{5.15}$$

Taking $v = \frac{T_K(u_n)}{\lambda}$ in (5.15) and using (5.13), one has

$$\int_Q M\left(\frac{T_K(u_n)}{\lambda}\right)dx\,dt \le CK, \tag{5.16}$$

where C is a constant independent of K and n. This implies that

$$\text{meas}\{(x,t)\in Q \,|\, |u_n| > K\} \le \frac{C'K}{M\left(\frac{K}{\lambda}\right)}, \tag{5.17}$$

where C' is a constant independent of K and n. Finally,

$$\lim_{K\to\infty} \text{meas}\{(x,t)\in Q \,|\, |u_n| > K\} = 0 \text{ uniformly with respect to } n. \quad \square$$

Now we turn to prove the almost every convergence of u_n and $b_n(u_n)$.

For that take a $C^2(\mathbb{R})$ non-decreasing function ξ_k such that $\xi_k(s) = s$ for $|s| \le \frac{k}{2}$ and $\xi_k(s) = k$ for $|s| \ge k$.

Multiplying the approximating equation by $\xi_k'(b_n(u_n))$, we get

$$\begin{aligned}\frac{\partial}{\partial t}&\big(\xi_k\big(b_n(u_n)\big)\big) - \operatorname{div}\big(a(x,t,u_n,\nabla u_n)\xi_k'\big(b_n(u_n)\big)\big)\\&+ a(x,t,u_n,\nabla u_n)\xi_k''\big(b_n(u_n)\big)b_n'(u_n)\nabla u_n - \operatorname{div}\big(\xi_k'\big(b_n(u_n)\big)\Phi_n(u_n)\big)\\&+ \xi_k''\big(b_n(u_n)\big)b_n'(u_n)\Phi_n(u_n)\nabla u_n = f_n\xi_k'\big(b_n(u_n)\big), \end{aligned} \tag{5.18}$$

which implies that

$$\xi_k\big(b_n(u_n)\big)\big) \text{ is bounded in } W_0^{1,x}L_M(Q), \tag{5.19}$$

and

$$\frac{\partial \xi_k\big(b_n(u_n)\big)\big)}{\partial t} \text{ is bounded in } L^1(Q) + W^{-1,x}L_{\overline{M}}(Q), \tag{5.20}$$

independently of n as soon as $k < n$. Due to Definition (3.1) and (5.1) of b_n, it is clear that

$$\{|b_n(u_n)| \le k\} \subset \{|u_n| \le k^*\}$$

as soon as $k < n$ and k^* is a constant independent of n. As a first consequence we have

$$\nabla\xi_k\big(b_n(u_n)\big)\big) = \xi_k'\big(b_n(u_n)\big)b_n'\big(T_{k^*}(u_n)\big)\nabla T_{k^*}(u_n) \text{ a.e. in } Q \tag{5.21}$$

as soon as $k < n$. Secondly, the following estimate holds true

$$\big\|\xi_k'\big(b_n(u_n)\big)b_n'\big(T_{k^*}(u_n)\big)\big\|_{L^\infty(Q)} \le \|\xi'\|_{L^\infty(\mathbb{R})}\big(\max_{|r|\le k^*}\big(b'(r)\big) + 1\big)$$

as soon as $k < n$.

As a consequence of (5.13), (5.21) we then obtain (5.19). To show that (5.20) holds true, due to (5.18) we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\big(\xi_k\big(b_n(u_n)\big)\big) = {}& \operatorname{div}\big(a(x,t,u_n,\nabla u_n)\xi_k'\big(b_n(u_n)\big)\big)\\&- a(x,t,u_n,\nabla u_n)\xi_k''\big(b_n(u_n)\big)b_n'(u_n)\nabla u_n\\&+ \operatorname{div}\big(\xi_k'\big(b_n(u_n)\big)\Phi_n(u_n)\big) - \xi_k''\big(b_n(u_n)\big)b_n'(u_n)\Phi_n(u_n)\nabla u_n\\&+ f_n\xi_k'\big(b_n(u_n)\big). \end{aligned} \tag{5.22}$$

Since $\operatorname{supp}\xi'$ and $\operatorname{supp}\xi''$ are both included in $[-k,k]$, u_n may be replaced by $T_{k^*}(u_n)$ in each of these terms. As a consequence, each term on the right-hand side of (5.22) is bounded either in $W^{-1,x}L_{\overline{M}}(Q)$ or in $L^1(Q)$. Hence lemma 2.5 (cf. [17]) allows us to conclude that $\xi_k\big(b_n(u_n)\big)$ is compact in $L^1(Q)$.

Due to the choice of ξ_k, we conclude that for each k, the sequence $T_k\big(b_n(u_n)\big)$ converges almost everywhere in Q, which implies that the sequence $b_n(u_n)$ converges almost everywhere to some measurable function v in Q. Thus by using the same argument as in ([5], [7], [6], [9]), we can show the following lemma.

Lemma 5.4. *Let u_n be a solution of the approximate problem* (5.5)–(5.7). *Then*

$$u_n \to u \text{ a.e. in } Q, \tag{5.23}$$

$$b_n(u_n) \to b(u) \text{ a.e. in } Q. \tag{5.24}$$

We now establish that $b(u)$ belongs to $L^\infty\big(0,T;L^1(\Omega)\big)$. First note that (5.23) makes it possible to pass to the limit-inf in (5.14) as n tends to $+\infty$. We obtain that

$$\frac{1}{K}\int_\Omega B_K(u)(\tau)\,dx \le (\|f\|_{L^1(Q)} + \|b(u_0)\|_{L^1(\Omega)}) \equiv C,$$

for almost any τ in $(0,T)$. Now the definition of $B_K(s)$ and the fact that $\frac{1}{K}B_K(u)$ converges pointwise to $b(u)$, as K tends to $+\infty$, shows that $b(u)$ belongs to $L^\infty\big(0,T;L^1(\Omega)\big)$, as claimed.

Now we prove the following result.

Lemma 5.5. *Let u_n be a solution of the approximate problem* (5.5)–(5.7). *Then*

$$\big(a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\big)_n \text{ is bounded in } \big(L_{\overline{M}}(Q)\big)^N. \tag{5.25}$$

for all $k \ge 0$.

Proof. Let $\varphi \in \big(E_M(Q)\big)^N$ with $\|\varphi\|_{M\cdot Q} = 1$. In view of the monotonicity of a, one easily has

$$\begin{aligned}\int_Q a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\varphi\,dx\,dt \\ \le \int_Q a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\nabla T_k(u_n)\,dx\,dt \\ + \int_Q a\big(x,t,T_k(u_n),\varphi\big)\big(\nabla T_k(u_n) - \varphi\big)\,dx\,dt,\end{aligned} \tag{5.26}$$

and

$$
\begin{aligned}
-\int_Q a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\varphi\,dx\,dt \\
\le \int_Q a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\nabla T_k(u_n)\,dx\,dt \\
-\int_Q a\big(x,t,T_k(u_n),-\varphi\big)\big(\nabla T_k(u_n)+\varphi\big)\,dx\,dt. \qquad (5.27)
\end{aligned}
$$

Since $T_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q)$, one easily deduces that $a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)$ is a bounded sequence in $\big(L_{\overline{M}}(Q)\big)^N$. Thus, up to a subsequence,

$$
a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) \rightharpoonup \varphi_k \text{ in } \big(L_{\overline{M}}(Q)\big)^N \text{ for } \sigma(\Pi L_{\overline{M}},\Pi E_M) \qquad (5.28)
$$

for some $\varphi_k \in \big(L_{\overline{M}}(Q)\big)^N$. □

Step 3. This step is devoted to introduce for $K \ge 0$ fixed a time regularization $w_{\mu,j}^i$ of the function $T_K(u)$ and to establish the following proposition:

Proposition 5.6. *Let u_n be a solution of the approximate problem* (5.5)–(5.7). *Then for any $k \ge 0$:*

$$
\nabla u_n \to \nabla u \text{ a.e. in } Q, \qquad (5.29)
$$

$$
a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) \rightharpoonup a\big(x,t,T_k(u),\nabla T_k(u)\big) \text{ weakly in } \big(L_{\overline{M}}(Q)\big)^N, \qquad (5.30)
$$

$$
M(|\nabla T_k(u_n)|) \to M(|\nabla T_k(u)|) \text{ strongly in } L^1(Q), \qquad (5.31)
$$

as n tends to $+\infty$.

Proof. This proof is devoted to introduce for $k \ge 0$ fixed, a time regularization of the function $T_k(u)$ in order to perform the monotonicity method. This kind of regularization has been first introduced by R. Landes (see Lemma 6 and Proposition 3, p. 230, and Proposition 4, p. 231, in [24]). More recently, it has been exploited in [10] and [15] to solve some nonlinear evolution problems with L^1 or measure data.

Let $v_j \in D(Q)$ be a sequence such that $v_j \to u$ in $W_0^{1,x}L_M(Q)$ for the modular convergence and let $\psi_i \in D(\Omega)$ be a sequence which converges strongly to u_0 in $L^1(\Omega)$.

Let $w_{\mu,j}^i = T_k(v_j)_\mu + e^{-\mu t}T_k(\psi_i)$ where $T_k(v_j)_\mu$ is the mollification with respect to time of $T_k(v_j)$. Note that $w_{\mu,j}^i$ is a smooth function having the following properties:

$$\frac{\partial w^i_{\mu,j}}{\partial t} = \mu\big(T_k(v_j) - w^i_{\mu,j}\big), \qquad w^i_{\mu,j}(0) = T_k(\psi_i), \qquad |w^i_{\mu,j}| \le k, \tag{5.32}$$

$$w^i_{\mu,j} \to T_k(u)_\mu + e^{-\mu t} T_k(\psi_i) \text{ in } W_0^{1,x} L_M(Q), \tag{5.33}$$

for the modular convergence as $j \to \infty$,

$$T_k(u)_\mu + e^{-\mu t} T_k(\psi_i) \to T_k(u) \text{ in } W_0^{1,x} L_M(Q), \tag{5.34}$$

for the modular convergence as $\mu \to \infty$.

Let now the function h_m defined on $\mathbb{R}$ by

$$h_m(s) = \begin{cases} 1 & \text{if } |s| \le m, \\ -|s| + m + 1 & \text{if } m \le |s| \le m+1, \\ 0 & \text{if } |s| \ge m+1, \end{cases}$$

for any $m \ge k$.

Using the admissible test function $\varphi^{\mu,i}_{n,j,m} = \big(T_k(u_n) - w^\mu_{i,j}\big) h_m(u_n)$ as test function in (5.5) leads to

$$\begin{aligned} &\left\langle \frac{\partial b_n(u_n)}{\partial t}, \varphi^{\mu,i}_{n,j,m} \right\rangle + \int_Q a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n) - \nabla w^\mu_{i,j}\big) h_m(u_n)\, dx\, dt \\ &\quad + \int_Q a(x,t,u_n,\nabla u_n)\big(T_k(u_n) - w^\mu_{i,j}\big) \nabla u_n h'_m(u_n)\, dx\, dt \\ &\quad + \int_{\{m \le |u_n| \le m+1\}} \Phi_n(u_n) \nabla u_n h'_m(u_n) \big(T_k(u_n) - w^\mu_{i,j}\big)\, dx\, dt \\ &\quad + \int_Q \Phi_n(u_n) h_m(u_n) \big(\nabla T_k(u_n) - \nabla w^\mu_{i,j}\big)\, dx\, dt = \int_Q f_n \varphi^{\mu,i}_{n,j,m}\, dx\, dt. \end{aligned} \tag{5.35}$$

Let $\varepsilon(n,j,\mu,i) > 0$ be a positive sequence such that

$$\lim_{i\to\infty} \lim_{\mu\to\infty} \lim_{j\to\infty} \lim_{n\to\infty} \varepsilon(n,j,\mu,i) = 0.$$

The very definition of the sequence $w^\mu_{i,j}$ makes it possible to establish the following lemma.

Lemma 5.7. *Let $\varphi^{\mu,i}_{n,j,m} = \big(T_k(u_n) - w^\mu_{i,j}\big) h_m(u_n)$. For any $k \ge 0$ we have*

$$\left\langle \frac{\partial b_n(u_n)}{\partial t}, \varphi^{\mu,i}_{n,j,m} \right\rangle \ge \varepsilon(n,j,\mu,i), \tag{5.36}$$

where $\langle\,,\rangle$ *denotes the duality pairing between* $L^1(Q)+W^{-1,x}L_{\overline{M}}(Q)$ *and* $L^\infty(Q)\cap W_0^{1,x}L_M(Q)$.

Proof. This lemma will be proved in Appendix. □

Now we turn to complete the proof of Proposition 5.6. First, it is easy to see that

$$\int_Q f_n\varphi_{n,j,m}^{\mu,i}\,dx\,dt=\varepsilon(n,j,\mu). \tag{5.37}$$

Indeed, by the almost everywhere convergence of u_n, we have that $\big(T_k(u_n)-w_{i,j}^\mu\big)h_m(u_n)$ converges to $\big(T_k(u)-w_{i,j}^\mu\big)h_m(u)$ in $L^\infty(Q)$ weak-* and then

$$\int_Q f_n\big(T_k(u_n)-w_{i,j}^\mu\big)h_m(u_n)\,dx\,dt\to\int_Q f_n\big(T_k(u)-w_{i,j}^\mu\big)h_m(u)\,dx\,dt$$

so that

$$\big(T_k(u)-w_{i,j}^\mu\big)h_m(u)\to\big(T_k(u)-T_k(u)_\mu-e^{-\mu t}T_k(\psi_i)\big)\text{ in }L^\infty(Q)\text{ weak-* as }j\to\infty.$$

Also

$$\big(T_k(u)-T_k(u)_\mu-e^{-\mu t}T_k(\psi_i)\big)\to 0\text{ in }L^\infty(Q)\text{ weak-* as }\mu\to+\infty.$$

Then we deduce that

$$\int_Q f_n\big(T_k(u_n)-w_{i,j}^\mu\big)h_m(u_n)\,dx\,dt=\varepsilon(n,j,\mu). \tag{5.38}$$

Similarly, Lebesgue's convergence theorem shows that

$$\Phi_n(u_n)h_m(u_n)\to\Phi(u)h_m(u)\text{ strongly in }\big(E_{\overline{M}}(Q)\big)^N\text{ as }n\to+\infty$$

and

$$\Phi_n(u_n)\chi_{\{m\le|u_n|\le m+1\}}\big(T_k(u_n)-w_{i,j}^\mu\big)\to\Phi(u)\chi_{\{m\le|u|\le m+1\}}\big(T_k(u)-w_{i,j}^\mu\big)$$

strongly in $\big(E_{\overline{M}}(Q)\big)^N$ as $n\to+\infty$.

Then by virtue of $\nabla T_k(u_n)\rightharpoonup\nabla T_k(u)$ weakly in $\big(L_M(Q)\big)^N$ and

$$\nabla u_n\chi_{\{m\le|u_n|\le m+1\}}=\nabla T_{m+1}(u_n)\chi_{\{m\le|u_n|\le m+1\}}$$

a.e. in Q, one has

$$\int_Q \Phi_n(u_n)h_m(u_n)\big(\nabla T_k(u_n) - \nabla w_{i,j}^{\mu}\big)\,dx\,dt \to \int_Q \Phi(u)h_m(u)\big(\nabla T_k(u) - \nabla w_{i,j}^{\mu}\big)\,dx\,dt$$

as $n \to +\infty$, and

$$\int_{\{m\le|u_n|\le m+1\}} \Phi_n(u_n)\nabla u_n\big(T_k(u_n) - w_{i,j}^{\mu}\big)\,dx\,dt \to \int_{\{m\le|u|\le m+1\}} \Phi(u)\nabla u\big(T_k(u) - w_{i,j}^{\mu}\big)\,dx\,dt$$

as $n \to +\infty$. On the other hand, by using the modular convergence of $w_{i,j}^{\mu}$ as $j \to +\infty$ and letting μ tend to infinity, we get

$$\int_Q \Phi_n(u_n)h_m(u_n)\big(\nabla T_k(u_n) - \nabla w_{i,j}^{\mu}\big)\,dx\,dt = \varepsilon(n,j,\mu), \tag{5.39}$$

and

$$\int_{\{m\le|u_n|\le m+1\}} \Phi_n(u_n)\nabla u_n\big(T_k(u_n) - w_{i,j}^{\mu}\big)\,dx\,dt = \varepsilon(n,j,\mu). \tag{5.40}$$

Concerning the third term of the right-hand side of (5.35) we obtain that

$$\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n h_m'(u_n)\big(T_k(u_n) - w_{i,j}^{\mu}\big)\,dx\,dt \le 2k\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt. \tag{5.41}$$

Then by (5.25) we deduce that

$$\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n h_m'(u_n)\big(T_k(u_n) - w_{i,j}^{\mu}\big)\,dx\,dt \le \varepsilon(n,\mu,m). \tag{5.42}$$

Finally, by means of (5.35)–(5.42), we obtain that

$$\int_Q a(x,t,u_n,\nabla u_n)\nabla u_n h_m'(u_n)\big(T_k(u_n) - w_{i,j}^{\mu}\big)h_m(u_n)\,dx\,dt \le \varepsilon(n,j,\mu,m). \tag{5.43}$$

Splitting the first integral on the left-hand side of (5.44) where $|u_n| \leq k$ and $|u_n| > k$, we can write

$$\begin{aligned}\int_Q & a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n) - \nabla w^\mu_{i,j}\big)h_m(u_n)\,dx\,dt \\ &= \int_Q a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\big(\nabla T_k(u_n) - \nabla w^\mu_{i,j}\big)h_m(u_n)\,dx\,dt \\ &\quad - \int_{\{|u_n|>k\}} a(x,t,u_n,\nabla u_n)\nabla w^\mu_{i,j}h_m(u_n)\,dx\,dt.\end{aligned}$$

Since $h_m(u_n) = 0$ if $|u_n| \geq m+1$, one has

$$\begin{aligned}\int_Q & a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n) - \nabla w^\mu_{i,j}\big)h_m(u_n)\,dx\,dt \\ &= \int_Q a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\big(\nabla T_k(u_n) - \nabla w^\mu_{i,j}\big)h_m(u_n)\,dx\,dt \\ &\quad - \int_{\{|u_n|>k\}} a\big(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n)\big)\nabla w^\mu_{i,j}h_m(u_n)\,dx\,dt = I_1 + I_2. \qquad (5.44)\end{aligned}$$

In the following we pass to the limit in (5.44): first we let n tend to $+\infty$, then we let j, then μ and finally m, tend to $+\infty$. Since $a\big(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n)\big)$ is bounded in $\big(L_{\overline{M}}(Q)\big)^N$, we have that $a\big(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n)\big) \rightharpoonup \varphi_m$ weakly in $L_{\overline{M}}(Q)$ in $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$ as n tends to infinity. Since $\nabla w^\mu_{i,j}h_m(u_n)_{\chi_{\{|u_n|>k\}}}$ converges to $\nabla w^\mu_{i,j}h_m(u)_{\chi_{\{|u|>k\}}}$ strongly in $E_M(\Omega)$ as n tends to infinity, it follows that

$$I_2 = \int_Q \varphi_m \nabla w^\mu_{i,j}h_m(u)_{\chi_{\{|u|>k\}}}\,dx\,dt + \varepsilon(n).$$

By letting $j \to \infty$, we get

$$I_2 = \int_Q \varphi_m\big(\nabla T_k(u)_\mu - e^{-\mu t}\nabla T_k(\psi_i)\big)h_m(u)_{\chi_{\{|u|>k\}}}\,dx\,dt + \varepsilon(n,j),$$

which, by letting $\mu \to +\infty$, implies that

$$I_2 = \int_Q \varphi_m \nabla T_k(u)_\mu h_m(u)_{\chi_{\{|u|>k\}}}\,dx\,dt + \varepsilon(n,j,\mu).$$

Using now the term I_1 of (5.44) one can easily show that

$$\begin{aligned}
&\int_Q a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\big(\nabla T_k(u_n)-\nabla w_{i,j}^{\mu}\big)h_m(u_n)\,dx\,dt \\
&\quad=\int_Q \big[a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)-a\big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\big)\big] \\
&\quad\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]h_m(u_n)\,dx\,dt \\
&\quad+\int_Q a\big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\big)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]h_m(u_n)\,dx\,dt \\
&\quad+\int_Q a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\nabla T_k(v_j)\chi_j^s h_m(u_n)\,dx\,dt \\
&\quad-\int_Q a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\nabla w_{i,j}^{\mu} h_m(u_n)\,dx\,dt=J_1+J_2+J_3+J_4, \qquad (5.45)
\end{aligned}$$

where χ_j^s denotes the characteristic function of the subset

$$\Omega_s^j=\{(x,t)\in Q\,|\,|\nabla T_k(v_j)|\le s\}.$$

As before, in the following we pass to the limit in (5.45): first we let $n\to+\infty$, then we let j, then μ and finally m, tend to $+\infty$. Starting with J_2, observe first that

$$\begin{aligned}
J_2=&\int_Q a\big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\big)\nabla T_k(u_n)h_m(u_n)\,dx\,dt \\
&-\int_Q a\big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\big)\nabla T_k(v_j)\chi_j^s h_m(u_n)\,dx\,dt.
\end{aligned}$$

Since $a\big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\big)h_m(u_n)\to a\big(x,t,T_k(u),\nabla T_k(v_j)\chi_j^s\big)h_m(u)$ strongly in $(E_{\overline{M}})^N$ and $\nabla T_k(u_n)\rightharpoonup\nabla T_k(u)$ weakly in $\big(L_M(Q)\big)^N$ for $\sigma(\Pi L_M,\Pi E_{\overline{M}})$. Moreover, it is easy to show that

$$\begin{aligned}
&\int_Q a\big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\big)\nabla T_k(v_j)\chi_j^s h_m(u_n)\,dx\,dt \\
&\quad\to\int_Q a\big(x,t,T_k(u),\nabla T_k(v_j)\chi_j^s\big)\nabla T_k(v_j)\chi_j^s h_m(u)\,dx\,dt
\end{aligned}$$

as n tends to $+\infty$. We get

$$J_2=\int_Q a\big(x,t,T_k(u),\nabla T_k(v_j)\chi_j^s\big)[\nabla T_k(u)-\nabla T_k(v_j)\chi_j^s]h_m(u)\,dx\,dt+\varepsilon(n).$$

Since $\nabla T_k(v_j)\chi_j^s h_m(u) \to \nabla T_k(u)\chi_s h_m(u)$ strongly in $(E_M(Q))^N$ as $j \to \infty$ and $a(x,t,T_k(u),\nabla T_k(v_j)\chi_j^s) \to a(x,t,T_k(u),\nabla T_k(u)\chi_s)$ strongly in $(L_{\overline{M}}(Q))^N$ as j goes to ∞, we have

$$J_2 = \varepsilon(n,j). \tag{5.46}$$

By letting $n \to \infty$ and since $a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varphi_k$ weakly in $(L_{\overline{M}}(Q))^N$ and $h_m(u_n) = 1$ in $\{(x,t) \mid |u_n| \le k\}$, we have

$$J_3 = \int_Q \varphi_k \nabla T_k(v_j)\chi_j^s \, dx\, dt + \varepsilon(n),$$

which gives

$$J_3 = \int_Q \varphi_k \nabla T_k(u)\chi_s \, dx\, dt + \varepsilon(n,j) \tag{5.47}$$

by letting $j \to \infty$.

Concerning J_4 we can write

$$J_4 = -\int_Q \varphi_k \nabla w_{i,j}^\mu \mu h_m(u)\, dx\, dt + \varepsilon(n), \tag{5.48}$$

which implies that, by letting $j \to \infty$,

$$J_4 = \int_Q \varphi_k [\nabla T_k(u) - e^{-\mu t}\nabla T_k(\psi_j)]\, dx\, dt + \varepsilon(n,j). \tag{5.49}$$

By letting $\mu \to \infty$ we obtain

$$J_4 = -\int_Q \varphi_k \nabla T_k(u)\, dx\, dt + \varepsilon(n,j,\mu,s). \tag{5.50}$$

We then conclude that

$$\begin{aligned} &\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n) - \nabla w_{i,j}^\mu] h_m(u_n)\, dx\, dt \\ &\quad = \int_Q \left[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\right] \\ &\qquad \cdot [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s] h_m(u_n)\, dx\, dt + \varepsilon(n,j,\mu,s). \end{aligned} \tag{5.51}$$

Now observe that

$$\begin{aligned}
&\int_Q \big[a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) - a\big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\big)\big] \\
&\qquad \cdot [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] h_m(u_n)\,dx\,dt \\
&\quad = \int_Q \big[a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) - a\big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\big)\big] \\
&\qquad \cdot [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s] h_m(u_n)\,dx\,dt \\
&\qquad + \int_Q a\big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\big)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s] h_m(u_n)\,dx\,dt \\
&\qquad - \int_Q a\big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\big)[\nabla T_k(u_n) - \nabla T_k(u)\chi^s] h_m(u_n)\,dx\,dt \\
&\qquad + \int_Q a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)[\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi_s] h_m(u_n)\,dx\,dt.
\end{aligned}$$

Passing to the limit in n and j in the last three terms on the right-hand side of the last equality, we get

$$\begin{aligned}
&\int_Q a\big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\big)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s] h_m(u_n)\,dx\,dt \\
&\quad - \int_Q a\big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\big)[\nabla T_k(u_n) - \nabla T_k(u)\chi_s] h_m(u_n)\,dx\,dt = \varepsilon(n,j)
\end{aligned}$$

and

$$\int_Q a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)[\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi_s] h_m(u_n)\,dx\,dt = \varepsilon(n,j). \quad (5.52)$$

This implies that

$$\begin{aligned}
&\int_Q \big[a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) - a\big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\big)\big] \\
&\qquad \cdot [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] h_m(u_n)\,dx\,dt \\
&\quad = \int_Q \big[a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) - a\big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\big)\big] \\
&\qquad \cdot [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s] h_m(u_n)\,dx\,dt + \varepsilon(n,j). \quad (5.53)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\int_Q \big[a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) - a\big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\big)\big][\nabla T_k(u_n) - \nabla T_k(u)\chi_s] \\
&\quad = \int_Q \big[a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) - a\big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\big)\big] \\
&\qquad \cdot [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] h_m(u_n)\,dx\,dt \\
&\qquad + \int_Q a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)[\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\big(1 - h_m(u_n)\big)\,dx\,dt \\
&\qquad - \int_Q a\big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\big)[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\big(1 - h_m(u_n)\big)\,dx\,dt.
\end{aligned} \tag{5.54}$$

Since $h_m(u_n) = 1$ in $\{|u_n| \le m\}$ and $\{|u_n| \le k\} \subset \{|u_n| \le m\}$ for m large enough, we deduce from (5.54) that

$$\begin{aligned}
&\int_Q \big[a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) - a\big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\big)\big][\nabla T_k(u_n) - \nabla T_k(u)\chi_s] \\
&\quad = \int_Q \big[a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) - a\big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\big)\big] \\
&\qquad \cdot [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] + \int_{\{|u_n|>k\}} a\big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\big) \\
&\qquad \cdot \nabla T_k(u)\chi_s\big(1 - h_m(u_n)\big)\,dx\,dt.
\end{aligned}$$

It is easy to see that the last terms of the last equality tend to zero as $n \to +\infty$, which implies that

$$\begin{aligned}
&\int_Q \big[a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) - a\big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\big)\big][\nabla T_k(u_n) - \nabla T_k(u)\chi_s] \\
&\quad = \int_Q \big[a\big(x,t,T_k(u),\nabla T_k(u)\big) - a\big(x,t,T_k(u),\nabla T_k(u)\chi_s\big)\big] \\
&\qquad \cdot [\nabla T_k(u) - \nabla T_k(u)\chi_s] h_m(u_n)\,dx\,dt + \varepsilon(n,j).
\end{aligned}$$

Combining (5.36), (5.45), (5.46), (5.47), (5.50) and (5.54) we obtain

$$\begin{aligned}
&\int_Q \big[a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) - a\big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\big)\big] \\
&\qquad \cdot [\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx\,dt \le \varepsilon(n,j,\mu,m,s).
\end{aligned} \tag{5.55}$$

To pass to the limit in (5.55) as n, j, m, s tend to infinity, we obtain

$$\lim_{s\to\infty}\lim_{n\to\infty}\int_Q \big[a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) - a\big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\big)\big] \cdot [\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx\,dt = 0. \tag{5.56}$$

This implies, by Lemma 3.2, the desired statement and finishes the proof of Proposition 5.6. □

Step 4. In this step we prove that u satisfies (4.2).

Lemma 5.8. *The limit u of the approximate solution u_n of* (5.5)–(5.7) *satisfies*

$$\lim_{m\to+\infty}\int_{\{m\le|u|\le m+1\}} a(x,t,u,\nabla u)\nabla u\,dx\,dt = 0. \tag{5.57}$$

Proof. Taking $T_1\big(u_n - T_m(u_n)\big)$ as test function in (5.5), we obtain

$$\left\langle \frac{\partial b_n(u_n)}{\partial t}, T_1\big(u_n - T_m(u_n)\big)\right\rangle + \int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt + \int_Q \operatorname{div}\Big[\int_0^{u_n}\Phi(r)T_1'\big(r - T_m(r)\big)\Big]\,dx\,dt = \int_Q f_n T_1\big(u_n - T_m(u_n)\big)\,dx\,dt. \tag{5.58}$$

Using the fact that $\int_0^{u_n}\Phi(r)T_1'\big(r-T_m(r)\big)\,dx\,dt \in W_0^{1,x}L_M(Q)$ and Stokes' formula, we get

$$\int_\Omega B_n^m\big(u_n(T)\big)\,dx + \int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt \le \int_Q \big|f_n T_1\big(u_n - T_m(u_n)\big)\big|\,dx\,dt + \int_\Omega B_n^m(u_{0n})\,dx, \tag{5.59}$$

where $B_n^m(r) = \int_0^r b_n'(s)T_1\big(s - T_m(s)\big)\,ds$.

In order to pass to the limit as n tends to $+\infty$ in (5.59), we use $B_n^m\big(u_n(T)\big) \ge 0$ and (5.3)–(5.4), we obtain that

$$\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt \le \int_{\{|u|>m\}} |f|\,dx\,dt + \int_{\{|u_0|>m\}} |b(u_0)|\,dx. \tag{5.60}$$

Finally by (3.7), (3.8) and (5.60) we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt = 0. \tag{5.61}$$

To this end, observe that for any fixed $m \ge 0$ one has

$$\begin{aligned}
&\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\
&\quad=\int_Q a(x,t,u_n,\nabla u_n)[\nabla T_{m+1}(u_n)-\nabla T_m(u_n)]\,dx\,dt\\
&\quad=\int_Q a\big(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n)\big)\nabla T_{m+1}(u_n)\,dx\,dt\\
&\qquad-\int_Q a\big(x,t,T_m(u_n),\nabla T_m(u_n)\big)\nabla T_m(u_n)\,dx\,dt.
\end{aligned}$$

According to (5.30)–(5.31), one is at liberty to pass to the limit as n tends to $+\infty$ for fixed $m \ge 0$ and to obtain

$$\begin{aligned}
&\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\
&\quad=\int_Q a\big(x,t,T_{m+1}(u),\nabla T_{m+1}(u)\big)\nabla T_{m+1}(u)\,dx\,dt\\
&\qquad-\int_Q a\big(x,t,T_m(u),\nabla T_m(u)\big)\nabla T_m(u)\,dx\,dt\\
&\quad=\int_{\{m\le|u|\le m+1\}} a(x,t,u,\nabla u)\nabla u\,dx\,dt.
\end{aligned}\tag{5.62}$$

Taking the limit as m tends to $+\infty$ in (5.62) and using the estimate (5.61) it possible to conclude that (5.57) holds true and the proof of Lemma 5.7 is complete. □

Step 5. In this step, u is shown to satisfy (4.3) and (4.4). Let S be a function in $W^{1,\infty}(\mathbb{R})$ such that S has a compact support. Let K be a positive real number such that $\operatorname{supp}(S) \subset [-K, K]$. Pointwise multiplication of the approximate equation (5.5) by $S(u_n)$ leads to

$$\begin{aligned}
&\frac{\partial B_S(u_n)}{\partial t} - \operatorname{div}\big(S(u_n)a(x,t,u_n,\nabla u_n)\big) + S'(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n\\
&\qquad - \operatorname{div}\big(S(u_n)\Phi(u_n)\big) + S'(u_n)\Phi(u_n)\nabla u_n = fS(u_n) \text{ in } D'(Q),
\end{aligned}\tag{5.63}$$

where $B_S(z) = \int_0^z b'(r)S(r)\,dr$.

It what follows we pass to the limit as n tends to $+\infty$ in each term of (5.63).

Limit of $\frac{\partial B_S^n(u_n)}{\partial t}$.

Since S is bounded and $B_S^n(u_n)$ converges to $B_S(u)$ a.e. in Q and in $L^\infty(Q)$ weak-*. Then $\frac{\partial B_S^n(u_n)}{\partial t}$ converges to $\frac{\partial B_S(u)}{\partial t}$ in $D'(Q)$ as n tends to $+\infty$.

Limit of $-\operatorname{div}\big(S(u_n)a(x,t,u_n,\nabla u_n)\big)$.

Since $\operatorname{supp} S \subset [-K,K]$, we have

$$S(u_n)a(x,t,u_n,\nabla u_n) = S(u_n)a\big(x,t,T_K(u_n),\nabla T_K(u_n)\big) \text{ a.e. in } Q.$$

The pointwise convergence of u_n to u as n tends to $+\infty$, the bounded character of S, (5.23) and (5.30) of Lemma 5.6 imply that

$$S(u_n)a\big(x,t,T_K(u_n),\nabla T_K(u_n)\big) \rightharpoonup S(u)a\big(x,t,T_K(u),\nabla T_K(u)\big) \text{ weakly in } \big(L_{\overline{M}}(Q)\big)^N$$

for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$ as n tends to $+\infty$ because $S(u)=0$ for $|u|\geq K$ a.e. in Q. Moreover,

$$S(u)a\big(x,t,T_K(u),\nabla T_K(u)\big) = S(u)a(x,t,u,\nabla u) \text{ a.e. in } Q.$$

Limit of $S'(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n$.

Since $\operatorname{supp} S' \subset [-K,K]$, we have

$$S'(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n = S'(u_n)a\big(x,t,T_K(u_n),\nabla T_K(u_n)\big)\nabla T_K(u_n) \text{ a.e. in } Q.$$

The pointwise convergence of $S'(u_n)$ to $S'(u)$ as n tends to $+\infty$, the bounded character of S' and (5.30)–(5.31) of Lemma 5.6 allow to conclude that

$$S'(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n \rightharpoonup S'(u)a\big(x,t,T_K(u),\nabla T_K(u)\big)\nabla T_K(u) \text{ weakly in } L^1(Q),$$

as n tends to $+\infty$. Moreover,

$$S'(u)a\big(x,t,T_K(u),\nabla T_K(u)\big)\nabla T_K(u) = S'(u)a(x,t,u,\nabla u)\nabla u \text{ a.e. in } Q.$$

Limit of $S(u_n)\Phi_n(u_n)$.

Since $\operatorname{supp} S \subset [-K,K]$, we have

$$S(u_n)\Phi_n(u_n) = S(u_n)\Phi_n\big(T_K(u_n)\big) \text{ a.e. in } Q.$$

As a consequence of (3.6), (5.2) and (5.23), it follows that

$$S'(u_n)\Phi_n(u_n) \to S'(u)\Phi\big(T_K(u)\big) \text{ strongly in } \big(E_M(Q)\big)^N,$$

as n tends to $+\infty$. The term $S'(u)\Phi\big(T_K(u)\big)$ is denoted by $S'(u)\Phi(u)$.

Limit of $S'(u_n)\Phi_n(u_n)\nabla u_n$.

Since $S \in W^{1,\infty}(\mathbb{R})$ with $\operatorname{supp} S \subset [-K, K]$, we have

$$S'(u_n)\Phi_n(u_n)\nabla u_n = \Phi_n\big(T_K(u_n)\big)\nabla S'(u_n) \text{ a.e. in } Q.$$

Moreover, $\nabla S'(u_n)$ converges to $\nabla S'(u)$ weakly in $L_M(Q)^N$ as n tends to $+\infty$, while $\Phi_n\big(T_K(u_n)\big)$ is uniformly bounded with respect to n and converges a.e. in Q to $\Phi\big(T_K(u)\big)$ as n tends to $+\infty$. Therefore

$$S'(u_n)\Phi_n(u_n)\nabla u_n \rightharpoonup \Phi\big(T_K(u)\big)\nabla S'(u) \text{ weakly in } L_M(Q).$$

Limit of $f_n S(u_n)$.

Due to (5.3) and (5.23), we have

$$f_n S(u_n) \to f S(u) \text{ strongly in } L^1(Q),$$

as n tends to $+\infty$.

As a consequence of the above convergence result, we are in a position to pass to the limit as n tends to $+\infty$ in equation (5.63) and to conclude that u satisfies (4.3).

It remains to show that $B_S(u)$ satisfies the initial condition (4.4). To this end, firstly note that, S being bounded, $B_S^n(u_n)$ is bounded in $L^\infty(Q)$. Secondly, (5.63) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial B_S^n(u_n)}{\partial t}$ is bounded in $L^1(Q) + W^{-1,x}L_{\overline{M}}(Q)$. Thus an Aubin type lemma (see, e.g., [32], Corollary 4) and (see also Lemma 2.6) implies that $B_S^n(u_n)$ lies in a compact set of $C^0\big([0,T]; L^1(\Omega)\big)$. It follows that on the one hand $B_S^n(u_n)(t=0) = B_S^n(u_{0n})$ converges to $B_S(u)(t=0)$ strongly in $L^1(\Omega)$. On the other hand, the smoothness of S implies that

$$B_S(u)(t=0) = B_S(u_0) \text{ in } \Omega.$$

As a conclusion of step 1 to step 5, the proof of theorem 5.1 is complete. □

6. Appendix

Proof Lemma 5.7. Integration by parts and the use of the properties of $w_{i,j}^{\mu}$ yield

$$\int_0^T \left\langle \frac{\partial b_n(u_n)}{\partial t}, \varphi_{n,j,m}^{\mu,i} \right\rangle dx\,dt = \int_0^T \left\langle \frac{\partial b_n(u_n)}{\partial t}, h_m(u_n)T_k(u_n) \right\rangle dx\,dt$$
$$- \int_0^T \left\langle \frac{\partial b_n(u_n)}{\partial t}, h_m(u_n)w_{i,j}^{\mu} \right\rangle dx\,dt = I_1^n + I_2^{n,\mu}. \quad (6.1)$$

We denote by

$$B_{m,k}^n(r) = \int_0^r b_n'(s)h_m(s)T_k(s)\,ds,$$
$$B_m^n(r) = \int_0^r b_n'(s)h_m(s)\,ds.$$

By a standard argument we can write the first term on the right-hand side of (6.1) as

$$I_1^n = \left[\int_\Omega B_{m,k}^n(u_n)\,dx\right]_0^T = \int_\Omega \left[B_{m,k}^n\big(u_n(T)\big) - B_{m,k}^n\big(u_n(0)\big)\right] dx$$
$$= \int_\Omega \left[B_{m,k}^n\big(u_n(T)\big) - B_{m,k}^n(u_{n0})\right] dx. \quad (6.2)$$

To pass to the limit in (6.2) as $n \to +\infty$, we first observe that $b_n'(u_n)h_m(u_n) = \big(b'\big(T_{m+1}(u_n)\big) + \frac{1}{n}\big)h_m(u_n)$ for n (with $n > m+1$) large enough. Then we deduce that

$$I_1^n = \int_\Omega \left[B_{m,k}\big(u(T)\big) - B_{m,k}(u_0)\right] dx + \varepsilon(n), \quad (6.3)$$

where $B_{m,k}(s) = \int_0^r b'(s)h_m(s)T_k(s)\,ds$.

The second term on the right-hand side of (6.1) can be written as

$$I_2^{n,\mu} = -\int_0^T \left\langle \frac{\partial b_n(u_n)}{\partial t}, h_m(u_n)w_{i,j}^{\mu} \right\rangle dt$$
$$- \int_\Omega [B_m^n(u_n)w_{i,j}^{\mu}]_0^T\,dx + \int_\Omega \int_0^T \alpha_m^n(u_n)\frac{\partial w_{i,j}^{\mu}}{\partial t}\,dx\,dt$$
$$= -\int_\Omega \left[B_m^n\big(u_n(T)\big)w_{i,j}^{\mu}(T) - B_m^n(u_{0n})T_k(\psi_i)\right]$$
$$+ \mu \int_\Omega \int_0^T B_m^n(u_n)\big(T_k(v_j) - w_{i,j}^{\mu}\big)\,dx\,dt. \quad (6.4)$$

By passing to the limit as n tends to infinity in (6.4) we get

$$I_2^{n,\mu} = -\int_\Omega B_m\big(u(T)\big)w_{i,j}^\mu(T) - B_m(u_0)T_k(\psi_i)\,dx + \mu\int_\Omega\int_0^T B_m(u)\big(T_k(v_j) - w_{i,j}^\mu\big)\,dx\,dt + \varepsilon(n),$$

where $B_m(s) = \int_0^r b'(s)h_m(s)\,ds$.

Now letting $j \to +\infty$, we get

$$I_2^{n,\mu} = -\int_\Omega \big[B_m\big(u(T)\big)\big(T_k(u)_\mu(T) + e^{-\mu T}T_k(\psi_i)\big) - B_m(u_0)T_k(\psi_i)\big]\,dx + \mu\int_\Omega\int_0^T B_m\big(u(T)\big)\big(T_k(u) - \big(T_k(u)_\mu - e^{-\mu T}T_k(\psi_i)\big)\,dx\,dt + \varepsilon(n,j). \tag{6.5}$$

Therefore, passing to the limit, first in μ and then in j, in the first terms on the right-hand side of the last equality, we deduce that

$$-\int_\Omega \big[B_m\big(u(T)\big)\big(T_k(u)_\mu(T) + e^{-\mu T}T_k(\psi_i)\big) - B_m(u_0)T_k(\psi_i)\big]\,dx = \int_\Omega \big[B_m\big(u(T)\big)\big(T_k\big(u(T)\big) - B_m(u_0)T_k(u_0)\big]\,dx + \varepsilon(\mu,i). \tag{6.6}$$

Let $w_\mu^i = \big(T_k(u)\big)_\mu + e^{-\mu t}T_k(\psi_i)$ and note that $\frac{\partial w_\mu^i}{\partial t} = \mu\big(T_k(u) - w_\mu^i\big)$. Then the second term on the right-hand side of (6.6) can be rewritten as

$$\begin{aligned}
&\mu\int_\Omega\int_0^T B_m(u)\big(T_k(u) - \big(T_k(u)_\mu - e^{-\mu t}T_k(\psi_i)\big)\,dx\,dt\\
&\quad= \mu\int_\Omega\int_0^T \big(B_m(u) - B_m\big(T_k(u)\big)\big)\big(T_k(u) - w_\mu^i\big)\,dx\,dt\\
&\qquad+ \mu\int_\Omega\int_0^T \big(B_m\big(T_k(u)\big) - B_m(w_\mu^i)\big)\big(T_k(u) - w_\mu^i\big)\,dx\,dt\\
&\qquad+ \mu\int_\Omega\int_0^T B_m(w_\mu^i)\big(T_k(u) - w_\mu^i\big)\,dt\,dx\\
&\quad= II_1^\mu + II_2^\mu + II_3^\mu,
\end{aligned} \tag{6.7}$$

where

$$II_1^\mu = \mu\int_{\{u>k\}}\int_0^T \big(B_m(u) - B_m(k)\big)(k - w_\mu^i)\,dx\,dt + \mu\int_{\{u<-k\}}\int_0^T \big(B_m(u) - B_m(-k)\big)(-k - w_\mu^i)\,dx\,dt \geq 0. \tag{6.8}$$

As $B_m(s)$ is non-decreasing for s and $-k \le w_\mu^i \le k$, it follows that

$$II_2^\mu \ge 0. \tag{6.9}$$

Moreover,

$$\begin{aligned} II_3^\mu &= \mu \int_\Omega \int_0^T B_m(w_\mu^i)\big(T_k(u) - w_\mu^i\big)\, dx\, dt \\ &= \int_\Omega \int_0^T B_m(w_\mu^i) \frac{\partial w_\mu^i}{\partial t}\, dx\, dt = \int_\Omega \big(\bar{B}\big(w_\mu^i(T)\big) - \bar{B}\big(w_\mu^i(0)\big)\, dx, \end{aligned} \tag{6.10}$$

where $\bar{B}_m(s) = \int_0^s B_m(r)\, dr$. Also $w_\mu^i \to T_k(u)$ a.e. in Q as μ, i tend to $+\infty$ and $|w_\mu^i| \le k$. Then Lebesque's convergence theorem shows that

$$II_3^\mu = \int_\Omega \big[\bar{B}\big(T_k(u)(T)\big) - \bar{B}\big(T_k(u_0)\big)\big]\, dx + \varepsilon(\mu, i). \tag{6.11}$$

In view of (6.5)–(6.11), one has

$$\begin{aligned} I_2^{n,\mu} = {} & \int_\Omega \big[B_m\big(u(T)\big)T_k\big(u(T)\big) - B_m(u_0)T_k(u_0)\big]\, dx \\ & + \int_\Omega \big[\bar{B}\big(T_k(u)(T)\big) - \bar{B}\big(T_k(u_0)\big)\big]\, dx + \varepsilon(\mu, i). \end{aligned} \tag{6.12}$$

As a consequence of (6.1), (6.3) and (6.12), we deduce that

$$\begin{aligned} \left\langle \frac{\partial b_n(u_n)}{\partial t}, \varphi_{n,j,m}^{\mu,i} \right\rangle \ge {} & \int_\Omega \big[B_{m,k}\big(u(T)\big) - B_{m,k}(u_0)\big]\, dx \\ & - \int_\Omega \big[B_m\big(u(T)\big)T_k\big(u(T)\big) - B_m(u_0)T_k(u_0)\big]\, dx \\ & + \int_\Omega \big[\bar{B}\big(T_k(u)(T)\big) - \bar{B}\big(T_k(u_0)\big)\big]\, dx + \varepsilon(n, j, \mu, i). \end{aligned} \tag{6.13}$$

Observe that for any $z \in \mathbb{R}$ we have

$$\begin{aligned} \bar{B}\big(T_k(z)\big) &= \int_0^{T_k(z)} B_m(r)\, dr \\ &= \left[r \int_0^r b'(t)h_m(t)\, dt\right]_0^{T_k(z)} - \int_0^{T_k(z)} rb'(r)h_m(r)\, dr \\ &= T_k(z) \int_0^{T_k(z)} b'(t)h_m(t)\, dt - \int_0^{T_k(z)} T_k(r)b'(r)h_m(r)\, dr \\ &= T_k(z)\alpha_m\big(T_k(z)\big) - \alpha_{m,k}\big(T_k(z)\big). \end{aligned} \tag{6.14}$$

Finally, we deduce that

$$\left\langle \frac{\partial b_n(u_n)}{\partial t}, \varphi^{\mu,i}_{n,j,m} \right\rangle \geq 0 + \varepsilon(n, j, \mu, i). \tag{6.15}$$

This is due to the fact that for $|r| < k$, we have

$$\bar{B}\big(T_k(r)\big) = T_k(r)B_m(r) - B_{m,k}(r),$$

and if $r > k$ we have

$$B_{m,k}(r) = \int_0^k b'(s)h_m(s)s\,ds + k\int_k^r b'(s)h_m(s)\,ds$$

$$-T_k(r)B_m(r) = -k\int_0^k b'(s)h_m(s)\,ds - k\int_k^r b'(s)h_m(s)\,ds$$

$$\bar{B}(k) = k\int_0^k b'(s)h_m(s)\,ds - \int_0^k b'(s)h_m(s)s\,ds.$$

The case $r < -k$ is similar to the previous one. This conclude the proof of Lemma 5.7. □

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