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A class of integral domains whose integral closures are small submodules of the quotient field

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Abstract. Let (V, M) be a valuation domain which is distinct from its quotient field K, and let $\pi : V \to V/M$ be the canonical surjection. Let D be a subring of V/M. It is proved that the pullback $R := \pi^{-1}(D)$ has the property that V (and hence each integral overring of R) is a small R-submodule of K. Applications include all classical $D + M$ constructions and locally pseudo-valuation domains.

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1. Introduction

All rings considered below are commutative with $1 \neq 0$; all subrings, subalgebras and modules are unital. Recall that if R is a ring and E is an R-submodule of an R-module F, then E is called a *small* R*-submodule of* F if, whenever an R-submodule G of F satisfies $E + G = F$, it must be the case that $G = F$. Consider a (commutative integral) domain R with quotient field K; let R' denote the integral closure of R (in K). Our main interest here is in finding a new sufficient condition that R' be a small R-submodule of K. Our main result, Theorem 2.1 (b), accomplishes somewhat more than this.

The motivation for this work was the following result of Zöschinger [15], Folgerung 1.8: if R is a Noetherian domain which is distinct from its quotient field K, then R', the integral closure of R, is a small R-submodule of K. It does not seem to have been noticed that each integrally closed domain which is not a field has this property. In fact, more generally, if R is *any* domain that is properly contained in its quotient field K, then R is a small R-submodule of K. This fact follows from work of Pareigis [13], it was generalized by Harada in [10], Theorem 1, and recovered in [10], Proposition 2, and it was extended to certain base rings that are not domains by Rayar [14], Proposition 1. (To see the relevance of [14], note that if R is any domain with quotient field K, then K is the injective envelope of the R-module R.) For another generalization due to Harada, see [9].

As explained following Corollary 2.2, many of the domains R satisfying the hypotheses of Theorem 2.1 are not Noetherian and, hence, are not susceptible to the methods of [15]. The rings addressed in Theorem 2.1 can be viewed as pullbacks of the kind that have proved to be useful in constructing counterexamples in multiplicative ideal theory for several decades. In Corollary 2.2, we isolate the application to the most special type of such a pullback, the classical $D + M$ construction, which essentially goes back to Krull. Corollary 2.4 gives an application to the pseudo-valuation domains of Hedstrom-Houston [11] and, more generally, to the locally pseudo-valuation domains that were introduced in [6].

Besides the conventions noted above, we adopt the following. If D is a domain with quotient field F, then an *overring of D* is any D-subalgebra of F, i.e., any ring S such that $D \subseteq S \subseteq F$; $\mathrm{Max}(A)$ denotes the set of all maximal ideals of a ring A, and $\subset$ denotes proper inclusion. Any unexplained material is standard, as in [8].

2. Results

Our main result shows that certain pullbacks that have received considerable attention (cf. [7], Section 2, and the B, I, D construction in [4]) satisfy the conclusion of the motivating result of Zöschinger [15], Folgerung 1.8.

Theorem 2.1. *Let R be a domain that is properly contained in its quotient field, K. Suppose that R has a nonzero ideal M which is the maximal ideal of some valuation overring V of R.* (*Equivalently, suppose that R has a valuation overring $V \neq K$ such that $V \times_{V/M} D = R$, where M is the maximal ideal of V and D is a subring of V/M.*) *Then:*

(a) *If E is an R-submodule of K, then either $E \subseteq V$ or $V \subseteq E$.*

(b) *V is a small R-submodule of K.*

(c) *Each integral overring of R is a small R-submodule of K.*

(d) *R', the integral closure of R, is a small R-submodule of K.*

Proof. The parenthetical comment is apparent, with $D := R/M$.

(a) The proof of (a) is inspired by the first part of the proof of [2], Theorem 3.1. Suppose that $E \not\subseteq V$. We shall prove that $V \subseteq E$. Choose $e \in E \backslash V$. Since V is a valuation domain, e^{-1} must be a nonunit of V, that is, an element of M. It follows that $1 = e^{-1}e \in ME$, so that

$$V \subseteq (ME)V = (MV)E = ME \subseteq RE = E,$$

as desired.

(b) Suppose that the assertion fails. Then there is an R-submodule E of K such that $E \subset K$ and $V + E = K$. As $V + E = K \not\subseteq V$, we must have $E \not\subseteq V$. Therefore, by (a), $V \subseteq E$, whence

$$K = V + E \subseteq E + E = E \subset K,$$

the desired contradiction.

(c), (d): Any submodule of a small submodule is itself a small submodule. Accordingly, since each integral overring of R is contained in V (cf. [8], Theorem 19.8), the assertions follow from (b), thus completing the proof. □

Notice that the rings R and V in Theorem 2.1 have the same quotient field since $M \neq 0$. Thus, the result of Pareigis(–Harada–Rayar) that was mentioned in the Introduction ensures that V is a small V-submodule of K. The above argumentation for parts (a) and (b) of Theorem 2.1 was necessary in order to replace "V-submodule" with "R-submodule".

Corollary 2.2 summarizes the import of parts (b) and (c) of Theorem 2.1 in case the given pullback R is a classical $D + M$ construction.

Corollary 2.2. *Let V be a valuation domain of the form $V = L + M$, where L is a field and $M \neq 0$ is the maximal ideal of V. Let D be a subring of L. Put $R := D + M$. Then V (and hence any integral overring of R) is a small R-submodule of the quotient field of R (that is, of the quotient field of V).*

Many non-Noetherian domains R arise in the context of Theorem 2.1 (cf. [7], Theorem 2.3). In fact, this is well known even in the context of Corollary 2.2, [2], Theorem 2.1 (m), where to ensure a non-Noetherian R, one need only arrange at least one of the following three conditions: V is not a discrete rank 1 valuation domain; D is not a field; D is a field such that $[L : D] = \infty$.

Before applying Theorem 2.1 to locally pseudo-valuation domains, we pause to show that the property being studied is a local property.

Corollary 2.3. (a) *Let R be a domain. Let S be an overring of R such that $S_{R \setminus M}$ is a small R_M-submodule of K for each maximal ideal M of R. Then S is a small R-submodule of K.*

(b) *Let R be a domain such that $(R_M)'$ is a small R_M-submodule of K for each maximal ideal M of R. Then R' is a small R-submodule of K.*

(c) *Let R be a domain that is properly contained in its quotient field, K. Suppose that for each maximal ideal M of R, there exists a valuation overring of R with* *maximal ideal MR_M. Then each integral overring of R (in particular, R') is a small R-submodule of K.*

Proof. (a) More generally, one has that if A is a ring and B is an A-submodule of an A-module C such that $B_{A\setminus M}$ is a small A_M-submodule of $C_{A\setminus M}$ for each $M \in \mathrm{Max}(A)$, then B is a small A-submodule of C. Indeed, if G is an A-submodule of C such that $B + G = C$, we shall prove that $G = C$. If $M \in \mathrm{Max}(A)$, then working with the canonical images inside $C_{A\setminus M}$, we have $B_{A\setminus M} + G_{A\setminus M} = (B+G)_{A\setminus M} = C_{A\setminus M}$. Since $B_{A\setminus M}$ is assumed small, it follows that $G_{A\setminus M} = C_{A\setminus M}$. Therefore, by globalization, $G = C$, as asserted.

(b) It is well known that if $M \in \mathrm{Max}(R)$, then $(R_M)' = (R')_{R\setminus M}$ (cf. [8], Proposition 10.2). Accordingly, the assertion follows from (a) by taking $S := R'$.

(c) Once again invoking the fact that any submodule of a small submodule is itself a small submodule, we see that it suffices to prove that R' is a small R-submodule of K. By (b), it is enough to show that if $M \in \mathrm{Max}(R)$, then $(R_M)'$ is a small R_M-submodule of K. This, in turn, follows by applying Theorem 2.1 (d) to the ring R_M. The proof is complete. □

Recall from [11] that a quasilocal domain (R, M) is called a pseudo-valuation domain (in short, a PVD) if there is a (necessarily uniquely determined) valuation overring V of R with maximal ideal M. According to [1], Proposition 2.6, the PVDs are precisely the pullback rings R as in the hypotheses of Theorem 2.1 for which D is a field. Recall from [6] that a domain R is called a locally pseudo-valuation domain (in short, an LPVD) if R_M is a PVD for each $M \in \mathrm{Max}(R)$. Of course, the most familiar examples of PVDs (resp. LPVDs) are the valuation (resp. Prüfer) domains.

Corollary 2.4. (a) *Let (R, M) be a PVD that is properly contained in its quotient field, K. Let V be the valuation overring of R with maximal ideal M. Then V (and hence any integral overring of R) is a small R-submodule of K.*

(b) *Let R be an LPVD that is properly contained in its quotient field, K. Then R' (and hence any integral overring of R) is a small R-submodule of K.*

Proof. (a) By the above remarks, one need only combine parts (b) and (c) of Theorem 2.1 with the characterization of PVDs in [1], Proposition 2.6.

(b) The assertion follows by combining the definitions of LPVD and PVD with Corollary 2.3 (c). (Notice that this leads to another proof of (a) since any PVD is also an LPVD.) For an alternate proof of (b), combine (a) with either Corollary 2.3 (a) or Corollary 2.3 (b). This completes the proof. □

We close with a couple of remarks.

Remark 2.5. (a) In view of [15], Folgerung 1.8, and Theorem 2.1 (d), one can ask whether there exists any domain R, properly contained in its quotient field K, for which R' is not a small R-submodule of K. We do not know the answer. If such a domain R exists, it follows from [10], Theorem 1, that R' cannot be a finitely generated R-module. By Corollary 2.3 (a), if such R exists, we may as well assume that it is quasilocal. We note that examples are known of quasilocal domains D such that D' has infinitely many maximal ideals. These are reasonable candidates since, for any such D, D' cannot be a finitely generated D-module by [3], Proposition 3, p. 329, and D cannot be a Noetherian ring by [12], Theorem 33.10 (2). However, we have been unable to determine for any of these D whether D' is a small D-submodule of its quotient field.

(b) The above focus should not be shifted from R' (the integral closure of a given domain R properly contained in its quotient field K) to R^*, the complete integral closure of R. Indeed, it is possible for such R to have $R^* = K$, which is, of course, not a small R-submodule of K. For an explicit example of an R with this behavior, consider any valuation domain R which is not a field and which has no prime ideal of height 1. The fact that $R^* = K$ follows from [5], Lemma 4.2 (which applies since R is a conducive domain, in the sense of [5], but not a G-domain).

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