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Stability criteria for certain third order nonlinear delay differential equations

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Abstract. In this paper we study the asymptotic stability of the trivial solution of third order nonlinear delay differential equations of the form

$$x'''(t) + f\big(x(t), x'(t)\big)x''(t) + g\big(x(t-r), x'(t-r)\big) + h\big(x(t-r)\big) = 0,$$

where $r > 0$ is a constant delay. In proving our result we make use of Lyapunov's second method by constructing a Lyapunov functional.

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1. Introduction

In an 1892 paper (for an English translation see [11]), the Russian mathematician Lyapunov introduced a novel approach, now called Lyapunov's second (or direct) method, to stability analysis. This approach consists in finding a scalar function, now called Lyapunov function, which depends on the system variables and satisfies certain preset conditions. Lyapunov showed that this function can be used to deduce the stability properties of solutions to the equations governing the system.

The stability in the sense of Lyapunov is based on the following physical stability definitions: Physically, we may have three basic stability aspects, depending on stability considerations of a motion on its orbit, of the orbit of a motion, and of the boundedness of the motion and its orbit. A motion or its orbit is considered as stable, if, by giving a small disturbance to the motion or to its orbit, the disturbed motion or its orbit, initially near to the unperturbed motion or to its orbit, remains near to it all time. More specifically, if for small disturbances the effect on the motion or on its orbit is small, we say that the motion or its orbit is in a "stable" situation. If for small disturbances the effect is considerable, the situation is "unstable". If for small disturbances the effect tends to disappear, the situation is "asymptotically stable". If, regardless of the magnitude of the disturbances, the effect tends to disappear, the situation is "asymptotically stable in the large". These three different stability aspects are of a qualitative type.

Lyapunov's second method is fairly general and can be applied to stability testing of linear and nonlinear, time-varying continuous systems and their differential equations. Since the use of Lyapunov's second method for the investigation of stability criteria of equations with delay encountered some principal difficulties, Krasovskii [10] carried out the use of functionals defined on equations' trajectories instead of Lyapunov functions. Later, the qualitative behavior of solutions of delay differential equations, or more generally of functional differential equations of first and second order, has been studied extensively and is still the object of intensive research.

There is a wide range of literature dealing with the theory of stability of solutions of first and second order linear and nonlinear differential equations with delay. In particular, in the ordinary case the reader is referred to the books of Burton ([1], [2]), Èl'sgol'ts [3], Èl'sgol'ts and Norkin [4], Gopalsamy [5], Hale [6], Hale and Verduyn Lunel [7], Kolmanovskii and Myshkis [8], Kolmanovskii and Nosov [9], Krasovskii [10] and Yoshizawa [16] and the references cited in these sources.

It should be noted that in spite of the existence of many results on the stability of solutions of delay differential equations of first and second order, there are only a few results on the same subject concerning nonlinear delay differential equations of third order. The reader is referred to the papers of Sadek [13], Sinha [14], Tunç [15], Zhu [17], and the references therein.

All these papers do not contain an explanatory example on the stability of solutions of third order nonlinear delay differential equations. That the stability of solutions of third order nonlinear delay differential equations is not dealt with in many papers is, perhaps, due to the construction of Lyapunov functionals for the delay differential equations of higher order. On the other hand, systems of delay differential equations do now attract a lot of interest in various areas of science. For many real systems have the after-effect property, i.e., the future states depend not only on the present but also on the past history. For instance, after-effect is believed to occur in mechanics, control theory, physics, chemistry, biology, medicine, economics, atomic energy, information theory, etc. This wide occurrence of after-effect is reason to regard it as a universal property of the surrounding world. For a comprehensive treatment of the subject we refer the reader to the book by Kolmanovskii and Myshkis [8], and the books mentioned above.

The aim of this article is to study the asymptotic stability of the trivial solution of the third order nonlinear delay differential equation

$$x'''(t) + f\big(x(t), x'(t)\big)x''(t) + g\big(x(t-r), x'(t-r)\big) + h\big(x(t-r)\big) = 0 \tag{1}$$

or its equivalent system

$$\begin{aligned}
x'(t) &= y(t), \qquad y'(t) = z(t), \\
z'(t) &= -f\big(x(t), y(t)\big)z(t) - g\big(x(t), y(t)\big) - h\big(x(t)\big) \\
&\quad + \int_{t-r}^{t} g_x\big(x(s), y(s)\big)y(s)\,ds + \int_{t-r}^{t} g_y\big(x(s), y(s)\big)z(s)\,ds \\
&\quad + \int_{t-r}^{t} h'\big(x(s)\big)y(s)\,ds.
\end{aligned} \tag{2}$$

Here r is a positive constant, that is, r is a constant delay; the functions f, g and h depend only on the arguments displayed explicitly and the primes in equation (1) denote differentiation with respect to $t \in \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$. It is generally assumed that the functions f, g and h are continuous for all values of their arguments on $\mathbb{R}^2$ and $\mathbb{R}$, respectively. Additionally it is supposed that the derivatives $h'(x) \equiv \frac{d}{dx}h(x)$, $g_x(x, y) \equiv \frac{\partial}{\partial x}g(x, y)$ and $g_y(x, y) \equiv \frac{\partial}{\partial y}g(x, y)$ exist and are continuous. Throughout the article $x(t)$, $y(t)$ and $z(t)$ are abbreviated as x, y and z, respectively. Moreover, it is assumed that solutions of equation (1) exist and are unique, and all solutions considered are real-valued.

It should be noted that, in 1965, Ponzo [12] described a technique involving integration by parts for constructing a Lyapunov function for the third order nonlinear differential equation without delay:

$$x'''(t) + f\big(x(t), x'(t)\big)x''(t) + g\big(x(t), x'(t)\big)x'(t) + h\big(x(t)\big) = 0.$$

Ponzo constructed a Lyapunov function and established sufficient conditions for the asymptotical stability of the trivial solution of this equation. He does, however, not give any explanatory example on the subject. The motivation for the present paper especially comes from the paper of Ponzo [12] and the works mentioned above. Our aim here is to improve the results by Ponzo [12] to the nonlinear delay equation (1) and to prove the asymptotic stability of trivial solution of that equation. We also give an explanatory example related to the equation (1). The equation discussed in Ponzo [12] is also a special case of equation (1) in the case of $r = 0$.

2. Preliminaries

We will start with some basic information on the general autonomous delay differential system; see also Burton [2], Èl'sgol'ts [3], Èl'sgol'ts and Norkin [4], Gopalsamy [5], Hale [6], Hale and Verduyn Lunel [7], Kolmanovskii and Myshkis [8], Kolmanovskii and Nosov [9], Krasovskii [10], and Yoshizawa [16].

Consider the general autonomous delay differential system

$$x' = f(x_t), \qquad x_t(\theta) = x(t+\theta), \qquad -r \le \theta \le 0,\ t \ge 0, \tag{3}$$

where $f : C_H \to \mathbb{R}^n$ is a continuous map, $f(0) = 0$, and we suppose that f takes closed bounded sets into bounded sets of $\mathbb{R}^n$. Here $(C, \|\cdot\|)$ is the Banach space of continuous functions $\phi : [-r, 0] \to \mathbb{R}^n$ with supremum norm, $r > 0$, C_H is the open H-ball in C; $C_H := \{\phi \in (C[-r,0], \mathbb{R}^n) \mid \|\phi\| < H\}$. Standard existence theory, see Burton [1], shows that if $\phi \in C_H$ and $t \ge 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ such that on $[t_0, t_0 + \alpha)$ satisfying equation (3) for $t > t_0$, $x_t(t, \phi) = \phi$ and α is a positive constant. If there is a closed subset $B \subset C_H$ such that the solution remains in B, then $\alpha = \infty$. Further, the symbol $|\cdot|$ will denote the norm in $\mathbb{R}^n$ with $|x| = \max_{1 \le i \le n} |x_i|$.

Definition 1 (See [1]). *Let $f(0) = 0$. The zero solution of equation* (3) *is*

(a) *stable if for each $t_1 \ge t_0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $[\|\phi\| \le \delta, t \ge t_1]$ imply that $|x(t, t_1, \phi)| < \varepsilon$;*

(b) *asymptotically stable if it is stable and if for each $t_1 \ge t_0$ there is an η such that $\|\phi\| \le \eta$ implies that $x(t, t_0, \phi) \to 0$ as $t \to \infty$.*

Definition 2 (See [14]). *If V is a continuous scalar function in C_H, we define the derivative of V along the solutions of* (3) *by the following relation*

$$\dot{V}_{(3)}(\phi) = \limsup_{h \to 0^+} \frac{V\big(x_h(\phi)\big) - V(\phi)}{h}.$$

Lemma (See [14]). *Suppose that $f(0) = 0$. Let V be a continuous functional defined on $C_H = C$ with $V(0) = 0$, and let $u(s)$ be a function, non-negative and continuous for $0 \le s < \infty$, $u(s) \to \infty$ as $u \to \infty$ with $u(0) = 0$. If for all $\phi \in C$, $u(|\phi(0)|) \le V(\phi)$, $V(\phi) \ge 0$, $\dot{V}_{(3)}(\phi) \le 0$, then the solution $x_t = 0$ of* (3) *is stable.*

If we define $Z = \{\phi \in C_H \mid \dot{V}_{(3)}(\phi) = 0\}$, then the solution $x_t = 0$ of (3) *is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.*

3. Main result

The main result of this paper is the following theorem.

Theorem. *In addition to the basic assumptions imposed on the functions f, g and h in equation* (1)*, we assume that there are positive constants a, b, c, c_1, λ, μ, L and M such that the following conditions hold for all x, y and z:*

(i) $f(x, y) \geq a + 2\lambda$ $(y \neq 0)$.

(ii) $g(x, 0) = 0$, $\frac{g(x,y)}{y} \geq b + 2\mu$ $(y \neq 0)$, $|g_x(x, y)| \leq L$ *and* $|g_y(x, y)| \leq M$.

(iii) $h(0) = 0$, $0 < c_1 \leq h'(x) \leq c$.

(iv) $ab - c > \frac{a}{y}\int_0^y f_x(x, \eta)\eta\, d\eta + \frac{1}{y}\int_0^y g_x(x, \eta)\, d\eta \geq 0$ $(y \neq 0)$.

Then the trivial solution of equation (1) *is asymptotically stable provided that*

$$r < \min\left\{\frac{4\mu a}{aL + aM + ac + (L + c)(1 + a)}, \frac{4\lambda}{L + M + c + M(1 + a)}\right\}.$$

Proof. Define the Lyapunov functional $V = V(x_t, y_t, z_t)$ by

$$\begin{aligned} V(x_t, y_t, z_t) = {} & \frac{1}{2}z^2 + ayz + a\int_0^y f(x, \eta)\eta\, d\eta + \int_0^y g(x, \eta)\, d\eta + h(x)y \\ & + a\int_0^x h(\xi)\, d\xi + \rho\int_{-r}^0 \int_{t+s}^t y^2(\theta)\, d\theta\, ds + \gamma\int_{-r}^0 \int_{t+s}^t z^2(\theta)\, d\theta\, ds, \end{aligned} \tag{4}$$

where the constants ρ and γ will be determined later in the proof. It will be shown that the above Lyapunov functional and its time derivative $\frac{d}{dt}V(x_t, y_t, z_t)$ satisfy the assumptions of the lemma, which implies asymptotic stability of the trivial solution of equation (1).

Now, using the assumptions $f(x, y) \geq a + 2\lambda$, $\frac{g(x,y)}{y} \geq b + 2\mu$, $(y \neq 0)$, and $0 < h'(x) \leq c$, it follows that

$$\int_0^y f(x, \eta)\eta\, d\eta \geq \left(\frac{a + 2\lambda}{2}\right)y^2,$$

$$\int_0^y g(x, \eta)\, d\eta = \int_0^y \frac{g(x, \eta)}{\eta}\eta\, d\eta \geq \left(\frac{b + 2\mu}{2}\right)y^2$$

and

$$h^2(x) = 2\int_0^x h(\xi)h'(\xi)\, d\xi \leq 2c\int_0^x h(\xi)\, d\xi.$$

Making use of the above inequalities, the Lyapunov functional $V = V(x_t, y_t, z_t)$ defined in (4) can be recast in the form

$$
\begin{aligned}
V(x_t, y_t, z_t) \geq{}& \frac{1}{2}z^2 + ayz + \left(\frac{a^2 + 2a\lambda}{2}\right)y^2 + \left(\frac{b + 2\mu}{2}\right)y^2 + h(x)y \\
&+ a\int_0^x h(\xi)\,d\xi + \rho\int_{-r}^0\int_{t+s}^t y^2(\theta)\,d\theta\,ds + \gamma\int_{-r}^0\int_{t+s}^t z^2(\theta)\,d\theta\,ds \\
\geq{}& \frac{1}{2}(z + ay)^2 + \left(\frac{ab - c}{2a}\right)y^2 + \frac{ac}{2}\left(\sqrt{2c^{-1}\int_0^x h(\xi)\,d\xi} - a^{-1}|y|\right)^2 \\
&+ a\lambda y^2 + \mu y^2 + \rho\int_{-r}^0\int_{t+s}^t y^2(\theta)\,d\theta\,ds + \gamma\int_{-r}^0\int_{t+s}^t z^2(\theta)\,d\theta\,ds \geq 0.
\end{aligned} \tag{5}
$$

Now it is clear from (5) that there exist some positive constants D_i $(i = 1, 2, 3)$ such that

$$
V \geq D_1x^2 + D_2y^2 + D_3z^2 + \rho\int_{-r}^0\int_{t+s}^t y^2(\theta)\,d\theta\,ds + \gamma\int_{-r}^0\int_{t+s}^t z^2(\theta)\,d\theta\,ds.
$$

Therefore, due to the above discussion, the existence of a continuous function $u(|\phi(0)|)$ with $u(|\phi(0)|) \geq 0$, which satisfies the inequality $u(|\phi(0)|) \leq V(\phi)$, can be easily verified, since the integrals $\rho\int_{-r}^0\int_{t+s}^t y^2(\theta)\,d\theta\,ds$ and $\gamma\int_{-r}^0\int_{t+s}^t z^2(\theta)\,d\theta\,ds$ are non-negative.

Now, by differentiating the functional $V(x_t, y_t, z_t)$ and using (4) and (2), we obtain that

$$
\begin{aligned}
\frac{d}{dt}V(x_t, y_t, z_t) ={}& -\Big[a\frac{g(x, y)}{y} - h'(x) - \frac{a}{y}\int_0^y f_x(x, \eta)\eta\,d\eta - \frac{1}{y}\int_0^y g_x(x, \eta)\,d\eta\Big]y^2 \\
&- \big(f(x, y) - a\big)z^2 + z\int_{t-r}^t g_x\big(x(s), y(s)\big)y(s)\,ds \\
&+ z\int_{t-r}^t g_y\big(x(s), y(s)\big)z(s)\,ds + ay\int_{t-r}^t g_x\big(x(s), y(s)\big)y(s)\,ds \\
&+ ay\int_{t-r}^t g_y\big(x(s), y(s)\big)z(s)\,ds + z\int_{t-r}^t h'\big(x(s)\big)y(s)\,ds \\
&+ ay\int_{t-r}^t h'\big(x(s)\big)y(s)\,ds + \rho y^2r - \rho\int_{t-r}^t y^2(s)\,ds \\
&+ \gamma z^2r - \gamma\int_{t-r}^t z^2(s)\,ds.
\end{aligned} \tag{6}
$$

By noting the assumptions (i)–(iv) of the theorem and the inequality $2|ab| \leq a^2 + b^2$, one can easily get the following inequalities:

$$-(f(x,y)-a)z^2 \leq -2\lambda z^2,$$

$$-\Big[a\frac{g(x,y)}{y} - h'(x) - \frac{a}{y}\int_0^y f_x(x,\eta)\eta\,d\eta - \frac{1}{y}\int_0^y g_x(x,\eta)\,d\eta\Big]y^2$$

$$\leq -\Big[ab - c - \frac{a}{y}\int_0^y f_x(x,\eta)\eta\,d\eta - \frac{1}{y}\int_0^y g_x(x,\eta)\,d\eta\Big]y^2 - 2\mu a y^2 \leq -2\mu a y^2,$$

$$z\int_{t-r}^t g_y\big(x(s),y(s)\big)z(s)\,ds \leq \frac{M}{2}rz^2(t) + \frac{M}{2}\int_{t-r}^t z^2(s)\,ds,$$

$$z\int_{t-r}^t g_x\big(x(s),y(s)\big)y(s)\,ds \leq \frac{L}{2}rz^2(t) + \frac{L}{2}\int_{t-r}^t y^2(s)\,ds,$$

$$ay\int_{t-r}^t g_x\big(x(s),y(s)\big)y(s)\,ds \leq \frac{aL}{2}ry^2(t) + \frac{aL}{2}\int_{t-r}^t y^2(s)\,ds,$$

$$ay\int_{t-r}^t g_y\big(x(s),y(s)\big)z(s)\,ds \leq \frac{aM}{2}ry^2(t) + \frac{aM}{2}\int_{t-r}^t z^2(s)\,ds,$$

$$z\int_{t-r}^t h'\big(x(s)\big)y(s)\,ds \leq \frac{c}{2}rz^2(t) + \frac{c}{2}\int_{t-r}^t y^2(s)\,ds,$$

$$ay\int_{t-r}^t h'\big(x(s)\big)y(s)\,ds \leq \frac{ac}{2}ry^2(t) + \frac{ac}{2}\int_{t-r}^t y^2(s)\,ds.$$

Substituting the inequalities obtained above into (6), it is easy to see that

$$\begin{aligned}\frac{d}{dt}V(x_t,y_t,z_t) \leq &-\Big[2a\mu - \Big(\frac{aL+aM+ac+2\rho}{2}\Big)r\Big]y^2\\ &-\Big[2\lambda - \Big(\frac{L+M+c+2\gamma}{2}\Big)r\Big]z^2\\ &+\Big[\frac{(L+aL+c+ac)}{2} - \rho\Big]\int_{t-r}^t y^2(s)\,ds\\ &+\Big[\frac{(M+aM)}{2} - \gamma\Big]\int_{t-r}^t z^2(s)\,ds. \end{aligned} \tag{7}$$

If we choose $\rho = \frac{(L+aL+c+ac)}{2}$ and $\gamma = \frac{(M+aM)}{2}$, we obtain from (7) that

$$\begin{aligned}\frac{d}{dt}V(x_t,y_t,z_t) \leq &-\Big[2a\mu - \Big(\frac{aL+aM+ac+2\rho}{2}\Big)r\Big]y^2\\ &-\Big[2\lambda - \Big(\frac{L+M+c+2\gamma}{2}\Big)r\Big]z^2. \end{aligned} \tag{8}$$

Taking into account (8), we can conclude for some positive constants α and σ that

$$\frac{d}{dt}V(x_t, y_t, z_t) \le -\alpha y^2 - \sigma z^2 \le 0,$$

provided that

$$r < \min\left\{\frac{4\mu a}{aL + aM + ac + (L + c)(1 + a)}, \frac{4\lambda}{L + M + c + M(1 + a)}\right\}.$$

It can also be easily shown that the largest invariant set in Z is $Q = \{0\}$, where $Z = \{\phi \in C_H \mid \dot{V}(\phi) = 0\}$. Namely, the only solution of equation (1) for which $\frac{d}{dt}V(x_t, y_t, z_t) = 0$ is the solution $x_t \equiv 0$. Thus, in view of the above discussion, one can say that the trivial solution of equation (1) is asymptotically stable. This completes the proof of theorem. □

Example. Consider third order nonlinear delay differential equation

$$x'''(t) + \left(8 + \left(x'(t)\right)^2\right)x''(t) + 4x'(t - r) + \sin x'(t - r) + 2x(t - r) = 0. \quad (9)$$

Now it can be seen that differential equation (9) has the form (1) and may be expressed as

$$\begin{aligned} x'(t) &= y(t), \qquad y'(t) = z(t), \\ z'(t) &= -\left(8 + y^2(t)\right)z(t) - \left(4y(t) + \sin y(t)\right) - 2x(t) \\ &\quad + 2\int_{t-r}^{t} y(s)\, ds + \int_{t-r}^{t} \left(4 + \cos y(s)\right)z(s)\, ds. \end{aligned} \quad (10)$$

Clearly, by comparing (10) with (1) and taking into account the assumptions of the Theorem, we have that

$$\begin{gathered} f(y) = 8 + y^2 \ge 8 = a + 2\lambda, \\ g(y) = 4y + \sin y, \qquad g(0) = 0, \\ \frac{g(y)}{y} = 4 + \frac{\sin y}{y} \qquad (y \neq 0), \\ 4 + \frac{\sin y}{y} \ge 3 = b + 2\mu, \\ h(x) = 2x, \qquad h(0) = 0, \qquad h'(x) = 2, \end{gathered}$$

$c_1 \in (0, 2]$, $c = 2$, $ab > 2$, $M = 5$ and $L = 0$ (or $L = \varepsilon$ for any $\varepsilon > 0$).

Hence, the above facts show that all conditions (i) to (iv) of the Theorem are satisfied.

Corollary. *Take any* $a \in (2/3, 8)$ *in* (4). *Applying the Theorem, one obtains that the trivial solution of* (9) *is asymptotically stable if the delay* r *satisfies*

$$r < \min\left\{\frac{2(3a-2)}{9a+2}, \frac{2(8-a)}{12+5a}\right\}.$$

For instance, with $a = 1$ the result follows for $r < 2/11$.

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