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The counting hierarchy in binary notation

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Abstract. We present a new recursion-theoretic characterization of FCH, the *hierarchy of counting functions*, in binary notation. Afterwards we introduce a theory of *bounded arithmetic*, TCA, that can be seen as a reformulation, in the binary setting, of Jan Johannsen and Chris Pollett's system D^0_2. Using the previous inductive characterization of FCH, we show that a strategy similar to the one applied to D^0_2 can be used in order to characterize FCH as the class of functions provably total in TCA.

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1. Introduction

For the past two decades several theories of *bounded arithmetic* were introduced because of their connection to *computational complexity theory*. We can, for instance, mention the work of Samuel Buss [1], where he introduces theories like S^1_2, U^1_2, and V^1_2 and proves they characterize, respectively, PTIME, PSPACE and EXPTIME as the class of functions provably total in these theories, with appropriate graphs. For related work in the area, see also [12], [13], [9], [10] and [2].

In this article we focus on a particular class, the *hierarchy of counting functions*, FCH. Since it was introduced by Wagner in 1986 [19], by means of some computational considerations over the class #P, several characterizations of this computational complexity class emerged. We present an inductive characterization of FCH which differs from other characterizations in virtue of the use of binary notation instead of numerical notation and of closing the class with respect to some counting operations instead of some sums (see [18]).

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Next, our purpose is to find a system of *bounded arithmetic*, in binary notation, that characterizes FCH in the sense that the algorithms the theory proves total are exactly the ones in the Counting Hierarchy. Our departure point is a paper of Johannsen and Pollett [10], published in the *Annual Symposium on LICS* in 1998, where an hierarchy of second-order bounded arithmetic theories D_k^0 for $k \geq 1$ is defined. In particular D_2^0 characterizes FCH. Based on D_2^0, we introduce a second-order theory TCA (an acronym for *theory for counting arithmetic*) in binary notation, and using the previous characterization of FCH we show that the provably total functions in this theory (having appropriate graphs) are still the functions in FCH.

Apropos bounded arithmetic theories related with counting, we should also mention the systems C_3^0, C_2^0 and Δ_1^b-CR ([10], [11]) by Johannsen and Pollett and the system VTC^0 ([14], [15], [16]) by Phuong Nguyen and Stephen Cook. All these systems were developed in order to capture the computational power of particular classes of thresholds circuits. As commented in [10]: "*A phenomenon that is commonly observed in Complexity Theory is that proofs of results about counting complexity classes* (#P, $\mathsf{Mod}_p\,P$ *etc.*) *can often be scaled down to yield results about small depth circuit classes with the corresponding counting gates*."

The systems above characterize classes of functions computable by constant-depth threshold circuits of polynomial or quasi-polynomial size (TC^0, qTC^0, respectively). The first-order theory C_3^0 characterizes qTC^0 and is isomorphic to D_2^0 via the so called RSUV-isomorphism [10]. The first-order theory C_2^0 characterizes TC^0 and is RSUV-isomorphic to D_1^0 [10]. It was noticed in [11] that to capture exactly the class TC^0 it is enough to consider the subsystem Δ_1^b-CR of C_2^0. Δ_1^b-CR is a "minimal" first-order theory to TC^0. Another way to characterize TC^0 via formal systems is considering the finitely axiomatizable second-order theory VTC^0 (see [14], [15], [16]). It is known that VTC^0 is RSUV-isomorphic to Δ_1^b-CR and the former system appears to be weaker than D_1^0 ([16]).

We summarize the prior work discussed above in the following scheme:

$$
\begin{array}{ccccccc}
\mathsf{qTC}^0 & \longleftarrow - - - & \mathsf{C}_3^0 & \xleftrightarrow{RSUV} & \mathsf{D}_2^0 & - - - \longrightarrow & \mathsf{FCA}(2^{\tau_2(n)}) = \mathsf{FCH} \\
\mathsf{TC}^0 & \longleftarrow - - - & \mathsf{C}_2^0 & \xleftrightarrow{RSUV} & \mathsf{D}_1^0 & - - - - \longrightarrow & \mathsf{FCA}(2^{\tau_1(n)}) \\
\nwarrow & & \vee\!\!/ & & & & \\
 & - - & \Delta_1^b - \mathsf{CR} & \xleftrightarrow{RSUV} & \mathsf{VTC}^0 & & \text{with } \tau_1(n){:=}O(n) \text{ and } \tau_{k+1}(n){:=}2^{\tau_k(log n)}
\end{array}
$$

The option (in TCA) for a language that directly describes finite sequences of zeros and ones, instead of the numerical notation adopted in the theories above, comes in the sequel of a similar choice done by Fernando Ferreira in [4] in the context of PTIME and seems more natural for dealing with sub-exponential computability. This way (and this was our main motivation), TCA is compatible with Ferreira's theories of feasible analysis [5] being a natural candidate (after properly improved) for pursuing new results in *weak analysis*.

Note that Σ_1^b-NIA $\subseteq$ TCA and the former theory was introduced as a base system to develop a theory for analysis, BTFA [5], where a basic portion of ordinary mathematics can be formalized [3].

The intent is that TCA works as a base theory for building a second-order system for analysis TCA^2, following the informal correspondence:

$$\frac{\mathsf{BTFA}}{\Sigma_1^b\text{-}\mathsf{NIA}} \sim \frac{\mathsf{TCA}^2}{\mathsf{TCA}}.$$

The enrichment of TCA and the questions concerning the formalization of analysis in this framework are mentioned here just as a motivation. They are not dealt with in this paper.

The interest in systems connected with FCH instead of PTIME is justified by [6], where *counting* was proved to be a consequence of *integration*, even in weak systems like BTFA. Therefore, a theory like TCA^2 seems to be the appropriate setting to formalize Riemann Integration (see [7] and [8]).

2. The hierarchy of counting functions

With the aim of studying the complexity of computing the permanent of a matrix, Valiant defined, in the late 1970s, the class #P [17] that consists of the functions f for which there exists a nondeterministic Turing Machine, M, working in polynomial time, such that for all x, $f(x)$ is the number of accepting computations, in M, induced by the input x.

Based in #P, Wagner introduced in [19][1] the class FCH, the *hierarchy of counting functions*, through the definition

$$\mathsf{FCH} = \bigcup_{i \geq 0} i\#\mathsf{P},$$

where $0\#\mathsf{P} = \mathsf{P}$ and $(i+1)\#\mathsf{P} = \#\mathsf{P}^{i\#\mathsf{P}}$, for $i \geq 0$, i.e., $(i+1)\#\mathsf{P}$ is the class of functions that "count" the number of accepting computations in a polynomial time nondeterministic Turing Machine, permitting a function in $i\#\mathsf{P}$ as an oracle. Some recursion-theoretic characterizations of FCH are already known. One, due to Vollmer and Wagner [18], states that

FCH *is the smallest class of functions that contains the arithmetic operations* 0, 1, $+$, $\dot{-}$, $\cdot$ *and the projections* P_j^n, *and is closed under composition and the sums* $\sum_{i=0}^{2^{p(|x|)}} g(x,i)$, *with* p *a polynomial and* $|x|$ *the length of* x.

[1] In this paper FCH is denoted by PHCF.

We give an alternative inductive characterization of FCH, in the line of the PTIME characterization introduced by Ferreira in [4]. Since it is presented in binary notation, we need to introduce some operations. Let $2^{<\omega}$ (also known as $\{0,1\}^*$) be the set of all finite sequences of 0's and 1's, where the empty sequence is denoted by ε. For x and y elements in $2^{<\omega}$, $x\hat{\ }y$ represents the *concatenation* of x by y (we usually omit the symbol ^ and just write xy); $x \subseteq y$ means that x is an *initial subword* of y (string prefix); $|x|$ denotes the *length* of x, i.e., the number of 0's and 1's in the word x; $x_{|_y}$ is the *truncation* of x by y defined by

$$x_{|_y} := \begin{cases} x, & \text{if } |x| \leq |y|, \\ z, & \text{if } z \subseteq x \wedge |z| = |y|; \end{cases}$$

$x \times y$ is the *product* of x by y defined as being the word x concatenated with itself length of y times; $x \preceq y$ (resp. $x \equiv y$) abbreviates $1 \times x \subseteq 1 \times y$ (resp. $1 \times x = 1 \times y$) meaning that the length of x is less than or equal (resp. equal) to the length of y; and $\leq_l$ is the linear order defined by $x \leq_l y :\Leftrightarrow \big(x \preceq y \wedge \neg(x \equiv y)\big) \vee \big(x \equiv y \wedge \exists z \subseteq x(z0 \subseteq x \wedge z1 \subseteq y)\big) \vee (x = y)$, i.e., it is defined first according to length and then, within the same length, lexicographically.

Remark 2.1.
- Between $2^{<\omega}$ and $\mathbb{N}$, we consider the natural bijection that respects the linear orders $\leq_l$ (in $2^{<\omega}$) and $\leq$ (in $\mathbb{N}$).
- Despite using the same symbol, from the context it will be clear if 0 refers to the word in $2^{<\omega}$ that corresponds to the natural number 1 by the previous bijection, or if it refers to the number 0 in $\mathbb{N}$ that corresponds to the empty word, ε, in $2^{<\omega}$. A similar consideration holds for 1.
- Again by the bijection between $2^{<\omega}$ and $\mathbb{N}$, given a function, we consider the function as the same and we denote it in the same way in spite of being presented in binary or numerical notation. From the context it will be clear in what notation the function is.

A new inductive characterization of FCH is presented in the proposition below.

Proposition 2.2. FCH *is the smallest class of functions that includes the initial functions*

(1) $C_0(x) = x0$,

(2) $C_1(x) = x1$,

(3) $P_i^n(x_1, \ldots, x_n) = x_i$ *for* $1 \leq i \leq n$,

(4) $Q(x, y) = \begin{cases} 0, & \textit{if } x \subseteq y, \\ \varepsilon, & \textit{otherwise}, \end{cases}$

and is closed under the following schemes:

- *composition*
  $f(\bar{x}) = g\big(h_1(\bar{x}), \ldots, h_k(\bar{x})\big)$,
- *bounded recursion on notation*
  $f(\bar{x}, \varepsilon) = g(\bar{x})$,
  $f(\bar{x}, y0) = h_0\big(\bar{x}, y, f(\bar{x}, y)\big)\big|_{t(\bar{x}, y)}$,
  $f(\bar{x}, y1) = h_1\big(\bar{x}, y, f(\bar{x}, y)\big)\big|_{t(\bar{x}, y)}$,
  where t is a bounding function[2]*, i.e., t belongs to the smallest class of functions that includes ε , 0, 1, ^, $\times$, P^n_j and is closed under composition*
- *cardinality*

$$c(\bar{x}, \varepsilon) = \begin{cases} 0, & \text{if } f(\bar{x}, \varepsilon) = 0, \\ \varepsilon, & \text{otherwise}, \end{cases} \qquad c\big(\bar{x}, S(y)\big) = \begin{cases} S\big(c(\bar{x}, y)\big), & \text{if } f\big(\bar{x}, S(y)\big) = 0, \\ c(\bar{x}, y), & \text{otherwise}, \end{cases}$$

where S is the successor function defined by $S(\varepsilon) = 0$, $S(x0) = x1$, $S(x1) = S(x)0$.

The idea of the last scheme is that $c(\bar{x}, y) = \#\{w \leq_l y : f(\bar{x}, w) = 0\}$, where $\#A$ denotes the number of elements in the set A.

Proof. Let $\mathscr{C}$ be the class of functions introduced in the proposition. We want to prove that $\mathsf{FCH} = \mathscr{C}$.

The initial functions of $\mathscr{C}$ are in P, which is contained in FCH. FCH is closed under composition and as $\mathsf{P} \subseteq \#\mathsf{P} \subseteq \mathsf{P}^{\#\mathsf{P}} \subseteq 2\#\mathsf{P} \subseteq \mathsf{P}^{2\#\mathsf{P}} \subseteq 3\#\mathsf{P} \subseteq \mathsf{P}^{3\#\mathsf{P}} \subseteq \cdots \subseteq \mathsf{FCH}$ and $\mathsf{P}^{k\#\mathsf{P}}$ is closed under bounded recursion on notation, we have that FCH is also closed under this kind of recursion. In order to prove that FCH is closed under cardinality take $f(\bar{x}, y)$ in FCH. We have to see that $c(\bar{x}, y)$ is in FCH. Since $c(\bar{x}, y) = \#\{w \leq_l y : f(\bar{x}, w) = 0\}$, we have $c(\bar{x}, y) = \sum_{i=0}^{y} g(\bar{x}, i)$, where

$$g(\bar{x}, y) = \begin{cases} 1, & \text{if } f(\bar{x}, y) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since $g \in \mathsf{FCH}$, we have that $c(\bar{x}, y)$ is in FCH. So, by definition of $\mathscr{C}$, $\mathscr{C} \subseteq \mathsf{FCH}$.

Conversely, considering the definition of FCH given by Vollmer and Wagner, already mentioned, let us prove that $\mathsf{FCH} \subseteq \mathscr{C}$. Since $\mathsf{P} \subseteq \mathscr{C}$ (note that in [4], P is characterized exactly as being $\mathscr{C}$ without the cardinality scheme), it is immediate that 0, 1, $+$, $\dot{-}$, $\cdot$, P^n_j are in $\mathscr{C}$. By definition, $\mathscr{C}$ is closed under composition, so we just have to prove that $\mathscr{C}$ is closed under the sums presented in the characterization

[2]The bounding functions ensure that the recursion scheme does not produce functions with exponential growth.

of FCH. Since $h(x) = 2^{p(|x|)} \in \mathsf{P} \subseteq \mathscr{C}$ and $\mathscr{C}$ is closed under composition, it is enough to prove that if $g \in \mathscr{C}$, then $f(x,y) := \sum_{i=0}^{y} g(x,i) \in \mathscr{C}$. Now

$$\begin{aligned} f(x,y) &= \sum_{i=0}^{y} g(x,i) = g(x,0) + g(x,1) + \cdots + g(x,y) \\ &= \#\{v : v < g(x,0)\} + \#\{v : v < g(x,1)\} + \cdots + \#\{v : v < g(x,y)\} \\ &= \#\{\langle v,i\rangle : i \le y \wedge v < g(x,i)\} \\ &= \#\big\{u : \exists v, i \preceq u\big(u = \langle v,i\rangle \wedge i \le y \wedge v < g(x,i)\big)\big\}, \end{aligned}$$

where $\langle v,i\rangle$ is the code of (v,i), done in PTIME, for instance in the following manner: $\langle v,i\rangle = \mathrm{cod}(v)11\,\mathrm{cod}(i)$, with $\mathrm{cod}(\varepsilon) = \varepsilon$, $\mathrm{cod}(x0) = \mathrm{cod}(x)01_{|_{11\times x1}}$, $\mathrm{cod}(x1) = \mathrm{cod}(x)10_{|_{11\times x1}}$. Consider

$$h(x,y,u) = \begin{cases} 0, & \text{if } \exists v, i \preceq u\big(u = \langle v,i\rangle \wedge i \le y \wedge v < g(x,i)\big), \\ \varepsilon, & \text{otherwise.} \end{cases}$$

Note that $\exists z \preceq x\varphi(x,z) \leftrightarrow \#\{z \le_l 1 \times x : \chi(x,z) = 0\} \ne \varepsilon \leftrightarrow c_\chi(x, 1 \times x) \ne \varepsilon$, where χ is the characteristic function of φ and c_χ is the function obtained by cardinality from χ. So it is easy to see that $h \in \mathscr{C}$. Moreover, $f(x,y) = \#\{u : h(x,y,u) = 0\} = c_h\big(x, y, t(x,y)y11t(x,y)y11\big)$, where t is a bounding function such that $g(x,i) \le t(x,i)$. We just have to prove that, in fact, there exists t a bounding function for g, because, as a consequence of that and of the monotonicity of t, we have that $t(x,y)y11t(x,y)y11$ majors u. Since $g \in \mathscr{C} \subseteq \bigcup_{i \ge 0} i\#\mathsf{P}$, there exists $k \ge 1$, $k \in \mathbb{N}$, such that $g \in k\#\mathsf{P} = \#\mathsf{P}^{(k-1)\#\mathsf{P}}$. So there exists a nondeterministic Turing machine M, working in polynomial time p with an oracle in $(k-1)\#\mathsf{P}$ such that $g(x,i)$ is the number of acceptance states of M with input x, i. Without loss of generality we may suppose that M has only two possibilities for the next move, so $g(x,i) \le 2^{p(|x|,|i|)}$. This way we prove that there exists t a bounding function for g. Therefore $f \in \mathscr{C}$. Being $\mathscr{C}$ closed under the previous sums we have that $\mathsf{FCH} \subseteq \mathscr{C}$. □

3. Theory for counting arithmetic (TCA)

Let $\mathscr{L}$ be the first-order language with equality which has the constants 0, 1, ε, the binary function symbols $\hat{}$, $\times$ and the binary relation symbols $=$, $\subseteq$.

Let $\mathscr{L}_2^b$ be the second-order language with equality that results from $\mathscr{L}$ adding second-order variables denoted by $X^t, Y^q, \ldots$ with t, q terms of $\mathscr{L}$ and a relation symbol $\in$ that infixes between a term of $\mathscr{L}$ and a second-order variable. The idea behind these variables is that in the standard model first-order variables are elements in $2^{<\omega}$, while second-order variables are subsets X^t of $2^{<\omega}$ verifying $x \in X^t \to x \preceq t$, where t is a term not depending on x.

The *terms* in $\mathscr{L}_2^b$ coincide with the terms in $\mathscr{L}$ and the *class of formulas* in $\mathscr{L}_2^b$ can be defined as the smallest class of expressions containing the atomic formulas $t_1 \subseteq t_2$, $t_1 = t_2$, $t_1 \in F^t$, with t_1, t_2 terms and F^t a second-order variable, and closed under the Boolean operations $\neg$, $\wedge$, $\vee$, $\to$, the first-order quantifications $\forall x$, $\exists x$, the bounded first-order quantifications $\forall x \preceq t$, $\exists x \preceq t$ and the second-order quantifications $\forall F^t$, $\exists F^t$. Note that in $\mathscr{L}_2^b$, $(\forall x \preceq t)P$ and $(\exists x \preceq t)P$ are treated as new formulas and not as abbreviations for $\forall x(x \preceq t \to P)$ and $\exists x(x \preceq t \wedge P)$, respectively. It is a technical detail that helps in some proofs in sequent calculus.

The theory we present next is closely connected with D_2^0, a theory introduced in [10] by Johannsen and Pollett.

Definition 3.1. TCA (Theory for counting arithmetic) is the second-order theory in the language $\mathscr{L}_2^b$, which has the following axioms:

- *Basic axioms*

(1) $x\varepsilon = x$,	(8) $x \subseteq y0 \leftrightarrow x \subseteq y \vee x = y0$,
(2) $x(y0) = (xy)0$,	(9) $x \subseteq y1 \leftrightarrow x \subseteq y \vee x = y1$,
(3) $x(y1) = (xy)1$,	(10) $x0 = y0 \to x = y$,
(4) $x \times \varepsilon = \varepsilon$,	(11) $x1 = y1 \to x = y$,
(5) $x \times y0 = (x \times y)x$,	(12) $x0 \neq y1$,
(6) $x \times y1 = (x \times y)x$,	(13) $x0 \neq \varepsilon$,
(7) $x \subseteq \varepsilon \leftrightarrow x = \varepsilon$,	(14) $x1 \neq \varepsilon$.

- $\forall y \forall F^t(y \in F^t \to y \preceq t)$, with t a term where y does not occur.
- *Induction on notation for* $\boldsymbol{\Sigma_0^{1,b}}$*-formulas*: $A(\varepsilon) \wedge \forall x\big(A(x) \to A(x0) \wedge A(x1)\big) \to \forall x A(x)$, with A a $\Sigma_0^{1,b}$*-formula* (i.e., with no quantifications of second-order and where all the first-order quantifications are bounded). Note that in the standard model these formulas define exactly the predicates in the polynomial hierarchy PH (also known as Meyer–Stockmeyer hierarchy).
- *PH bounded comprehension*: $\exists F^t \forall y \preceq t\big(y \in F^t \leftrightarrow A(y)\big)$, with A a $\Sigma_0^{1,b}$-formula that may have other free variables other than y, but where the variable F^t does not occur, and t a term in which y does not occur.
- *Replacement for* $\boldsymbol{\Sigma_0^{1,b}}$*-formulas*[3]:

$$\forall x \preceq t \exists F^q \varphi(x, F^q) \to \exists G^{(tq'1)(tq'1)} \forall x \preceq t \bar{\varphi}(x, G^{(tq'1)(tq'1)}),$$

[3] In [7], we call this scheme "Substitution for $\boldsymbol{\Sigma_0^{1,b}}$-formulas". It was pointed out to us that "substitution" is not the standard designation for this scheme. This is the reason why we are using the more traditional epithet "replacement" ([1], [10] and [15]).

with φ a $\Sigma_0^{1,b}$-formula, t a term where x does not occur, $q' := q[t/x]$, i.e., q' is the term that results from q replacing all the occurrences of x by the term t, and $\bar{\varphi}$ results from φ replacing all the occurrences of $s \in F^q$ by $\langle x, s\rangle \in G^{(tq'1)(tq'1)}$, where $\langle\, ,\rangle$ is the pairing function that results from the coding introduced in the proof of Proposition 2.2. (This is a technical axiom that permits a kind of "permutations" between bounded first-order universal quantifications and second-order existential quantifications.)

- *Counting axiom*:

$$\exists C^v\, \mathrm{Count}(C^v, F^t), \qquad \text{where } v := (tt11)(tt11) \text{ and } \mathrm{Count}(C^v, F^t)$$

abbreviates the conjunction of $\forall x \preceq t \exists^1 j \preceq v \langle x, j\rangle \in C^v$—a clause which states the functionality of C^v—together with

$$(\varepsilon \notin F^t \to \langle \varepsilon, \varepsilon\rangle \in C^v) \wedge (\epsilon \in F^t \to \langle \varepsilon, 0\rangle \in C^v),$$

and

$$\forall x <_l 1 \times t\big[\big(S(x) \notin F^t \to \forall j \preceq v(\langle x, j\rangle \in C^v \to \langle S(x), j\rangle \in C^v)\big) \\ \wedge \big(S(x) \in F^t \to \forall j \preceq v(\langle x, j\rangle \in C^v \to \langle S(x), S(j)\rangle \in C^v)\big)\big],$$

where $\exists^1 j \preceq v\varphi(j)$ abbreviates the formula $\exists j \preceq v\big(\varphi(j) \wedge \forall k \preceq v\big(\varphi(k) \to k = j\big)\big)$, S is the successor function and t is a term where x does not occur.

In the last scheme, the idea behind the formula Count is that C^v counts the number of elements in F^t. Given $x \preceq t$, we have that $\langle x, j\rangle \in C^v$ if and only if there exists j elements less than or equal to x (by the order $\leq_l$) in F^t.

In order to present some properties in the theory TCA, we define some classes of formulas. A $\Sigma_1^{1,b}$*-formula* (resp. $\Pi_1^{1,b}$*-formula*) is a formula in the language $\mathscr{L}_2^b$ of the form: $\exists F_1^{t_1} \ldots \exists F_k^{t_k}\, \varphi(F_1^{t_1}, \ldots, F_k^{t_k}, \bar{p}, \overline{G}^r)$ (resp. $\forall F_1^{t_1} \ldots \forall F_k^{t_k}\, \varphi(F_1^{t_1}, \ldots, F_k^{t_k}, \bar{p}, \overline{G}^r)$), where φ is a $\Sigma_0^{1,b}$-formula. A $\Sigma_1^{1,b}$*-extended formula* (resp. $\Pi_1^{1,b}$*-extended formula*) is a formula that can be built in a finite number of steps, starting with $\Sigma_0^{1,b}$-formulas and allowing conjunctions, disjunctions, bounded first-order quantifications and second-order existential (resp. universal) quantifications. A formula is $\Delta_1^{1,b}$ (resp. $\Delta_1^{1,b}$*-extended*) *in* TCA if it is equivalent in TCA to both a $\Sigma_1^{1,b}$-formula (resp. a $\Sigma_1^{1,b}$-extended formula) and a $\Pi_1^{1,b}$-formula (resp. a $\Pi_1^{1,b}$-extended formula).

Proposition 3.2. *The following is provable in* TCA*:*

(1) *replacement for* $\Sigma_1^{1,b}$*-extended formulas.*

(2) *bounded comprehension for* $\Delta_1^{1,b}$*-extended formulas.*

(3) *induction on notation for* $\Delta_1^{1,b}$*-extended formulas.*

(4) *minimization scheme for* $\Delta_1^{1,b}$*-extended formulas, i.e.,* $\mathsf{TCA} \vdash \exists x\varphi(x) \to \exists x(\varphi(x) \wedge \forall y <_l x \neg\varphi(y))$*, with* φ *a* $\Delta_1^{1,b}$*-extended formula.*

Proof. (1) Replacement for $\Sigma_1^{1,b}$-formulas follows immediately since two second-order existential quantifiers can be transformed into one. In fact, $\exists F^t \exists H^q \varphi(F^t, H^q) \leftrightarrow \exists G^{(1tq1)(1tq1)} \bar{\varphi}(G^{(1tq1)(1tq1)})$, where $\bar{\varphi}$ results from φ, replacing all the occurrences of $s \in F^t$ and $s \in H^q$ by $\langle \varepsilon, s\rangle \in G$ and $\langle 0, s\rangle \in G$, respectively. By replacement, every $\Sigma_1^{1,b}$-extended formula is equivalent to a $\Sigma_1^{1,b}$-formula, so we have (1).

(2) Take $\varphi(x)$ a $\Delta_1^{1,b}$-extended formula and consider $\psi(x, X^\varepsilon)$ the formula $\varphi(x) \leftrightarrow \varepsilon \in X^\varepsilon$.

In TCA, we know that $\forall x \preceq t \exists X^\varepsilon \psi(x, X^\varepsilon)$. By replacement for $\Sigma_1^{1,b}$-extended formulas, we have $\exists G^{(tq'1)(tq'1)} \forall x \preceq t(\varphi(x) \leftrightarrow \langle x, \varepsilon\rangle \in G^{(tq'1)(tq'1)})$.

Consequently, by comprehension for $\Sigma_0^{1,b}$-formulas, we know that $\exists F^t \forall x \preceq t(x \in F^t \leftrightarrow \varphi(x))$.

(3) Suppose that

$$\varphi(\varepsilon) \wedge \forall x(\varphi(x) \to \varphi(x0) \wedge \varphi(x1)), \qquad (*)$$

with φ a $\Delta_1^{1,b}$-extended formula.

Take a. Applying the bounded comprehension scheme for $\Delta_1^{1,b}$-extended formulas, we have $\exists F^{a0} \forall x \preceq a0(x \in F^{a0} \leftrightarrow \varphi(x))$. From $(*)$ we know, in particular, that $\varphi(\varepsilon) \wedge \forall x \subseteq a(\varphi(x) \to \varphi(x0) \wedge \varphi(x1))$. So $\varepsilon \in F^{a0} \wedge \forall x \subseteq a(x \in F^{a0} \to x0 \in F^{a0} \wedge x1 \in F^{a0})$. By induction on notation for $\Sigma_0^{1,b}$-formulas, we have $\forall x \subseteq a(x \in F^{a0})$. So $a \in F^{a0}$, which implies $\varphi(a)$. Since a is arbitrary, we conclude that $\forall x\varphi(x)$.

(4) Take φ a $\Delta_1^{1,b}$-extended formula and consider $\psi(x)$ the $\Delta_1^{1,b}$-extended formula defined by $\forall y <_l x\neg\varphi(y)$. It can easily be proved that slow induction for $\Sigma_0^{1,b}$-formulas is valid in TCA, i.e., $\theta(\varepsilon) \wedge \forall x(\theta(x) \to \theta(S(x))) \to \forall x\theta(x)$, with θ a $\Sigma_0^{1,b}$-formula. Using a strategy similar to the one adopted in (3), we can expand the result and prove that slow induction for $\Delta_1^{1,b}$-extended formulas is valid in TCA. So $\mathsf{TCA} \vdash \psi(\varepsilon) \wedge \forall x(\psi(x) \to \psi(S(x))) \to \forall x\psi(x)$. Suppose that $\exists x\varphi(x)$. Take b such that $\varphi(b)$. If $\forall y <_l b\neg\varphi(y)$, the result follows. If not, i.e, $\neg\psi(b)$, because we have $\psi(\varepsilon)$ we know that $\neg\forall x(\psi(x) \to \psi(S(x)))$. Thus $\exists x(\psi(x) \wedge \neg\psi(S(x)))$, i.e., $\exists x((\forall y <_l x\neg\varphi(y)) \wedge (\exists y <_l S(x)\varphi(y)))$, and we obtain that $\exists x(\forall y <_l x\neg\varphi(y) \wedge \varphi(x))$. □

Theorem 3.3. *A function* f *in* FCH *can be defined in* TCA *by* φ_f *a* $\Delta_1^{1,b}$*-extended formula satisfying* $\mathsf{TCA} \vdash \forall \bar{x} \exists z \preceq b_f(\bar{x})\varphi_f(\bar{x}, z)$ *and* $\mathsf{TCA} \vdash \varphi_f(\bar{x}, z) \wedge \varphi_f(\bar{x}, y) \to z = y$*, with* b_f *a term.*

Proof. The proof can be done by induction on the complexity of the description of the function f, according to Proposition 2.2, in a very similar way to the proof of an analogous result in the context of PTIME presented in [4] (see also [1]). Even knowing that in the present environment the φ_f formulas we obtain have a different complexity ($\Delta_1^{1,b}$-extended formulas), it causes no problem since TCA permits induction on notation for formulas with the complexity above. The only scheme not appearing in [4] we have to consider is cardinality. The $\Delta_1^{1,b}$-extended formula φ_f, when f results from g by cardinality, is obtained applying bounded comprehension to φ_g and then the counting axiom. More precisely, since $\varphi_g(\bar{x}, y, 0)$ is a $\Delta_1^{1,b}$-extended formula, applying the bounded comprehension scheme, we know that $\mathsf{TCA} \vdash \forall \bar{x} \forall y \exists F^y \forall w \preceq y\big(w \in F^y \leftrightarrow \varphi_g(\bar{x}, w, 0)\big)$. Fix $\bar{x}$, y. By the counting axiom (applied to F^y) we have $\exists C^v \operatorname{Count}(C^v, F^y)$ with $v := (yy11)(yy11)$. Take $\varphi_f'(\bar{x}, y, z)$ the $\Sigma_1^{1,b}$-extended formula defined by $\exists F^y \exists C^v \big(\forall w \preceq y\big(w \in F^y \leftrightarrow \varphi_g(\bar{x}, w, 0)\big) \wedge \operatorname{Count}(C^v, F^y) \wedge \exists s \preceq v(s \in C^v \wedge s = \langle y, z \rangle)\big)$, $b_f(\bar{x}, y) := v$ and $\varphi_f''(\bar{x}, y, z)$ the $\Pi_1^{1,b}$-extended formula defined by $\forall w \preceq b_f(\bar{x}, y)\big(\varphi_f'(\bar{x}, y, w) \rightarrow w = z\big)$.

We can show that $\mathsf{TCA} \vdash \varphi_f'(\bar{x}, y, z) \leftrightarrow \varphi_f''(\bar{x}, y, z)$ and with a meticulous but quite straightforward work it can be proved that $\mathsf{TCA} \vdash \forall \bar{x} \forall y \exists z \preceq b_f(\bar{x}, y) \varphi_f'(\bar{x}, y, z)$; $\mathsf{TCA} \vdash \varphi_f'(\bar{x}, y, z) \wedge \varphi_f'(\bar{x}, y, z') \rightarrow z = z'$; $\mathsf{TCA} \vdash \varphi_g(\bar{x}, \varepsilon, 0) \leftrightarrow \varphi_f'(\bar{x}, \varepsilon, 0)$; $\mathsf{TCA} \vdash \varphi_f'(\bar{x}, \varepsilon, 0) \vee \varphi_f'(\bar{x}, \varepsilon, \varepsilon)$; $\mathsf{TCA} \vdash \varphi_g\big(\bar{x}, S(y), 0\big) \wedge \varphi_f'(\bar{x}, y, r) \rightarrow \varphi_f'\big(\bar{x}, S(y), S(r)\big)$ and $\mathsf{TCA} \vdash u \neq 0 \wedge \varphi_g\big(\bar{x}, S(y), u\big) \wedge \varphi_f'(\bar{x}, y, r) \rightarrow \varphi_f'\big(\bar{x}, S(y), r\big)$. Therefore φ_f' is the $\Delta_1^{1,b}$-extended formula in the conditions of Theorem 3.3 we were looking for. □

Though in a different formulation and with some changes in order to characterize the provably total functions in TCA, the main strategy we use is similar to the one adopted in [10] concerning D_k^0 and in [1] concerning some second-order theories. So we just sketch the proof omitting some details and stressing the differences, focusing precisely on the axioms we introduce in a different manner.

It is possible to formulate the theory TCA in a version of *Gentzen's sequent calculus*, denoted by $\mathsf{LK_{FCH}}$ and defined in the following way: in addition to the initial sequents of the form $A \Rightarrow A$, with A an atomic formula and the sequents for equality, $\mathsf{LK_{FCH}}$ has also the following axioms:

(1) $\Rightarrow A(\bar{s})$, with A a basic axiom of TCA and $\bar{s}$ terms,

(2) $s \in F^t \Rightarrow s \preceq t$, where the variables in the term s do not occur in t,

(3) $\Rightarrow A(\varepsilon) \wedge \forall x \prec s\big(A(x) \rightarrow A(x0) \wedge A(x1)\big) \rightarrow \forall x \preceq s A(x)$, with A a $\Sigma_0^{1,b}$-formula, where the variables in s do not occur,

(4) $\Rightarrow \exists F^s \forall y \preceq s\big(y \in F^s \leftrightarrow A(y)\big)$, with A a $\Sigma_0^{1,b}$-formula where F^s does not occur,

(5) $\Rightarrow \exists C^v \operatorname{Count}(C^v, F^t)$,

and all the second-order inference rules (like the ones presented in [1] with the obvious modifications to our language), complemented with the following *replacement rule*:

$$\frac{\Gamma, a \preceq t \Rightarrow \exists F^q \varphi(a, F^q)}{\Gamma \Rightarrow \exists G^{\bar{q}} \forall x \preceq t \bar{\varphi}(x, G^{\bar{q}})}.$$

Here a is a proper variable, φ a $\Sigma_0^{1,b}$-formula, $\bar{q} = (tq'1)(tq'1)$ and q' and $\bar{\varphi}$ are presented in the replacement scheme.

In order to simplify notation, in the next theorems we omit the bounding term in the second-order variables and we abbreviate as usual $F_1, \ldots, F_n$, $n \in \mathbb{N}$, by $\bar{F}$ and $\exists F_1, \ldots, \exists F_n$ (resp. $\forall F_1, \ldots, \forall F_n$) by $\exists \bar{F}$ (resp. $\forall \bar{F}$).

Theorem 3.4. *Suppose that* $\mathsf{LK}_{\mathsf{FCH}} \vdash \Gamma \Rightarrow \Delta$, *where* Γ *and* Δ *are formed by* $\Sigma_1^{1,b}$*-formulas. Consider* $\Gamma := \exists \bar{X} \varphi_1(\bar{x}, \bar{F}, \bar{X}), \ldots, \exists \bar{X} \varphi_n(\bar{x}, \bar{F}, \bar{X})$ *and* $\Delta := \exists \bar{Y} \psi_1(\bar{x}, \bar{F}, \bar{Y}), \ldots, \exists \bar{Y} \psi_m(\bar{x}, \bar{F}, \bar{Y})$ *where* φ_i *and* ψ_i *are* $\Sigma_0^{1,b}$*-formulas,* $\bar{Y} = Y_1, \ldots, Y_k$ *and* $\varphi_1, \ldots, \varphi_n$, $\psi_1, \ldots, \psi_m$ *have different components of* $\bar{X}$, $\bar{Y}$, *respectively.*

Consider $\varphi(\bar{x}, \bar{F}, \bar{X}) := \bigwedge_{j=1}^n \varphi_j$ *and* $\psi(\bar{x}, \bar{F}, \bar{Y}) := \bigvee_{i=1}^m \psi_i$ *and denote by* $\theta(\bar{x}, \bar{F}, \bar{X}, \bar{Y})$ *the formula* $\varphi(\bar{x}, \bar{F}, \bar{X}) \to \psi(\bar{x}, \bar{F}, \bar{Y})$.

Then there are terms $t_i(\bar{x})$ $(1 \le i \le k)$, $\Sigma_1^{1,b}$*-extended formulas* $M_i^{\Sigma}(w, \bar{x}, \bar{F}, \bar{X})$ *and* $\Pi_1^{1,b}$*-extended formulas* $M_i^{\Pi}(w, \bar{x}, \bar{F}, \bar{X})$ $(1 \le i \le k)$ *such that*

$$\begin{aligned}\mathsf{TCA} \vdash \forall \bar{x} \forall \bar{F} \forall \bar{X} \theta(\bar{x}, \bar{F}, \bar{X}, \{w \preceq t_1(\bar{x}) : M_1^{\Sigma}(w, \bar{x}, \bar{F}, \bar{X})\}, \ldots, \\ \{w \preceq t_k(\bar{x}) : M_k^{\Sigma}(w, \bar{x}, \bar{F}, \bar{X})\})\end{aligned}$$

and

$$\mathsf{TCA} \vdash \forall \bar{x} \forall \bar{F} \forall \bar{X} \forall w \preceq t_i(\bar{x}) \big(M_i^{\Sigma}(w, \bar{x}, \bar{F}, \bar{X}) \leftrightarrow M_i^{\Pi}(w, \bar{x}, \bar{F}, \bar{X})\big) \qquad (1 \le i \le k),$$

and the predicates $w \preceq t_i(\bar{x}) \wedge M_i^{\Delta}(w, \bar{x}, \bar{F}, \bar{X})$ *are in* $\mathsf{CH}^{\bar{X}, \bar{F}}$ *i.e., in the class of predicates whose characteristic functions are in* $\mathsf{FCH}^{\bar{X}, \bar{F}}$.[4]

In the previous result M_i^{Δ} is either M_i^{Σ} or M_i^{Π} according to our conveniences (e.g., to have the formulas in the right classes) and we denote by $\theta(\bar{x}, \bar{F}, \bar{X}, \{w \preceq t_1(\bar{x}) : M_1^{\Sigma}(w, \bar{x}, \bar{F}, \bar{X})\}, \ldots)$ the formula $\theta(\bar{x}, \bar{F}, \bar{X}, G, \ldots)$, where the occurrences of $s \in G$ are replaced with $s \preceq t_1(\bar{x}) \wedge M_1^{\Sigma}(s, \bar{x}, \bar{F}, \bar{X})$.

Proof. Let P be a $\mathsf{LK}_{\mathsf{FCH}}$-proof of $\Gamma \Rightarrow \Delta$. By the free cut elimination theorem we can suppose that P has just $\Sigma_1^{1,b}$-formulas. Let us prove, by induction on the

[4] $\mathsf{FCH}^{\bar{X}, \bar{F}}$ is the class of functions resulting from FCH by adding (in the characterization of FCH given by Proposition 2.2) $\bar{X}$, $\bar{F}$, considered as functions of $2^{<\omega}$ in $\{\epsilon, 0\}$, as initial functions.

number of lines in P, that for every sequent $\Pi \Rightarrow \Lambda$ in P there exist the terms and the formulas described in the theorem.

If $\Pi \Rightarrow \Lambda$ is an initial sequent other than (4) and (5) (including the axiom for induction), nothing has to be proved. For axiom (4), considering that $\Pi \Rightarrow \Lambda$ is $\Rightarrow \exists F^s \forall y \preceq s(y \in F^s \leftrightarrow A(y))$ with A a $\Sigma_0^{1,b}$-formula, we define $M_1^{\Sigma}(w, \bar{x}, \overline{X})$ and $M_1^{\Pi}(w, \bar{x}, \overline{X})$ as being $A(w, \bar{x}, \overline{X})$ and $t_1 := s$. For axiom (5), $\Pi \Rightarrow \Lambda$ is $\Rightarrow \exists C^{(tt11)(tt11)}\,\mathrm{Count}(C, F^t)$. If

$$\chi_{F^t}(x) = \begin{cases} \varepsilon, & \text{if } x \notin F^t, \\ 0, & \text{if } x \in F^t, \end{cases}$$

we have that $\chi_{F^t} \in \mathsf{FCH}^{F^t}$, so $c_{\chi_{F^t}} \in \mathsf{FCH}^{F^t}$. In an obvious way the result in Theorem 3.3 can be extended to FCH^{F^t}, so that $j = c_{\chi_{F^t}}(i)$ can be defined in TCA by a $\Delta_1^{1,b}$-extended formula we denote by $\varphi^{\Delta}_{c_{\chi_{F^t}}}(i, j)$. Then we define $M_1^{\Sigma}(w, \bar{x}, F^t)$ as being $\exists i \preceq t \exists j \preceq t1\big(w = \langle i, j\rangle \wedge \varphi^{\Sigma}_{c_{\chi_{F^t}}}(i, j)\big)$. Similarly $M_1^{\Pi}(w, \bar{x}, F^t)$ is defined, replacing $\varphi^{\Sigma}_{c_{\chi_{F^t}}}(i, j)$ with $\varphi^{\Pi}_{c_{\chi_{F^t}}}(i, j)$ and $t_1(\bar{x}) := (tt11)(tt11)$.

In the induction step we just sketch the case of the replacement rule. Suppose that $\Pi \Rightarrow \Lambda$ is $\Gamma' \Rightarrow \exists G \forall x \preceq t\overline{\varphi}(x, G)$, obtained from $\Gamma', a \preceq t \Rightarrow \exists F \varphi(a, F)$, with φ a $\Sigma_0^{1,b}$-formula (the rule is only applied to formulas of this complexity). By induction hypotheses for this last sequent there exist $(M')_1^{\Sigma}(w, \bar{x}, y, \overline{X})$, $(M')_1^{\Pi}(w, \bar{x}, y, \overline{X})$ and $t_1'(\bar{x}, y)$ in the desired conditions. For $\Gamma' \Rightarrow \exists G \forall x \preceq t\overline{\varphi}(x, G)$ define $M_1^{\Sigma}(w, \bar{x}, \overline{X})$ as being $\exists y \preceq t \exists s \preceq t_1'\big((M')_1^{\Sigma}(s, \bar{x}, y, \overline{X}) \wedge w = \langle y, s\rangle\big)$, and define $M_1^{\Pi}(w, \bar{x}, \overline{X})$ as being $\exists y \preceq t \exists s \preceq t_1'\big((M')_1^{\Pi}(s, \bar{x}, y, \overline{X}) \wedge w = \langle y, s\rangle\big)$ and $t_1(\bar{x}) := \big(t(\bar{x})t_1'\big(\bar{x}, t(\bar{x})\big)\big)1 \times 11$. □

From the previous theorem we get the following result.

Theorem 3.5. *If* $\mathsf{TCA} \vdash \forall \bar{x} \exists y \varphi(\bar{x}, y)$, *with* φ *a* $\Sigma_1^{1,b}$*-formula, then there exists a function* $f \in \mathsf{FCH}$ *such that, for all* $\bar{\sigma} \in 2^{<\omega}$, *we have* $\varphi\big(\bar{\sigma}, f(\bar{\sigma})\big)$.

Moreover, there exist a $\Delta_1^{1,b}$*-extended formula* θ *in* TCA *and a term* $t(\bar{x})$ *such that*

(1) $f(\bar{\sigma}) = \tau \leftrightarrow \theta(\bar{\sigma}, \tau)$,

(2) $\mathsf{TCA} \vdash \forall \bar{x} \forall y\big(\theta(\bar{x}, y) \to \varphi(\bar{x}, y)\big)$,

(3) $\mathsf{TCA} \vdash \forall \bar{x} \exists y \preceq t\theta(\bar{x}, y)$,

(4) $\mathsf{TCA} \vdash \forall \bar{x} \exists^1 y\theta(\bar{x}, y)$.

Proof. Suppose that $\mathsf{TCA} \vdash \forall \bar{x} \exists y \varphi(\bar{x}, y)$. Since φ is a $\Sigma_1^{1,b}$-formula it is possible to prove that there exists a term $t(\bar{x})$ such that $\mathsf{TCA} \vdash \forall \bar{x} \exists y \preceq t(\bar{x})\varphi(\bar{x}, y)$. Hence there exists an $\mathsf{LK}_{\mathsf{FCH}}$-proof of $\Rightarrow \exists y \preceq t(\bar{x})\varphi(\bar{x}, y)$. Being φ a $\Sigma_1^{1,b}$-formula, suppose it is $\exists U_1 \ldots \exists U_k \tilde{\varphi}$, with $\tilde{\varphi}$ a $\Sigma_0^{1,b}$-formula. Then there exists an

$\mathsf{LK}_{\mathsf{FCH}}$-proof of $\Rightarrow \exists U_1 \ldots \exists U_k \exists y \preceq t\tilde{\varphi}$. The result follows applying Theorem 3.4 and defining $f(\bar{\sigma})$ as being $\mu y \preceq t(\bar{\sigma})\tilde{\varphi}(\bar{\sigma}, y, \{w \preceq t_1(\bar{\sigma}) : M_1^{\Sigma}(w, \bar{\sigma})\}, \ldots, \{w \preceq t_k(\bar{\sigma}) : M_k^{\Sigma}(w, \bar{\sigma})\})$ and $\theta(\bar{x}, y) :\leftrightarrow \tilde{\varphi}(\bar{x}, y, \{w \preceq t_1(\bar{x}) : M_1^{\Delta}(w, \bar{x})\}, \ldots, \{w \preceq t_k(\bar{x}) : M_k^{\Delta}(w, \bar{x})\}) \wedge \forall y' <_l y \neg\tilde{\varphi}(\bar{x}, y', \{w \preceq t_1(\bar{x}) : M_1^{\Delta}(w, \bar{x})\}, \ldots, \{w \preceq t_k(\bar{x}) : M_k^{\Delta}(w, \bar{x})\})$, where M_i^{Σ}, M_i^{Δ} come from Theorem 3.4. To verify that this formula works, use Proposition 3.2. To confirm that $f \in \mathsf{FCH}$, notice that it is possible to prove that for any predicate $\tau(\bar{x}, y, \overline{X})$ in $\mathsf{CH}^{\overline{X}}$ satisfying $\forall \bar{x} \forall \overline{X} \exists y \preceq t(\bar{x})\tau(\bar{x}, y, \overline{X})$ we have $f(\bar{x}, \overline{X}) := \mu y \preceq t(\bar{x})\tau(\bar{x}, y, \overline{X}) \in \mathsf{FCH}^{\overline{X}}$ (in the present case we do not have second-order variables $\overline{X}$). □

Thus the provably total functions in TCA with $\Sigma_1^{1,b}$-graphs are exactly the functions of FCH.

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