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Multilinear functionals of Schatten class type and approximation numbers

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Abstract. In this article we investigate the relationship between multilinear functionals of Schatten class type and approximation numbers, and obtain some results that are similar to the those we know in the linear theory.

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1. Introduction

The Schatten–von Neumann classes for linear operators were introduced in 1946 and 1948 by R. Schatten and J. von Neumann ([11], [12]). For $1 \le p < \infty$ and H_1, H_2 Hilbert spaces, the *p-th Schatten–von Neumann class* $S_p(H_1, H_2)$ consists of all linear operators $T : H_1 \to H_2$ which admit a representation of the form

$$Tx = \sum_{n=1}^{\infty} \lambda_n (x \mid h_n) g_n,$$

where $(\lambda_n)_n \in l_p$ and $(h_n)_n$ and $(g_n)_n$ are orthonormal sequences in H_1 and H_2, respectively (for details see [4]).

For $n \in \mathbb{N}$, we define the *n-th approximation number* of T as

$$a_n(T) = \inf\{\|T - U\| \mid U \in \mathscr{L}(H; G), \dim U(H) < n\}.$$

We can show that $T \in S_p(H_1; H_2)$ when $\big(a_n(T)\big)_n \in l_p$ ([9], 15.5).

The multilinear mappings of Schatten class type were studied by H. A. Braunss and H. Junek ([3]) based on ideas presented by Pietsch in [10]. The purpose of this article is to study the relationship between multilinear functionals of Schatten class type and approximation numbers as defined by Pietsch in [10], and to obtain similar results to those in linear theory. We will see that the sequence formed by the approximation numbers of a multilinear functional of Schatten class type $\mathcal{S}_p$ ($p \geq 1$) is in l_p, but the converse is not true in general.

Some comments about notation are in order. Throughout this article, H, G, H_j, G_j indicate Hilbert spaces, and E, F, E_j, F_j denote Banach spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). By E' we mean the topological dual space of E; $\mathcal{L}(E_1, \ldots, E_m; F)$ denotes the space of m-linear continuous mappings from $E_1 \times \cdots \times E_m$ into F. In the case $E_1 = \cdots = E_m = E$, we write $\mathcal{L}(^m E; F)$. The inner product is denoted by $(\cdot \mid \cdot)$ and e_j is the j-th vector of the l_2 standard basis. The p-th Schatten class for linear operators is denoted by $\mathcal{S}_p(H; G)$. The space formed by the m-linear absolutely $(r; s_1, \ldots, s_m)$-summing mappings is denoted by $\mathcal{L}_{\mathrm{as},(r;s_1,\ldots,s_m)}(E_1, \ldots, E_m; F)$. In the case of $s_1 = \cdots = s_m = s$, we write $\mathcal{L}_{\mathrm{as},(r,s)}(E_1, \ldots, E_m; F)$ (see [2]). Finally, $l_{w,2}(E)$ denotes the space of the weakly 2-summing sequences in E.

2. Definitions, examples and properties

Definition 2.1. Let $0 < p < \infty$. A multilinear mapping $T \in \mathcal{L}(H_1, \ldots, H_n; F)$ is of *Schatten class type* $\mathcal{S}_p$ if, for each $i = 1, \ldots, n$, there exist a Hilbert space K_i, an operator $T_i \in \mathcal{S}_p(H_i; K_i)$ and $S \in \mathcal{L}(K_1, \ldots, K_n; F)$ such that $T = S \circ (T_1, \ldots, T_n)$. We denote the space of such mappings as $\mathcal{L}(\mathcal{S}_p) \cdot (H_1, \ldots, H_n; F)$. A norm (or $\frac{p}{n}$ norm if $p < 1$) for that space is $\|T\|_{\mathcal{S}_p} = \inf_{T = S \circ (T_1, \ldots, T_n)} \|S\| \prod_{j=1}^n \sigma_p(T_j)$.

Here we will consider only the case $F = \mathbb{K}$. More details can be found in [6] and [8].

Definition 2.2. A *rank* ρ is a law which associates to each m-linear functional a non-negative integer number $0, 1, \ldots, \infty$ such that

(a) $\rho(i_n) = n$, $n = 0, 1, 2, \ldots$, where $i_n \in \mathcal{L}(^m l_2^n; \mathbb{K})$ is defined by $i_n(x_1, \ldots, x_m) = \sum_{k=1}^n (x_1 \mid e_k) \ldots (x_m \mid e_k)$;

(b) $\rho(S + T) \leq \rho(S) + \rho(T)$ when $S, T \in \mathcal{L}(E_1, \ldots, E_m; \mathbb{K})$;

(c) $\rho\big(S \circ (T_1, \ldots, T_m)\big) \leq \rho(S)$ when $S \in \mathcal{L}(F_1, \ldots, F_m; \mathbb{K})$, $T_j \in \mathcal{L}(E_j; F_j)$, $j = 1, \ldots, m$.

Example 2.3. (1) Maximal rank:

$$\nu(T) = \min\Big\{n \mid T = \sum_{i=1}^n a_{1i}(\cdot) \ldots a_{mi}(\cdot), a_{ji} \in E_j', j = 1, \ldots, m, i = 1, \ldots, n\Big\},$$
$$T \in \mathcal{L}(E_1, \ldots, E_m; \mathbb{K}).$$

(2) Minimal rank:

$$\mu(T) = \max\{n \mid \text{there exists } T_j \in \mathscr{L}(l_2^n; E_j), j = 1, \dots, m, \text{ such that } i_n = T \circ (T_1, \dots, T_m)\}.$$

Definition 2.4. Consider a rank ρ. For $T \in \mathscr{L}(E_1, \dots, E_m; \mathbb{K})$ and $n = 1, 2, \dots,$ the *n-th approximation number* is defined by

$$a_n^\rho(T) = \inf\{\|T - U\| \mid U \in \mathscr{L}(E_1, \dots, E_m; \mathbb{K}), \rho(U) < n\}.$$

Properties 2.5. (1) $\|T\| = a_1^\rho(T) \geq a_2^\rho(T) \geq \cdots \geq 0$ *for* $T \in \mathscr{L}(E_1, \dots, E_m; \mathbb{K})$.

(2) $a_{n+k-1}^\rho(S + T) \leq a_n^\rho(S) + a_k^\rho(T)$ *for* $T, S \in \mathscr{L}(E_1, \dots, E_m; \mathbb{K})$.

(3) $a_n^\rho\big(S \circ (T_1, \dots, T_m)\big) \leq a_n^\rho(S)\|T_1\| \dots \|T_m\|$ *for* $T_j \in \mathscr{L}(E_j; F_j)$, $j = 1, \dots, m$ *and* $S \in \mathscr{L}(F_1, \dots, F_m; \mathbb{K})$.

(4) *If* $\rho(T) < n$, *then* $a_n^\rho(T) = 0$.

We can show that the following holds.

Proposition 2.6. *Suppose that ρ is a rank defined for multilinear functionals $T \in \mathscr{L}(E_1, \dots, E_m; \mathbb{K})$. Then $\rho(T) \leq \nu(T)$.*

More details about approximation numbers can be found in [10].

3. The Schatten classes and approximation numbers

Theorem 3.1. *Let $T \in \mathscr{L}(\mathscr{S}_p)(H_1, \dots, H_m; \mathbb{K})$, $1 \leq p < +\infty$. Then $\big(a_k^\rho(T)\big) \in l_p$ for all ranks ρ.*

Proof. Let $T \in \mathscr{L}(\mathscr{S}_p)(H_1, \dots, H_m; \mathbb{K})$. By definition, there exist Hilbert spaces $K_1, \dots, K_m$, linear operators $S_j \in \mathscr{S}_p(H_j; K_j)$, $j = 1, \dots, m$, and an m-linear application $R \in \mathscr{L}(K_1, \dots, K_m; \mathbb{K})$ such that $T = R \circ (S_1, \dots, S_m)$.

Given $0 < \varepsilon < 1$, for each $n \in \mathbb{N}$ and each $j = 1, \dots, m$, there exists $S_j^{(n)} \in \mathscr{L}(H_j; K_j)$ such that $\dim S_j^{(n)} < n$ and $\|S_j - S_j^{(n)}\| < a_n(S_j) + \varepsilon$. Let $c \geq \max\{2\|S_j\| + 1 \mid j = 1, \dots, m\}$.

If

$$S_j^{(n)} = \sum_{k_j=1}^{n-1} (\cdot \mid h_{k_j}^{(j)}) g_{k_j}^{(j)},$$

where $h_{k_j}^{(j)} \in H_j$, $g_{k_j}^{(j)} \in K_j$, $k_j = 1, \dots, n-1$, $j = 1, \dots, m$, we have

$$R\circ(S_1^{(n)},\dots,S_m^{(n)})(x_1,\dots,x_m)=R\Big(\sum_{k_1=1}^{n-1}(x_1\,|\,h_{k_1}^{(1)})g_{k_1}^{(1)},\dots,\sum_{k_n=1}^{n-1}(x_m\,|\,h_{k_m}^{(m)})g_{k_m}^{(m)}\Big)$$
$$=\sum_{k_1,\dots,k_m=1}^{n-1}(x_1\,|\,h_{k_1}^{(1)})\dots(x_m\,|\,h_{k_m}^{(m)})\xi_{k_1,\dots,k_m}.$$

Here $\xi_{k_1,\dots,k_m}=R(g_{k_1}^{(1)},\dots,g_{k_m}^{(m)})$.

Since

$$\rho\Big(\sum_{k_1,\dots,k_m=1}^{n-1}(\cdot\,|\,h_{k_1}^{(1)})\dots(\cdot\,|\,h_{k_m}^{(m)})\xi_{k_1,\dots,k_m}\Big)$$
$$\le\nu\Big(\sum_{k_1,\dots,k_m=1}^{n-1}(\cdot\,|\,h_{k_1}^{(1)})\dots(\cdot\,|\,h_{k_m}^{(m)})\xi_{k_1,\dots,k_m}\Big)\le m(n-1),$$

and

$$\|S_j^{(n)}\|\le a_n(S_j)+\varepsilon+\|S_j\|<2\|S_j\|+1,\qquad j=1,\dots,m,$$

it follows that

$$a_{mn}^{\rho}(T)\le\|T-R(S_1^{(n)},\dots,S_m^{(n)})\|$$
$$=\sup_{\substack{\|x_j\|\le1\\ j=1,\dots,m}}\|R(S_1x_1,\dots S_mx_m)-R(S_1^{(n)}x_1,\dots,S_m^{(n)}x_m)\|$$
$$\le\sup_{\substack{\|x_j\|\le1\\ j=1,\dots,m}}\sum_{j=1}^{m}\|R(S_1^{(n)}x_1,\dots S_{j-1}^{(n)}x_{j-1},S_jx_j-S_j^{(n)}x_j,S_{j+1}x_{j+1},\dots,S_mx_m)\|$$
$$\le\sup_{\substack{\|x_j\|\le1\\ j=1,\dots,m}}\sum_{j=1}^{m}\|R\|\,\|S_1^{(n)}x_1\|\dots\|S_{j-1}^{(n)}x_{j-1}\|\,\|S_jx_j-S_j^{(n)}x_j\|\,\|S_{j+1}x_{j+1}\|\dots\|S_mx_m\|$$
$$\le\sum_{j=1}^{m}\|R\|c^{m-1}\|S_j-S_j^{(n)}\|$$
$$\le\sum_{j=1}^{m}\|R\|c^{m-1}\big(a_n(S_j)+\varepsilon\big).$$

As $0<\varepsilon<1$, we have $a_{mn}^{\rho}(T)\le\sum_{j=1}^{m}\|R\|c^{m-1}a_n(S_j)$. Then

$$\sum_{n=1}^{\infty}\big(a_{mn}^{\rho}(T)\big)^p\le\|R\|^pc^{(m-1)p}\sum_{n=1}^{\infty}\Big(\sum_{j=1}^{m}a_n(S_j)\Big)^p.$$

We study now the convergence of the series on the right-hand side.

$$\sum_{n=1}^{\infty}\Big(\sum_{j=1}^{m}a_n(S_j)\Big)^p=\sum_{n=1}^{\infty}\big(a_n(S_1)+\cdots+a_n(S_m)\big)^p$$
$$\leq\bigg(\Big(\sum_{n=1}^{\infty}a_n^p(S_1)\Big)^{1/p}+\cdots+\Big(\sum_{n=1}^{\infty}a_n^p(S_m)\Big)^{1/p}\bigg)^p<+\infty$$

because $\big(a_n(S_j)\big)_{n=1}^{\infty}\in l_p$, $j=1,\dots,m$.

Hence

$$\sum_{n=1}^{\infty}\big(a_{mn}^{\rho}(T)\big)^p<\infty.$$

If $\sum_{n=1}^{\infty}\big(a_{mn}^{\rho}(T)\big)^p=A$, $k\in\mathbb{N}$, $lm\leq k\leq(l+1)m$, and we take into account that $a_1^{\rho}(T)\geq a_2^{\rho}(T)\geq\cdots\geq 0$, then we have

$$S_k=|a_1^{\rho}(T)|^p+\cdots+|a_k^{\rho}(T)|^p\leq|a_1^{\rho}(T)|^p+\cdots+|a_{m-1}^{\rho}(T)|^p+mA.$$

This means that the sequence of partial sums of $\sum_n|a_n^{\rho}(T)|^p$ is bounded. Thus, $\sum_n|a_n^{\rho}(T)|^p<+\infty$. □

The converse of the Theorem 3.1 is not true in general. Before we give some examples, we need to prove the following result.

Proposition 3.2. *Let* $T_\lambda=T_\lambda(x_1,\dots,x_m)=\sum_{j=1}^{\infty}\lambda_j(x\,|\,e_j)\dots(x_m\,|\,e_j)\in\mathscr{L}(^m l_2;\mathbb{K})$, $m\geq 2$. *If* $\lambda=(\lambda_j)_j$ *is a non-increasing sequence of non-negative real numbers, then* $a_k^{\nu}(T_\lambda)\leq\lambda_k$ *for all* $k\in\mathbb{N}$.

Proof. We define

$$T^m: l_m\times\cdots\times l_m\to\mathbb{K},$$
$$T^m(x_1,\dots,x_m)=\sum_{i=1}^{\infty}x_{1i}\dots x_{mi},\qquad x_j=(x_{ji})_i,\ j=1,\dots,m.$$

It follows (using Hölder's inequality) that

$$\|T^m\|\leq\sup_{\substack{\|x_j\|\leq 1\\ j=1,\dots,m}}\sum_{i=1}^{\infty}|x_{1i}\dots x_{mi}|$$
$$\leq\sup_{\substack{\|x_j\|\leq 1\\ j=1,\dots,m}}\Big(\sum_{i=1}^{\infty}|x_{1i}|^m\Big)^{1/m}\dots\Big(\sum_{i=1}^{\infty}|x_{mi}|^m\Big)^{1/m}\leq 1.$$

Let $\sigma_i = \lambda_i^{1/m}$ and $\sigma = (\sigma_i)_{i=1}^{\infty}$. Let

$$D^{(k+1)} \in \mathscr{L}(l_2; l_m), \qquad D^{(k+1)}(x) = \sum_{i>k} \sigma_i (x \,|\, e_i) e_i, \qquad x \in l_2.$$

We have

$$\Big(\sum_{i>k} |\sigma_i (x \,|\, e_i)|^m\Big)^{1/m} = \Big(\sum_{i>k} |\lambda_i| \, |(x \,|\, e_i)|^m\Big)^{1/m}$$
$$\leq \big(\sup_{i>k} |\lambda_i|\big)^{1/m} \Big(\sum_{i>k} |(x \,|\, e_i)|^2\Big)^{1/2} < \infty.$$

For each $k \in \mathbb{N}$, we define:

$$R_k : l_2 \times \cdots \times l_2 \to \mathbb{K},$$
$$R_k(x_1, \ldots, x_m) = \sum_{i=1}^{k} \lambda_i (x_1 \,|\, e_i) \ldots (x_m \,|\, e_i).$$

Since $v(R_k) \leq k$, we can write

$$a_k^v(T_\lambda) \leq \|T_\lambda - R_{k-1}\| = \Big\| \sum_{i>k-1} \lambda_i (\cdot \,|\, e_i) \ldots (\cdot \,|\, e_i) \Big\|$$
$$= \|T^m \circ (D^{(k)}, \ldots, D^{(k)})\|$$
$$\leq \|T^m\| \, \|D^{(k)}\|^m.$$

We calculate now the value of $\|D^{(k)}\|$. Let $x = (\xi_i)_i \in l_2$, $\|x\| \leq 1$. Then

$$\|D^{(k)} x\| = \Big(\sum_{i=k}^{\infty} |\sigma_i \xi_i|^m\Big)^{1/m} \leq \Big(\sum_{i=k}^{\infty} |\sigma_i \xi_i|^2\Big)^{1/2} \leq \sigma_k \Big(\sum_{i=k}^{\infty} |\xi_i|^2\Big)^{1/2} \leq \sigma_k \|x\| \leq \sigma_k.$$

We conclude that

$$a_k^v(T_\lambda) \leq \|T^m\| \, \|D^{(k)}\|^m \leq \|T^m\| \sigma_k^m = \lambda_k. \qquad \square$$

As we have already pointed out, the converse of Theorem 3.1 is not true in general. The examples below show this.

Example 3.3. Let $T \in \mathscr{L}(^m l_2; \mathbb{K})$, $T(x_1, \ldots, x_m) = \sum_{j=1}^{\infty} \left(\frac{1}{j}\right)^{m/2} (x_1 \,|\, e_j) \ldots (x_m \,|\, e_j)$, $m \geq 3$. Suppose that $T \in \mathscr{L}(\mathscr{S}_1)(^m l_2; \mathbb{K}) \subset \mathscr{L}(\mathscr{S}_2)(^m l_2; \mathbb{K})$. As $\mathscr{L}(\mathscr{S}_2)(^m l_2; \mathbb{K}) =$

$\mathscr{L}_{\mathrm{as},(2/m;2)}(^m l_2;\mathbb{K})$ (see [8], 3.6), we have $T \in \mathscr{L}_{\mathrm{as},(2/m;2)}(^m l_2;\mathbb{K})$. In view of $(e_j)_j \in l_{w,2}(l_2)$, we have

$$\sum_{j=1}^{\infty} |T(e_j,\dots,e_j)|^{2/m} = \sum_{j=1}^{\infty} \left|\left(\frac{1}{j}\right)^{m/2}\right|^{2/m} = \sum_{j=1}^{\infty} \left|\frac{1}{j}\right| = +\infty,$$

which is a contradiction. Hence, $T \notin \mathscr{L}(\mathscr{S}_1)(^m l_2;\mathbb{K})$. As $a_k^{\nu}(T) \le \lambda_k = \left(\frac{1}{k}\right)^{m/2}$ (using Proposition 3.2), we have

$$\sum_{k=1}^{\infty} |a_k^{\nu}(T)| \le \sum_{k=1}^{\infty} \left|\frac{1}{k}\right|^{m/2} < +\infty.$$

In other words, $\left(a_k^{\nu}(T)\right)_k \in l_1$.

Example 3.4. Let $1 < p \le 2$ e $m \ge 2$. Consider $T \in \mathscr{L}(^m l_2;\mathbb{K})$ defined by $T(x_1,\dots,x_m) = \sum_{j=1}^{\infty} \frac{1}{j}(x_1 \mid e_j)\dots(x_m \mid e_j)$. We have that $T \notin \mathscr{L}(\mathscr{S}_2)(^m l_2;\mathbb{K}) = \mathscr{L}_{\mathrm{as},(2/m;2)}(^m l_2;\mathbb{K})$.

In fact, we can see that

$$\sum_{j=1}^{\infty} |T(e_j,\dots,e_j)|^{2/m} = \sum_{j=1}^{\infty} \left|\frac{1}{j}\right|^{2/m} = +\infty,$$

with $(e_j)_j \in l_{w,2}(l_2)$. Then $T \notin \mathscr{L}(\mathscr{S}_p)(^m l_2;\mathbb{K}) \subset \mathscr{L}(\mathscr{S}_2)(^m l_2;\mathbb{K})$.

Now, since $\lambda = \left(\frac{1}{k}\right)$ is a non-increasing sequence, we have $a_k^{\nu}(T) \le \lambda_k$ for all $k \in \mathbb{N}$. We note that $\lambda = (\lambda_k) = \left(\frac{1}{k}\right) \in l_p$ for all $1 < p \le 2$. Hence

$$\sum_{k=1}^{\infty} |a_k^{\nu}(T)|^p \le \sum_{k=1}^{\infty} |\lambda_k|^p < +\infty.$$

To produce an example for $p > 2$, we need the next proposition. Its proof is simple and uses the theorem below ([9], 17.5.3).

Theorem 3.5. *For all $r \ge 2$, we have $\mathscr{L}_{\mathrm{as},(r;2)}(H;G) = \mathscr{S}_r(H;G)$.*

Proposition 3.6. *Suppose that $p \ge 2$. If $T \in \mathscr{L}(\mathscr{S}_p)(H_1,\dots,H_m;\mathbb{K})$, then $T \in \mathscr{L}_{\mathrm{as},(p/m;2)}(H_1,\dots,H_m;\mathbb{K})$.*

Proof. Let $T \in \mathscr{L}(\mathscr{S}_p)(H_1,\dots,H_m;\mathbb{K})$. By definition, there exist Hilbert spaces $K_1,\dots,K_m$, operators $S_j \in \mathscr{S}_p(H_j;K_j)$, $j = 1,\dots,m$, and $R \in \mathscr{L}(K_1,\dots,K_m;\mathbb{K})$ such that $T = R \circ (S_1,\dots,S_m)$.

Let $x_1^{(k)}, \dots, x_s^{(k)} \in H_k$, $k = 1, \dots, m$. Then

$$\begin{aligned}
\Big(\sum_{j=1}^{s} |T(x_j^{(1)}, \dots, x_j^{(m)})|^{p/m}\Big)^{m/p} \\
&= \Big(\sum_{j=1}^{s} |R(S_1 x_j^{(1)}, \dots, S_m x_j^{(m)})|^{p/m}\Big)^{m/p} \\
&\le \|R\| \Big(\sum_{j=1}^{s} \|S_1 x_j^{(1)}\|^{p/m} \dots \|S_m x_j^{(m)}\|^{p/m}\Big)^{m/p} \\
&\le \|R\| \Big(\sum_{j=1}^{s} \|S_1 x_j^{(1)}\|^{p}\Big)^{1/p} \dots \Big(\sum_{j=1}^{s} \|S_m x_j^{(m)}\|^{p}\Big)^{1/p} \\
&\le \|R\|\, \|S_1\|_{\mathrm{as},(p,2)} \dots \|S_m\|_{\mathrm{as},(p,2)} \|(x_j^{(1)})_{j=1}^s\|_{w,2} \dots \|(x_j^{(m)})_{j=1}^s\|_{w,2}.
\end{aligned}$$

Therefore, $T \in \mathscr{L}_{\mathrm{as},(p/m;2)}(H_1, \dots, H_m; \mathbb{K})$. □

Example 3.7. Let $p > 2$ and $m \ge 2$. We define

$$T \in \mathscr{L}(^m l_2; \mathbb{K}), \qquad T(x_1, \dots, x_m) = \sum_{j=1}^{\infty} \lambda_j (x_1 \,|\, e_j) \dots (x_m \,|\, e_j)$$

where $(\lambda_j)_j \in l_p \backslash l_{p/m}$, $(\lambda_j)_j$ is non-increasing and $\lambda_j \ge 0$ for all $j \in \mathbb{N}$ (for example, $\left(\frac{1}{j^a}\right)_j$, with $a = \frac{1}{p} + \varepsilon$, $0 < \varepsilon < \frac{m-1}{p}$).

If $T \in \mathscr{L}(\mathscr{S}_p)(^m l_2; \mathbb{K})$, by Proposition 3.6 we would have

$$T \in \mathscr{L}_{\mathrm{as},(p/m,2)}(^m l_2; \mathbb{K}).$$

But

$$\sum_{j=1}^{\infty} |T(e_j, \dots, e_j)|^{p/m} = \sum_{j=1}^{\infty} |\lambda_j|^{p/m} = +\infty,$$

where $(e_j)_j \in l_{w,2}(l_2)$. Therefore, T is not in $\mathscr{L}(\mathscr{S}_p)(^m l_2; \mathbb{K})$. On the other hand, T is such that $(a_k^{\nu})_k \in l_p$—use Proposition 3.2 and the fact that $(\lambda_k)_k \in l_p$.

A question one could ask here is the following: is the converse of Theorem 3.1 true for any rank ρ? The answer is no. In fact, suppose that the converse is true for some rank ρ. Let ν be the maximal rank. Then

$$\big(a_k^{\nu}(T)\big) \in l_p \implies \big(a_k^{\rho}(T)\big) \in l_p \implies T \in \mathscr{L}(\mathscr{S}_p)(H_1, \dots, H_m; \mathbb{K}),$$

which is not true in general.

4. The Schatten classes and nuclear applications

In the linear case, the space of nuclear operators $N(H_1, H_2)$ coincides with the space $\mathscr{S}_1(H_1, H_2)$. We now investigate the relation between the multilinear functionals of nuclear type (see [1]) and the Schatten classes.

Definition 4.1. Let $T \in \mathscr{L}(E_1, \dots, E_m; F)$. T is *nuclear* if there are sequences $(\varphi_n^{(j)})_{n=1}^{\infty}$ in E_j', $j = 1, \dots, m$, and a sequence $(c_n)_{n=1}^{\infty}$ in F such that

$$T(x_1, \dots, x_m) = \sum_{n=1}^{\infty} \varphi_n^{(1)}(x_1) \dots \varphi_n^{(m)}(x_m) c_n$$

with $\sum_{n=1}^{\infty} \|\varphi_n^{(1)}\| \dots \|\varphi_n^{(m)}\| \, \|c_n\| < +\infty$.

We denote the space of all m-linear nuclear mappings by $\mathscr{L}_N(E_1, \dots, E_m; F)$, endowed with the norm

$$\|T\|_N = \inf \sum_{n=1}^{\infty} \|\varphi_n^{(1)}\| \dots \|\varphi_n^{(m)}\| \, \|c_n\|,$$

where the infimum is taken over all sequences $(\varphi_n^{(j)})_{n=1}^{\infty}$, $j = 1, \dots, n$, and $(c_n)_{n=1}^{\infty}$ which satisfy the definition.

Proposition 4.2. *If $T \in \mathscr{L}(\mathscr{S}_1)(H_1, \dots, H_m; \mathbb{K})$, then T is nuclear.*

Proof. Let $T \in \mathscr{L}(\mathscr{S}_1)(H_1, \dots, H_m; \mathbb{K})$. Then $T = S \circ (S_1, \dots, S_m)$, where $S_j \in \mathscr{S}_1(H_j; G_j)$, $j = 1, \dots, m$, and $S \in \mathscr{L}(G_1, \dots, G_m; \mathbb{K})$. For each $j = 1, \dots, m$, we can write

$$S_j(x) = \sum_{n=1}^{\infty} \lambda_n^{(j)} (x \mid h_n^{(j)}) g_n^{(j)},$$

where $(\lambda_n^{(j)})_n \in l_1$, $(h_n^{(j)})_n$ and $(g_n^{(j)})_n$, $j = 1, \dots, m$, are orthonormal sequences in H_j and G_j, respectively, $j = 1, \dots, m$.

Then

$$\begin{aligned} T(x_1, \dots, x_m) &= S \circ (S_1, \dots, S_m)(x_1, \dots, x_m) \\ &= S\Big(\sum_{n_1=1}^{\infty} \lambda_{n_1}^{(1)} (x_1 \mid h_{n_1}^{(1)}) g_{n_1}^{(1)}, \dots, \sum_{n_m=1}^{\infty} \lambda_{n_m}^{(m)} (x_m \mid h_{n_m}^{(m)}) g_{n_m}^{(m)} \Big) \\ &= \sum_{n_1=1}^{\infty} \dots \sum_{n_m=1}^{\infty} \lambda_{n_1}^{(1)} \dots \lambda_{n_m}^{(m)} (x_1 \mid h_{n_1}^{(1)}) \dots (x_m \mid h_{n_m}^{(m)}) S(g_{n_1}^{(1)}, \dots, g_{n_m}^{(m)}). \end{aligned}$$

Moreover,

$$\sum_{n_1=1}^{\infty}\cdots\sum_{n_m=1}^{\infty}|\lambda_{n_1}^{(1)}|\dots|\lambda_{n_m}^{(m)}|\,\|h_{n_1}^{(1)}\|\dots\|h_{n_m}^{(m)}\|\,\|S(g_{n_1}^{(1)},\dots,g_{n_m}^{(m)})\|$$
$$\leq\sum_{n_1=1}^{\infty}\cdots\sum_{n_m=1}^{\infty}|\lambda_{n_1}^{(1)}|\dots|\lambda_{n_m}^{(m)}|\,\|S\|<+\infty.$$

Therefore, $T\in\mathscr{L}_N(H_1,\dots,H_m;\mathbb{K})$. □

The converse is not true in general, as the following example shows.

Example 4.3. Let $T\in\mathscr{L}(^m l_2;\mathbb{K})$, $T(x_1,\dots,x_m)=\sum_{j=1}^{\infty}\left(\frac{1}{j}\right)^{m/2}(x_1\,|\,e_j)\dots(x_m\,|\,e_j)$, $m\geq 3$. We have $T\in\mathscr{L}_N(^m l_2;\mathbb{K})$ because

$$\sum_{j=1}^{\infty}\left|\frac{1}{j}\right|^{m/2}\|e_j\|\dots\|e_j\|\leq\sum_{j=1}^{\infty}\left|\frac{1}{j}\right|^{m/2}<+\infty.$$

But T is not in $\mathscr{L}(\mathscr{S}_1)(^m l_2;\mathbb{K})$; see Example 3.3.

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